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Original research article

A New Family of Distribution with Application on Two Real Datasets on Survival Problem

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ABSTRACT

In this paper we introduce a new Modi family of continuous probability distributions with application on patients suffering from disease and their survival times. The proposed distribution possesses a density function with three parameters and an inverted J-shape hazard rate function. We studied the nature of proposed distribution with the help of its mathematical and statistical properties. The probability density function of order statistics for this distribution is also obtained. We perform classical estimation of parameters by using the technique of maximum likelihood estimate. We apply it to two real datasets and show that it provides better fit than other well known distributions.

Keywords: Exponential distribution; Maximum likelihood estimation; Order statistics; Probability weighted moments; Renyi entropy

1. Introduction

Mathematical modeling of a real world problem enables us to better explain and understand it, and possibly enables us to regenerate it, either on a large scale, or on a simplified scale that exhibits only the vital parts of that phenomenon. Probability distribution models are helpful to capture these real life phenomena, which are learned or extracted directly from data gathered about them. The lifetime models exhibiting monotone and non-monotone failure rate properties have wide applications in the fields of engineering, life-sciences, finance and insurance, environmental sciences, medical sciences, biological studies, demography, actuaries and economics.

In the modern era the distribution theory has stressed solving problems faced by the researchers and developing differ-

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ent models so that lifetime data sets can be better analyzed and applied in different fields. Azzalini [1], Marshall and Olkin [2] and Gupta et al. [3] were the pioneers in this area. Eugene et al. [4] proposed a beta-generated method. Cooray and Ananda [5] proposed the composite method by combining two distributions. Shaw and Buckley [6] generated a new transformation method by adding a parameter. Cordeiro et al. [7] proposed a new class of distribution with the addition of two more shape parameters. This introduction of additional parameter(s) has been proved helpful in studying tail properties and also for providing a better goodness-of-fit of the distribution under study. Alzaatreh et al. [8] introduced the Transformed-Transformer method that allows for using any continuous probability density function as the generator. Modi and Gill [9] derived the Length-biased Weighted Maxwell Distribution. Tahir et al. [10] propose a new family of continuous distributions called the odd generalized exponential family. An extended-G geometric family was generated by Cordeiro, Silva and Ortega [11]. Cordeiro et al. [12] proposed the Exponentiated Weibull-H Family of distributions. Modi and Joshi [13] calculated distribution of product and ratio of t and Rayleigh random variables. Power Lindley-G Family of distributions is generated by Hassan and Nasr [14]. A compounding approach was given by Berreto-Souza et al. [15]. Ferreira and Steel [16] introduced the inverse probability integral transformation method. Pappas [17] presented the Alpha Logarithmic transformed method. Logarithmic transformed method was developed by Maurya et al. [18].

Realizing the need for more flexible lifetime distributions, we have proposed the Modi family of distribution. In particular, we have discussed Modi exponential distribution with three parameters which can be used to fit and analyse data in different fields. The organization of the paper is as follows: In Section 2, we define the Modi generator and in Section 3, we define the Exponential distribution. In Section 4, we provide the cumulative distribution function (CDF) and the probability density function (PDF) of the Modi exponential distribution. In Section 5, we find the expression for the survival function and the hazard rate function of the derived distribution. Mode and median of the proposed distribution are given in Section 6. Formulas to calculate moments and probability weighted moments of the Modi exponential distribution are given in Section 7 and Section 8, respectively. The Renyi entropy and distribution of the order statistics for the new distribution are discussed in Section 9 and Section 10, respectively. The method of maximum likelihood estimation is used to estimate the parameters of the derived distribution in Section 11. In Section 12, we show the application of the Modi exponential distribution on two real datasets and compare them with some well known distributions. We will need the following lemmas to derive the result of the derivations:

Lemma 1.1 (Eq. (1.110) in [19]). *If α is a positive real non integer and $|x| \leq 1$, then by binomial series expansion we have:*

$$(1-x)^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} x^j.$$

Lemma 1.2 (Eq. (3.423.3) in [19]). *For $a < 1$, $q > -1$, $Re(p) > 0$,*

$$\int_0^{\infty} \frac{x^q e^{-px}}{(1-ae^{-px})^2} dx = \frac{\Gamma(q+1)}{ap^{(q+1)}} \sum_{k=1}^{\infty} \frac{a^k}{k^q}.$$

2. Modi Family

We introduce a new Modi family of probability distributions to model lifetime data or survival data. The CDF $F(x)$ and PDF $f(x)$ of the Modi generator are, respectively, given by:

$$F(x) = \frac{\left(1+\alpha^{\beta}\right) S(x)}{\alpha^{\beta} + S(x)} \tag{2.1}$$

and

$$f(x) = \frac{\left(1+\alpha^{\beta}\right)\left(\alpha^{\beta} s(x)\right)}{\left\{\alpha^{\beta} + S(x)\right\}^{2}} \tag{2.2}$$

for all $x > 0$, where $\alpha > 0, \beta > 0$.

