Asymptotic Confidence Ellipses of Parameters for the Inverse Gaussian Distribution

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Abstract

In this article, we derive maximum likelihood equations and find Fisher information matrix to construct asymptotic confidence ellipses for parameters of the inverse Gaussian distribution. We use the coverage probabilities to compare with the confidence coefficient of 0.98. The investigation of the accuracy of the confidence ellipses are fulfilled via the Monte Carlo method. Four cases of sample sizes n (30, 100, 500, and 1,000) and six cases of λ are investigated at parameter μ which is set to be 1. R (2.15.2) software is used for our simulation study with 10,000 iterations. The results are as follows. The coverage probabilities of confidence ellipses for parameters of the inverse Gaussian distribution increase when sample size n is increased. They are also close to the confidence coefficient of 0.98 for all values of both parameters. In addition, the various values of the parameter λ when μ is 1 of the inverse Gaussian distribution give high coverage probabilities when n is large.

Keywords: Maximum Likelihood Estimate; Simultaneous Confidence Interval; Skewed Distribution.

1. Introduction

The inverse Gaussian distribution has been useful in statistics for a long time. Much progress has been made by a group of researchers, for instance, Erwin Schrodinger, Smoluchowski, Tweedie, Wald, Chhikara, Folk, Seshadri, etc. [1]. The inverse Gaussian distribution has two parameters. It is a continuous probability distribution. The probability density function of an inverse Gaussian distribution with parameters μ and λ (where μ is the mean and λ is the scale parameter) is given by [2]:,

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi}}x^{-3/2}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\},\ x>0.,\mu,\lambda>0.$$

This distribution has been used in a wide range of applications, most of which are based on the idea of the first passage time of Brownian motion. It is logical to use inverse Gaussian distribution as a lifetime model in studying life testing and reliability of a product or device [3]. The inverse Gaussian distribution is a useful statistical tool for biology, physics, engineering, finance and many other applications. For example, in tracer dynamics, emptiness of a dam, a purchase incidence model, the distribution of strike duration and many other [4]. Moreover, the

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Inverse Gaussian is the most appropriate statistical distribution when skewed data analysis is needed. Folks and Chhikara (1989) [5] explain that the application of the inverse Gaussian can meet part of the need for skewed data analysis [5].

There are two types of estimates for population parameters: point estimate and interval estimate. A point estimate calculate statistics from the sample data and can be considered a single number estimate of an unknown population parameter. A confidence interval estimate is a range of values for the population parameter with a predefined level of confidence that provides an upper and lower bound for a specific unknown population parameter [6]. Moreover, in the past decade, we see an increase of interest in research concerning confidence intervals in one-dimensional space. In two dimensions the confidence intervals are called confidence ellipses.

Consequently, in this research, we study the estimation for parameters of the inverse Gaussian distribution. We are going to construct confidence ellipses at 98 percent confidence level. Next, we investigate the accuracy of the confidence ellipses by the Monte Carlo method.

2. The Inverse Gaussian distribution

The classical parametrization of the inverse Gaussian distribution ($IG(\mu,\lambda)$) is a two-parameter family of continuous probability distribution with support on (0, ∞). According to Chhikara and Folks (1989) [5], the probability density function of an inverse Gaussian random variable X is

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi}}x^{-3/2}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\},\ \ x>0.$$

where $\mu>0$ and $\lambda>0$. The parameter μ is the mean of the distribution, and λ is a scale parameter. The characteristic function of an inverse Gaussian random variable X denoted by $C_X(t)$, is given by

$$C_X(t)=\exp\left\{\frac{\lambda}{\mu}\left[1-\left(\frac{2i\mu^2 t}{\lambda}\right)^{1/2}\right]\right\}.$$

The first four moments about zero are

$$\mu$$

$$\mu^2+\frac{\mu^2}{\lambda}$$

$$\mu^3+3\frac{\mu^4}{\lambda}+3\frac{\mu^5}{\lambda^2}\ \text{, and}$$

$$\mu^4+6\frac{\mu^5}{\lambda}+15\frac{\mu^6}{\lambda^2}+15\frac{\mu^7}{\lambda^3}.$$

3. Theoretical Results

3.1 Maximum Likelihood Method

For a random sample $X_1,X_2,\ldots,X_n$ from an inverse Gaussian population $IG(\mu,\lambda)$, the likelihood function followed by Folks and Chhikara (1989) [5] is

$$L(\mu,\lambda;x)=\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\left(\prod_1^n x_i^{-\frac{3}{2}}\right)\exp\left[-\lambda\sum_1^n\frac{(x_i-\mu)^2}{2\mu^2 x_i}\right],\ \mu>0,\ \lambda>0.$$