3. Exponential Distribution

The exponential distribution is a well-known distribution and has its importance in the study of growth, lifetime data, etc. Its modification and generalization in the form of exponentiated exponential (EE) distribution and generalized exponential distribution (GED) has been given by different mathematicians and statisticians. A continuous random variable X has exponential distribution, if its PDF $s(x)$ and CDF $S(x)$ are, respectively, given by:

$$S(x) = 1 - e^{-px} \tag{3.1}$$

and

$$s(x) = pe^{-px} \tag{3.2}$$

for all $x > 0$, where $p > 0$.

4. CDF and PDF of Modi Exponential Distribution

For the Modi generator using the CDF and PDF defined in Eq. (3.1) and Eq. (3.2), respectively, we propose a new Modi exponential distribution. Thus the CDF of the Modi exponential distribution can be defined as:

$$F(x) = \frac{\left(1+\alpha^{\beta}\right)\left(1-e^{-px}\right)}{\alpha^{\beta} + 1 - e^{-px}} \tag{4.1}$$

and its corresponding PDF is given by:

$$f(x) = \frac{p\alpha^{\beta} e^{-px}}{\left(1 - \frac{e^{-px}}{1+\alpha^{\beta}}\right)^{2}\left(1+\alpha^{\beta}\right)} \tag{4.2}$$

for all $x > 0$, where $\alpha > 0, \beta > 0, p > 0$.

Graphs of a distribution function and a density function of Modi exponential distribution for different combinations of values of its parameters p, β and α are shown in Fig. 1 and Fig. 2.

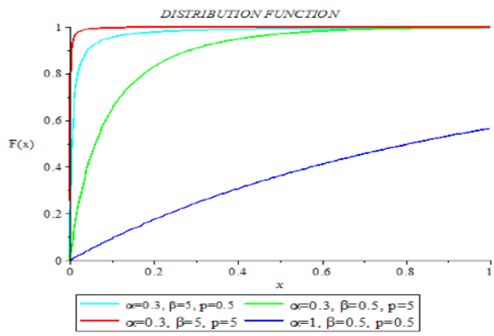

Fig. 1. Graph of a distribution function of Modi exponential distribution for different combinations of values of its parameters p, β and α.

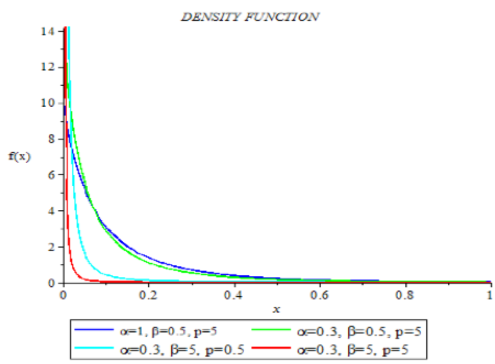

Fig. 2. Graph of a density function of Modi exponential distribution for different combinations of values of its parameters p, β and α.

5. Hazard Rate Function and Survival Function

The hazard rate function given by $h(x) = \dfrac{f(x)}{1 - F(x)}$ is an important measure to characterize life phenomenon. For the PDF and CDF given in Eq. (4.2) and Eq. (4.1), respectively, $h(x)$ takes the form:

$$h(x) = \frac{(1+\alpha^{\beta})\, p}{\alpha^{\beta} + 1 - e^{-px}} \qquad (5.1)$$

and its survival function is given by:

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= 1 - \frac{(1+\alpha^{\beta})(1-e^{-px})}{\alpha^{\beta} + 1 - e^{-px}} \\ &= \frac{\alpha^{\beta} e^{-px}}{\alpha^{\beta} + 1 - e^{-px}}. \end{aligned} \qquad (5.2)$$

Graphs of a hazard rate function and a survival function of Modi exponential distribution for different combinations of values of its parameters p, β and α are shown in Fig. 3 and Fig. 4.