Therefore, the maximum likelihood estimators (MLE) $\hat{\mu}^{(MLE)}$ and $\hat{\lambda}^{(MLE)}$ of μ and λ are

$$\hat{\mu}=\bar{X}\ \ ,\ \bar{X}=\frac{1}{n}\sum_1^n X_i$$

$$\hat{\lambda} = \frac{n}{\sum_{1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)}.$$

3.2 The Fisher Information of Parameters for the Inverse Gaussian distribution

Let a random variable X be distributed as the $IG(\mu,\lambda)$. The Fisher information matrix of θ, where $\theta = (\mu,\lambda)$, is a two-dimensional vector of parameters, denoted by $I(\theta)$. We illustrate the Fisher information matrix about μ,λ as

$$I(\theta) = I(\mu,\lambda) = -E\begin{bmatrix} \frac{\partial^2}{\partial\mu^2}\ln f(x;\mu,\lambda) & \frac{\partial^2}{\partial\mu\partial\lambda}\ln f(x;\mu,\lambda) \\ \frac{\partial^2}{\partial\lambda\partial\mu}\ln f(x;\mu,\lambda) & \frac{\partial^2}{\partial\lambda^2}\ln f(x;\mu,\lambda) \end{bmatrix}$$

Therefore, the Fisher information matrix follows:

$$I(\theta) = I(\mu,\lambda) = \begin{bmatrix} \frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{1}{2\lambda^2} \end{bmatrix}$$

where θ is a two-dimensional parameter vector, and $\theta' = (\mu,\lambda)$. The Fisher information matrix for a sample size n is

$$I_n(\theta) = I_n(\mu,\lambda) = \begin{bmatrix} \frac{\lambda n}{\mu^3} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{bmatrix}.$$

3.3 The Covariance Matrix

The covariance matrix which equals the inverse of the Fisher information matrix is denoted by $\Lambda = I_n^{-1}(\theta)$. That is, as $n \to \infty$,

$$\hat{\theta}_n^{(MLE)} \sim N_2\left(\theta, I_n^{-1}(\theta)\right).$$

The inverse of the Fisher information matrix is computed as

$$\Lambda = I_n^{-1}(\mu,\lambda) = \begin{bmatrix} \frac{\mu^3}{\lambda n} & 0 \\ 0 & \frac{2\lambda^2}{n} \end{bmatrix}.$$

3.4 Asymptotic Normal Distribution

We consider the sequence of random variables $\hat{\theta}_n^{(MLE)}$. By the Delta method theorem, we can state that

$$\sqrt{n}\left(\hat{\theta}_n^{(MLE)} - \theta\right) \xrightarrow{d} X$$

where X has the bivariate normal distribution, denoted by $N_2\left(0, I^{-1}(\theta)\right)$

$$\sqrt{n}\left(\hat{\theta}_n^{(MLE)} - \theta\right) \xrightarrow{d} X \sim N_2\left(0, I^{-1}(\theta)\right)$$

where $\xrightarrow{d}$ denotes the convergence in distribution and

$$I^{-1}(\theta) = I^{-1}(\mu,\lambda) = \begin{bmatrix} \frac{\mu^3}{\lambda} & 0 \\ 0 & 2\lambda^2 \end{bmatrix}. \quad (1)$$

From (1) and because the inverse of the Fisher information matrix equals the covariance matrix, we get

$$\sqrt{n}\left(\hat{\theta}_n^{(MLE)} - \theta\right) \xrightarrow{d} X \sim N_2\left(0, I^{-1}(\theta)\right),$$

$$\frac{\sqrt{n}\left(\hat{\theta}_n^{(MLE)} - \theta\right)}{\sqrt{I^{-1}(\theta)}} \xrightarrow{d} Z \sim N_2\left(0, I_2\right), \text{ and}$$

$$\frac{\left(\hat{\theta}_n^{(MLE)} - \theta\right)}{\sqrt{I_n^{-1}(\theta)}} \xrightarrow{d} Z \sim N_2\left(0, I_2\right).$$