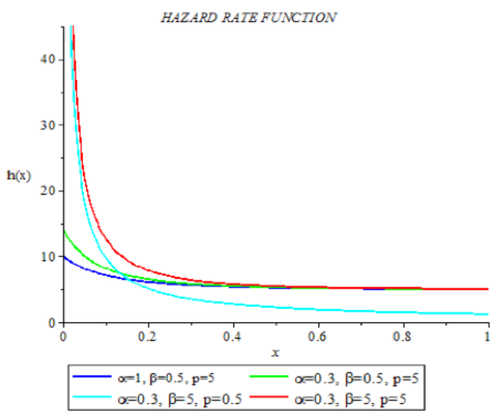

Fig. 3. Graph of a hazard rate function of Modi exponential distribution for different combinations of values of its parameters p, β and α.

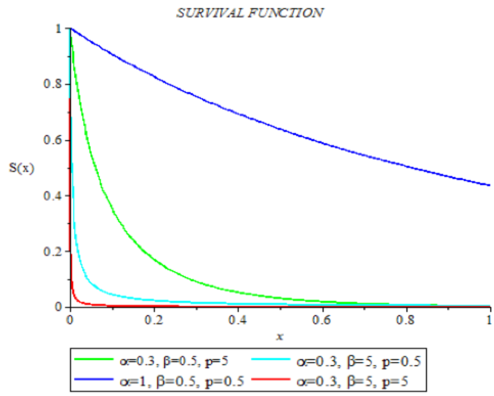

Fig. 4. Graph of a survival function of Modi exponential distribution for different combinations of values of its parameters p, β and α.

6. Mode and Median

If a random variable X has the PDF given by equation Eq. (4.2),

$$f(x) = \frac{p\alpha^{\beta} e^{-px}(1+\alpha^{\beta})}{(1 - e^{-px} + \alpha^{\beta})^2},$$

then the corresponding mode is given by $f'(x) = 0$, thus we obtain

$$\begin{aligned} f'(x) &= \frac{p\alpha^{\beta}(-pe^{-px})(1 + e^{-px} + \alpha^{\beta})}{(1 - e^{-px} + \alpha^{\beta})^3} = 0 \\ &\Rightarrow 1 + \alpha^{\beta} + e^{-px} = 0 \\ &\Rightarrow Mode = x = -\frac{1}{p}\ln\left(-1 - \alpha^{\beta}\right). \end{aligned} \qquad (6.1)$$

Median is obtained by:

$$\begin{aligned} \int_0^m f(x)dx &= \frac{1}{2} p\alpha^{\beta}\left(1 + \alpha^{\beta}\right) \\ &\quad \int_0^m \frac{e^{-px}}{(1 + \alpha^{\beta} - e^{-px})^2} dx \\ &= \frac{1}{2}\int_0^m \frac{e^{-px}}{(1 - e^{-px} + \alpha^{\beta})^2} dx \\ &= \frac{1}{2(1+\alpha^{\beta})\, p\alpha^{\beta}}. \end{aligned} \qquad (6.2)$$

Putting $1-e^{-px}+\alpha^{\beta}=t \Rightarrow pe^{-px}dx=dt$, we obtain

$$\int_{\alpha^{\beta}}^{1+\alpha^{\beta}-e^{-pm}} \frac{1}{pt^2}dt = \frac{1}{2p\alpha^{\beta}\left(1+\alpha^{\beta}\right)}$$
$$1+\alpha^{\beta}-e^{-pm} = \frac{2\alpha^{\beta}\left(1+\alpha^{\beta}\right)}{\left(1+2\alpha^{\beta}\right)}$$
$$e^{-pm} = \frac{\left(1+\alpha^{\beta}\right)}{\left(1+2\alpha^{\beta}\right)}. \quad (6.3)$$

7. Moments

If the PDF given by a random variable X in Eq. (4.2), then the corresponding r^{th} moment is given by:

$$\mu_r' = E\left(x^r\right) = \int_0^{\infty} x^r f(x)dx = \frac{p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)}\int_0^{\infty} x^r \frac{e^{-px}}{\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^2}dx.$$

Using Lemma 1.2, we obtain

$$\mu_r' = \frac{p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)}\frac{\Gamma(r+1)\left(1+\alpha^{\beta}\right)}{p^{(r+1)}}\sum_{k=1}^{\infty}\frac{1}{\left(1+\alpha^{\beta}\right)^k k^r}. \quad (7.1)$$