Therefore,

$$\left(\Lambda^{\frac{1}{2}}\right)^{-1}\left(\hat{\theta}_n^{(MLE)}-\theta\right) \xrightarrow{d} Z \sim N_2\left(0, I_2\right),$$

and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If Z_1, Z_2 are independent $N(0,1)$ random variables, then $Z' = (Z_1, Z_2)$ has the bivariate normal distribution $N_2(0, I_2)$ as

$$\left(\hat{\theta}_n^{(MLE)}-\theta\right)'\left(\Lambda\right)^{-\frac{1}{2}}\left(\Lambda\right)^{-\frac{1}{2}}\left(\hat{\theta}_n^{(MLE)}-\theta\right)=\left(\hat{\theta}_n^{(MLE)}-\theta\right)'\Lambda^{-1}\left(\hat{\theta}_n^{(MLE)}-\theta\right)$$

$$= Z_1^2 + Z_2^2 = \sum_1^2 Z_i^2$$

and is distributed as a Chi-square distribution with two degrees of freedom, denoted by χ_2^2.

3.5 Confidence Region for Parameters of the Inverse Gaussian Distribution

The $N_2\left(0, I^{-1}(\theta)\right)$ distribution assigns the probability $1-\alpha$ to the ellipse

$$\left\{\hat{\theta}_n^{(MLE)} : \left(\hat{\theta}_n^{(MLE)}-\theta\right)'\Lambda^{-1}\left(\hat{\theta}_n^{(MLE)}-\theta\right) \le \chi_2^2(\alpha)\right\}$$

where $\chi_2^2(\alpha)$ denotes the upper (100α)-th percentile of the χ_2^2 distribution. A $100(1-\alpha)\%$ confidence region for parameters $\theta' = (\mu, \lambda)$ of a two-dimensional normal distribution is the ellipse determined by all θ such that

$$\left(\hat{\theta}_n^{(MLE)}-\theta\right)'\Lambda^{-1}\left(\hat{\theta}_n^{(MLE)}-\theta\right) \le \chi_2^2(\alpha).$$

Since the inverse of covariance matrix is

$$\Lambda^{-1} = I_n(\mu,\lambda) = \begin{bmatrix} \frac{\lambda n}{\mu^3} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{bmatrix},$$ then

$$\left(\hat{\theta}_n^{(MLE)}-\theta\right)' I_n(\theta)\left(\hat{\theta}_n^{(MLE)}-\theta\right) \le \chi_2^2(\alpha)$$

where

$$\begin{bmatrix} \hat{\mu}_n^{MLE}-\mu & \hat{\lambda}_n^{MLE}-\lambda \end{bmatrix}_{1\times 2} \begin{bmatrix} \frac{\lambda n}{\mu^3} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{bmatrix}_{2\times 2} \begin{bmatrix} \hat{\mu}_n^{MLE}-\mu \\ \hat{\lambda}_n^{MLE}-\lambda \end{bmatrix}_{2\times 1} \le \chi_2^2(\alpha).$$

The $100\ (1-\alpha)\%$ confidence region for θ consist of all value (μ, λ) satisfying

$$\frac{\lambda n}{\mu^3}\left(\hat{\mu}_n^{MLE}-\mu\right)^2 + \frac{n}{2\lambda^2}\left(\hat{\lambda}_n^{MLE}-\lambda\right)^2 \le \chi_2^2(\alpha).$$

4. Computational Results

The simulation study is conducted for asymptotic confidence ellipses constructed for parameters of the inverse Gaussian distribution. We compare the coverage probabilities for confidence ellipses of parameters for inverse Gaussian distribution with the confidence coefficient of 0.98 and present the results in Table 4.1

The results of this study for different values of the parameters of the inverse Gaussian distribution and different sample sizes are shown only with high coverage probabilities for each sample size.

4.1 Sample Size n = 30

For $n = 30$, the confidence ellipses for $\mu = 1$ *and* $\lambda = 15$ of the inverse Gaussian distribution are shown in Fig. 1.

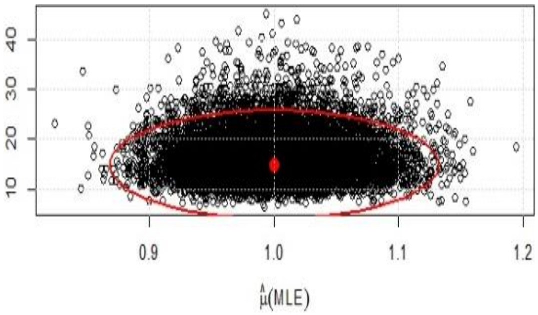

Fig.1. Confidence ellipses for parameters of the inverse Gaussian distribution when $n = 30$.

For $n = 30$, the maximum of the coverage probabilities is 0.9357 which occurs at $\mu = 1$ *and* $\lambda = 15$ as presented in Table 4.1. The coverage probabilities for $n = 30$ are lower than other coverage probabilities for the sample sizes $n = 100$, $n = 500$, and $n = 1000$.