8. Probability Weighted Moments

Let a random variable X has the PDF given by Eq. (4.2), then the corresponding probability weighted moments are given by:

$$\rho = E\left(x^a (F(x))^b\right) = \int_o^{\infty} x^a (F(x))^b f(x)dx$$
$$= \frac{p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)}\int_0^{\infty} x^a \left[\frac{\left(1+\alpha^{\beta}\right)\left(1-e^{-px}\right)}{\alpha^{\beta}+1-e^{-px}}\right]^b \frac{e^{-px}}{\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^2}dx$$
$$= \frac{p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)}\sum_{k=0}^{\infty}\frac{(r+1+k)!}{(r+1)!k!}\int_0^{\infty} x^a \left(1-e^{-px}\right)^b \frac{e^{-px(k+1)}}{\left(1+\alpha^{\beta}\right)^k}dx$$
$$= \frac{p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)}\sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\frac{(b+1+k)!}{(b+1)!k!}\frac{(-1)^q}{\left(1+\alpha^{\beta}\right)^k}\binom{b}{q}\int_0^{\infty} x^a e^{-px(k+q+1)}dx. \quad (8.1)$$

Using Lemma 1.2, we obtain

$$\rho = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\frac{(b+1+k)!}{(b+1)!k!}\frac{(-1)^q p\alpha^{\beta}}{\left(1+\alpha^{\beta}\right)^{k+1}}\binom{b}{q}\frac{\Gamma(a+1)}{(kp+qp+p)^{(a+1)}}. \quad (8.2)$$

9. Renyi Entropy

The deviation of the uncertainty is measured by the entropy of a random variable. The Renyi entropy is defined by Renyi [20] as follows:

$$E_X(\nu) = \frac{1}{(1-\nu)}\ln\left(\int_{-\infty}^{\infty}\left(f_X(x)\right)^{\nu}dx\right),$$

$\nu > 0, \nu \neq 1$. Using the PDF defined in equation

$$E_X(\nu) = \frac{1}{(1-\nu)}\ln\left(\int_0^{\infty}\left(\frac{p\alpha^{\beta}e^{-px}}{\left(1+\alpha^{\beta}\right)\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^2}\right)^{\nu}dx\right)$$
$$= \frac{1}{(1-\nu)}\ln\left(\left(\frac{p\alpha^{\beta}}{1+\alpha^{\beta}}\right)^{\nu}\int_0^{\infty}\left(e^{-px}\right)^{\nu}\left\{1-\frac{e^{-px}}{1+\alpha^{\beta}}\right\}^{-2\nu}dx\right)$$

$$= \frac{1}{(1-\nu)} \ln \left(\left(\frac{p\alpha^{\beta}}{1+\alpha^{\beta}} \right)^{\nu} \frac{1}{(2\nu-1)!} \sum_{l=0}^{\infty} \frac{(2\nu-1+l)!}{l!} \int_0^{\infty} e^{-p\nu x} \left\{ \frac{e^{-px}}{1+\alpha^{\beta}} \right\}^{l} dx \right)$$

$$= \frac{1}{(1-\nu)} \ln \left(\left(\frac{p\alpha^{\beta}}{1+\alpha^{\beta}} \right)^{\nu} \frac{1}{(2\nu-1)!} \sum_{l=0}^{\infty} \frac{(2\nu-1+l)!}{(1+\alpha^{\beta})^{l} l!} \int_0^{\infty} e^{-px(\nu+l)} dx \right).$$

Solving this integral, we obtain

$$E_X(\nu) = \frac{1}{(1-\nu)} \ln \left(\left(\frac{p\alpha^{\beta}}{1+\alpha^{\beta}} \right)^{\nu} \frac{1}{(2\nu-1)!} \sum_{l=0}^{\infty} \frac{(2\nu-1+l)!}{(1+\alpha^{\beta})^{l} p(\nu+l) l!} \right). \quad (9.1)$$

10. Order Statistics

In this section, we derive a compact form expression for the PDF of the ith order statistic of the Modi exponential distribution.

Let $X_1, X_2, X_3, \ldots, X_n$ be a simple random sample from the Modi exponential distribution with CDF and PDF given by Eq. (4.1) and Eq. (4.2), respectively.