4.2 Sample Size n = 100

For $n = 100$, the confidence ellipses for $\mu = 1$ *and* $\lambda = 15$ of the inverse Gaussian distribution are shown in Fig. 2.

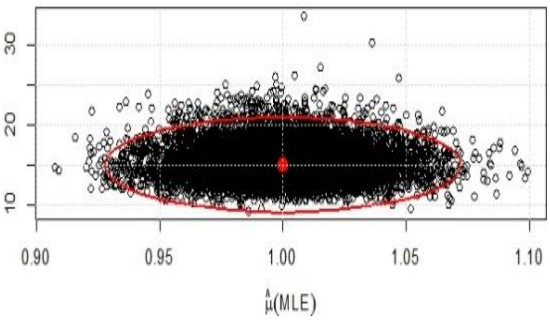

Fig.2. Confidence ellipses for parameters of the inverse Gaussian distribution when $n = 100$.

For $n = 100$, the maximum of the coverage probabilities is 0.9675 which occurs at $\mu = 1$ *and* $\lambda = 15$ as presented in Table 4.1.

4.3 Sample Size n = 500

For $n = 500$, the confidence ellipses for $\mu = 1$ *and* $\lambda = 1$ of the inverse Gaussian distribution are shown in Fig. 3.

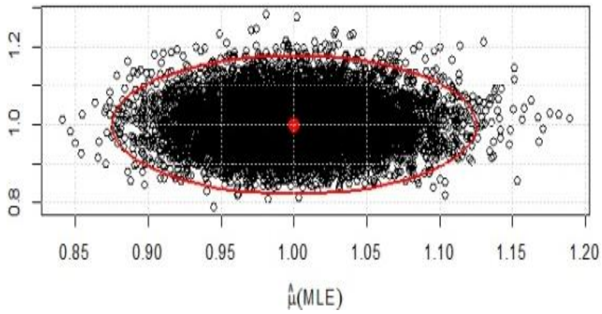

Fig.3. Confidence ellipses for parameters of the inverse Gaussian distribution when $n = 500$.

For $n = 500$, the maximum of the coverage probabilities is 0.9781 which occurs at $\mu = 1$ *and* $\lambda = 1$ as presented in Table 4.1. However, other values of the coverage probability are close to the confidence coefficient of 0.98 for all values of parameter λ.

4.3 Sample Size n = 1000

For $n = 1000$, the confidence ellipses for $\mu = 1$ *and* $\lambda = 20$ of the inverse Gaussian distribution are shown in Fig. 4.

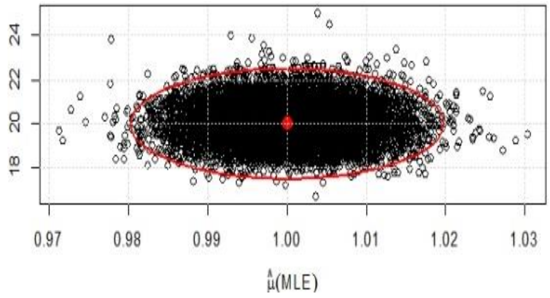

Fig.4. Confidence ellipses for parameters of the inverse Gaussian distribution when $n = 1000$.

For $n = 1000$, the maximum coverage probabilities is 0.9803 which occurs at $\mu = 1$ *and* $\lambda = 20$ as presented in Table 4.1. Other values of the coverage probability are closer to the confidence coefficient of 0.98 than other sample sizes for all values of parameter λ.

Table1. Maximum likelihood estimates of μ and λ and coverage probabilities for confidence ellipses at 98% confidence level.

n	μ	λ	Maximum Likelihood Estimates $\hat{\mu}_n^{MLE}$	$\hat{\lambda}_n^{MLE}$	Coverage Probabilities
30	1	1	1.00097	1.115050	0.9316
		3	0.99970	3.338670	0.9345
		5	1.00063	5.552180	0.9322
		10	0.99962	11.12991	0.9355
		15	0.99980	16.68642	**0.9357**
		20	0.99916	22.23127	0.9341
100	1	1	1.00074	4.031880	0.9654
		3	0.99972	3.099740	0.9635
		5	0.99924	5.153420	0.9670
		10	0.99940	10.31810	0.9657
		15	0.99973	15.44892	**0.9675**
		20	1.00018	20.64193	0.9659
500	1	1	0.99951	1.00536	**0.9781**
		3	1.00009	3.018610	0.9777
		5	1.00001	5.034990	0.9779
		10	1.00003	10.07004	0.9741
		15	0.9990	15.08658	0.9773
		20	1.00003	20.13996	0.9746
1000	1	1	1.00033	1.002680	0.9782
		3	1.00320	3.009740	0.9790
		5	1.00019	5.012180	0.9770
		10	1.00018	10.03262	0.9796
		15	1.00013	15.04985	0.9784
		20	1.00017	20.06536	**0.9803**