Let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample. We now give the PDF of $X_{r:n}$, say $f_{r:n}(x)$ and the p^{th} moments of $X_{r:n}$, $i = 1, 2, \ldots, n$. The PDF of the r^{th} order statistics $X_{r:n}$, $r = 1, 2, \ldots, n$ is given by (see, David [21])

$$f_{r:n}(x) = C_{r:n}[F(x;\alpha,\beta,p)]^{r-1} [1 - F(x;\alpha,\beta,p)]^{n-r} f(x;\alpha,\beta,p) \quad (10.1)$$

for all $x > 0$, where F and f are given by Eq. (4.1) and equation Eq. (4.2), respectively, and $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$. Thus, using Lemma 1.1, we obtain

$$f_{r:n}(x) = C_{r:n} \sum_{s=0}^{\infty} (-1)^s \binom{n-r}{s} [F(x;\alpha,\beta,p)]^{r+s-1} f(x;\alpha,\beta,p)$$

$$= C_{r:n} \sum_{s=0}^{\infty} (-1)^s \binom{n-r}{s} \left[\frac{\left(1+\alpha^{\beta}\right)\left(1-e^{-px}\right)}{\alpha^{\beta}+1-e^{-px}} \right]^{(r+s-1)} \frac{p\alpha^{\beta} e^{-px}}{(1+\alpha^{\beta})\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^2}$$

$$= C_{r:n} \sum_{s=0}^{\infty} (-1)^s \binom{n-r}{s} \frac{p\alpha^{\beta}}{(1+\alpha^{\beta})} e^{-px} (1-e^{-px})^{r+s-1} \left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^{-(r+s+1)}. \quad (10.2)$$

11. Maximum Likelihood Estimators

Let X be a random variable with the PDF of Modi exponential distribution defined as:

$$f(x) = \frac{p\alpha^{\beta} e^{-px}}{(1+\alpha^{\beta})\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^2}.$$

Then its log-likelihood function is defined by:

$$L(x;\alpha,\beta,p) = n \ln p + n\beta \ln \alpha - n \ln\left(1+\alpha^{\beta}\right) - p \sum_{i=0}^{\infty} x_i - 2 \sum_{i=0}^{\infty} \ln \left\{ 1 - \frac{e^{-px_i}}{1+\alpha^{\beta}}. \right\} \quad (11.1)$$

Thus the non-linear normal equations are given as follows:

$$\frac{\partial L(x;\alpha,\beta,p)}{\partial p} = \frac{n}{p} - \sum_{i=0}^{\infty} x_i - 2 \sum_{i=0}^{\infty} \frac{x_i e^{-px_i}}{1+\alpha^{\beta}-e^{-px_i}}, \quad (11.2)$$

$$\frac{\partial L(x;\alpha,\beta,p)}{\partial \beta} = n \ln \alpha - \frac{n\alpha^{\beta} \ln \alpha}{1+\alpha^{\beta}} - \sum_{i=0}^{\infty} x_i$$

$$- 2\sum_{i=0}^{\infty}\frac{\alpha^{\beta}\ln\alpha.e^{-px_i}}{(1+\alpha^{\beta})(1+\alpha^{\beta}-e^{-px_i})}, \quad (11.3)$$

$$\frac{\partial L(x;\alpha,\beta,p)}{\partial\alpha} = \frac{n\beta}{\alpha}-\frac{n\beta\alpha^{\beta-1}}{1+\alpha^{\beta}} - 2\sum_{i=0}^{\infty}\frac{\beta\alpha^{\beta-1}.e^{-px_i}}{(1+\alpha^{\beta})(1+\alpha^{\beta}-e^{-px_i})}. \quad (11.4)$$

We can estimate the unknown parameter by the method of maximum likelihood by equating non-linear Eqs. (11.2)-(11.4) to zero and solving them simultaneously.

12. Application to Real Life Data

In this section, the proposed Modi exponential distribution is applied to two real data sets. We observe its flexibility over some well known existing distributions. The results for the analysis in this present study are obtained using R software. We have also calculated the Akaike Information Criteria (AIC) and p-value for the considered distributions to observe their fit. Meanwhile, the distribution with the highest log-likelihood value or the lowest AIC is considered the best. The PDF of the distributions taken are as follows:

- Burr-XII distribution:

$$f(x)=\frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-(k+1)}$$

- Log-logistic distribution:

$$f(x)=\frac{\beta\left(\frac{x}{\alpha}\right)^{\beta-1}}{\alpha\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}}$$

- Dagum distribution:

$$f(x)=\beta\lambda\delta x^{-(1+\delta)}\left(1+\lambda x^{-\delta}\right)^{-(1+\beta)}$$