5. Examples

The following maintenance data were reported on active repair times (hours) for an airborne communication transceiver (Von Alven, 1964, p.156 cited in [5]) with samples sizes equal 45:

0.2	0.3	0.5	0.5	0.5	0.6	0.6
0.7	0.7	0.7	0.8	0.8	1.0	1.0
1.0	1.0	1.1	1.3	1.5	1.5	1.5
1.5	2.0	2.0	2.2	2.5	2.7	3.0
3.0	3.3	3.3	4.0	4.0	4.5	4.7
5.0	5.4	5.4	7.0	7.5	8.8	9.0
10.3	22.0	24.5				

The inverse Gaussian model was considered for these repair times and showed that it provides a good fit to the data. The observed values of the Kolmogorov-Smirnov test are 0.053 for the inverse Gaussians, indicating that model provides equally good fits.

The Maximum Likelihood Estimate of μ and λ are given by $\bar{x} = 3.67556$ and $\hat{\lambda} = 1.714833$. The covariance matrix of the outbreaks data is

$$\Lambda = \begin{bmatrix} 0.7111111 & 0 \\ 0 & 0.1777778 \end{bmatrix}.$$

Two pairs of the eigenvalue and eigenvector for Λ are

$$\lambda_1^{eigen} = 0.7111111, \quad e_1^{eigen} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

$$\lambda_2^{eigen} = 0.1777778, \text{ and}$$

$$e_2^{eigen} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The joint confidence ellipse is plotted in Fig. 5. The center is at $\theta' = (\ 4\ ,\ 2)$, and the half-lengths of the major and minor axes are given by

$$\sqrt{\lambda_1^{eigen}}\sqrt{\chi_2^2(0.02)} = \quad 2.358764$$

and

$$\sqrt{\lambda_2^{eigen}}\sqrt{\chi_2^2(0.02)} = 1.179382,$$

respectively. The axes lie along $e_1'^{eigen} = (\ -1\ ,\ 0\ )$ and $e_2'^{eigen} = (\ 0\ ,\ -1\ )$ when these vectors are plotted with θ as the origin. An indication of the elongation of the confidence ellipse is provided by the ratio of the lengths of the major and minor axes. This ratio is

$$\frac{2\sqrt{\lambda_1^{eigen}}\sqrt{\chi_2^2(0.02)}}{2\sqrt{\lambda_2^{eigen}}\sqrt{\chi_2^2(0.02)}} = \frac{\sqrt{\lambda_1^{eigen}}}{\sqrt{\lambda_2^{eigen}}} = 2.$$

Confidence Ellipse of Parameters for the Inverse Gaussian distribution

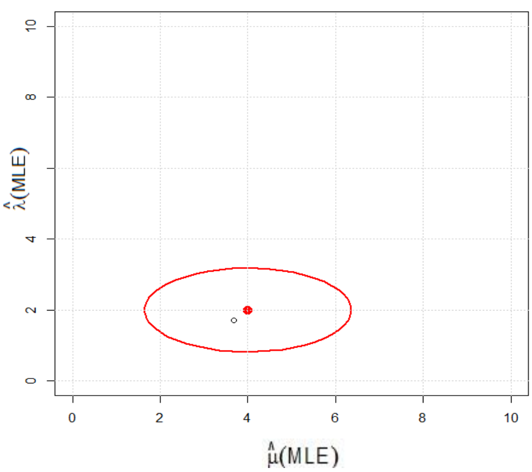

Fig. 5. A 98% confidence ellipse for μ and λ based on active repair times.

6. Conclusions

The coverage probabilities of confidence ellipses for parameters of the inverse Gaussian distribution increase when the sample size (n) increases. In addition, the coverage probabilities for confidence ellipses of parameters for various values of parameters λ are different only at the second or the third decimal place, and they are also close to the confidence coefficient of 0.98 for all values of parameters λ when the sample size increases.

The distribution we considered here has two parameters, and hence for simultaneous confidence set estimation we use ellipses. It would be interesting to consider a three-parameter distribution and try to construct confidence ellipsoids.

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8. References

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