- Modi exponential distribution:

$$f(x)=\frac{p\alpha^{\beta}e^{-px}}{(1+\alpha^{\beta})\left(1-\frac{e^{-px}}{1+\alpha^{\beta}}\right)^{2}}$$

- Weibull distribution:

$$f(x)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}.\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)$$

Data Set 1: This data set represents the survival times (in weeks) of 33 patients suffering from acute myelogeneous leukemia. These data have been studied by Feigl and Zelen [22]. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, and 43. For this data, we shall compare the proposed Modi exponential distribution to some well-known distributions. Let us assume the hypothesis at α=1% LOS,

H_0: The data follow the Modi exponential distribution.

H_1: The data do not follow the Modi exponential distribution.

Data Set 2: This dataset consists of 50 failure times of devices analyzed by Aarset [23]. The data are 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85,85, 86, and 86.

The ML estimate of the Modi exponential distribution parameters and AIC value for Data Set 1 and Data Set 2 are given in Tables 1, 2.

For both sets of data, the Modi exponential distribution has the highest log-likelihood value among other well known distributions and the lowest AIC value, thus making it fit better than the Weibull distribution, Dagum distribution, Log-Logistic distribution and Burr-XII distribution. The AIC can be calculated using AIC $= -2\log eL + 2k$, where logeL denotes the log-likelihood function calculated using maximum likelihood estimates; k is the number of parameters. Since p-value $> \alpha$, we cannot reject the null hypothesis and hence assume that data follows Modi exponential distribution.

13. Conclusion

In this paper, we have introduced a new family of distributions named the Modi family. The expressions for the CDF and PDF of exponential distribution are calculated. Further the

Table 1. The ML estimate of the Modi exponential distribution parameters and AIC value for Data Set 1.

Distributions	Estimates	Log-likelihood	D	p-value	AIC
Modi Exponential distribution	$p = 0.01278204$ $\beta = 0.75471975$ $\alpha = 3.55540465$	-153.6036	0.26426	0.01992	313.2072
Weibull distribution	$\alpha = 0.7293247$ $\beta = 26.9335397$	-155.2139	0.15165	0.4338	314.4278
Log-Logistic distribution	$\alpha = 17.758404$ $\beta = 1.084977$	-155.3724	0.13744	0.5612	314.7448
Dagum distribution	$\beta = 1.5283123$ $\lambda = 7.7974312$ $\delta = 0.9400119$	-155.4593	0.12719	0.6598	316.9186
Burr-XII distribution (3-parameter)	$\alpha = 12.2087094$ $c = 1.1616230$ $k = 0.8157016$	-155.7403	0.9995	2.2e-16	317.4806

Table 2. The ML estimate of the Modi exponential distribution parameters and AIC value for Data Set 2.

Distributions	Estimates	Log-likelihood	D	p-value	AIC
Modi exponential distribution	$p = 0.02019555$ $\beta = 1.97985533$ $\alpha = 3.25480171$	-241.4026	0.18124	0.07488	488.8052
Weibull distribution	$\alpha = 0.8960972$ $\beta = 4.5187333$	-243.3942	0.27867	0.0008482	490.7884
Log-logistic distribution	$\alpha = 29.965533$ $\beta = 1.088868$	-251.1021	0.24086	0.006048	508.2042
Dagum distribution	$\beta = 0.9899246$ $\lambda = 19.8390406$ $\delta = 0.9340654$	-252.0279	0.23424	0.008283	510.0558
Burr-XII distribution (3-parameter)	$\alpha = 19.3996776$ $c = 1.0170287$ $k = 0.9688708$	-252.7251	0.99821	2.2e-16	511.4502

expressions for the CDF and PDF of Modi exponential distribution are also derived. We have studied the mathematical and statistical properties for the derived distribution. From graphs drawn for PDF of derived distribution for different combinations of the parameters, we observe that it has reverse "J" shape density function. The graphs for the survival function and hazard rate function for new distribution are also drawn. Its mode, median and the probability weighted moments are calculated. The expression for its r^{th} moment of derived distribution is given in Eq. (7.1). We have also derived the expressions for the Renyi entropy and the PDF of its r^{th} order statistics. The estimation of parameters is done using the method of MLE. Moreover, the distribution is fitted to two real data sets and compared with the other well known distributions. Results show that the Modi Exponential distribution provides a better fit than some other well known distributions.

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