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Statistical Deferred Euler Summability Mean and Associated Korovokin-type Approximation Theorem

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ABSTRACT

Statistical convergence has recently attracted the wide-spread attention of researchers due mainly to the fact that it is more general than the classical convergence. As far as the recent research on the theory and applications of summability is concerned, two basic concepts, namely statistical convergence and Korovokin-type approximation theorems play a very vital role. In the present paper, we introduce the notions of deferred Euler statistical convergence as well as statistical deferred Euler summability means and establish some inclusion relations between them. Furthermore, we prove a Korovokin-type approximation theorem based on our proposed mean, and we also show that our theorem is stronger than the classical versions of Korovokin-type approximation theorem.

Keywords: Banach space; Deferred Euler statistical convergence; Euler mean; Korovokin-type approximation theorem; Statistical deferred Euler summability

1. Introduction and Motivation

The theory of summability arose from the process of summation of series and the significance of the concept of summability has been rightly demonstrated in varying contexts, e.g. in Fourier analysis, theory of functions, sequence space and many other fields. In the study of sequence space, classical convergence has got innumerable applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition, that is, every element of the sequence needs to be in some neighborhood (arbitrarily small) of the limit. However, such restrictions are relaxed in statistical convergence, where a set

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having a few elements that are not in the neighborhood of the limit is discarded subject to the condition that the natural density of the set to be discarded is zero, and at the same time the condition of convergence is valid for the majority of the other elements. The notion of statistical convergence was initially studied by Fast [1] and Steinhaus [2]. Recently, statistical convergence has been a dynamic research area due basically to the fact that it is broader than classical convergence and such theory is discussed for the study in the areas of (for instance) Number Theory, Functional Analysis and Approximation Theory.

The theory of approximation of functions has been originated from a well-known theorem of Weierstrass, and it has become an exciting interdisciplinary field of study for last 130 years. Later, the theory of approximation was enriched by Korovkin-type approximation results. Korovkin-type theorems furnish simple and useful tools for ascertaining whether a given sequence of positive linear operators, acting on some function space, is an approximation process or, equivalently, converge strongly to the identity operator. Roughly speaking, these theorems exhibit a variety of test subsets of functions which guarantee that the approximation (or the convergence) property holds in the whole space provided it holds for them. The custom of calling these kinds of results "Korovkin-type theorems" refers to Korovkin who in 1953 discovered such a property for the functions 1, x and x^2 in the space $C([0,1])$ of all continuous functions on the real interval $[0,1]$ as well as for the functions 1, $\cos x$ and $\sin x$ in the space of all continuous 2π-periodic functions on the real line.

After this discovery, several mathematicians have undertaken the programme of extending Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. Such developments generated a theory which is now-a-days referred to as Korovkin-type approximation theory. This theory has fruitful connections with real analysis, functional analysis, harmonic analysis, measure theory and probability theory, summability theory, etc. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool. Even today, the development of Korovkin-type approximation theory is far from complete, especially in those parts of it that concern limit operators different from the identity operator. Recently, the Korovkin-type approximation results based on statistical convergence and statistical summability under different mean have been a field of interest for many researchers. For more details, see the recent works [3–9].

We now present below some basic concepts and definitions which are needed in our present study. Let N be the set of natural numbers and also let $K \subseteq N$. Let

$$K_n = \{m : m \leq n \text{ and } m \in K\}$$

and suppose that the cardinality of K_n is $|K_n|$. Then the asymptotic or natural density of K_n is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n} \tag{1.1}$$

provided the limit exists.

We may recall here that, a given sequence (x_n) is convergent statistically to a number ℓ if, for each $\varepsilon > 0$,

$$K_\varepsilon = \{m \in \mathbb{N} : |x_m - \ell| \geq \varepsilon\}$$

has zero asymptotic density (see [2, 5]). This means that, for each $\varepsilon > 0$, we have

$$d(K_\varepsilon) = \lim_{n\to\infty} \frac{|K_\varepsilon|}{n} = 0. \tag{1.2}$$

It is then written as $stat \lim_{n\to\infty} x_n = \ell$. We present below an example to illustrate that every convergent sequence is statistically convergent but not conversely.

Example 1.1. Let us consider a sequence (x_n) by

$$x_n = \begin{cases} \frac{1}{6}, & m = n^2 \text{ for all } m \in \mathbb{N}, \\ \frac{1}{n+1}, & \text{otherwise.} \end{cases}$$

Here, the sequence (x_n) is statistically convergent to 0 but not classically convergent.

In the year 2018, Srivastava et al. [10], introduced the concept of deferred weighted statistical convergence and later they also established the idea of deferred Norlund equi-statistical statistical convergence (see [11]). In this direction we may refer some current results, see [12–14].

Motivated essentially by the above mentioned works, in the present paper we introduce the concept of deferred Euler statistical convergence as well as statistical deferred Euler summability means, and establish some inclusion relations between them. Moreover, we prove a Korovokin-type approximation type theorem based upon our proposed mean.

2. Preliminaries and Definitions

Recall that the Euler mean of order q, (E, q)- summability mean [15] is given by

$$t_n = \frac{1}{(1+q)^n} \sum_{m=1}^{n} \binom{n}{m} q^{n-m} x_m$$

for all $n \in \mathbb{N}$. We now introduce the following definitions.

Definition 2.1. A sequence (x_n) is said to be Euler statistically convergent to ℓ if, for each $\varepsilon > 0$,

$$\{m : m \leq (1+q)^n \text{ and } q^{n-m}|x_m - \ell| \geq \varepsilon\}$$

has zero natural density. This means that, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(1+q)^n} |\{m : m \leq (1+q)^n \text{ and } q^{n-m}|x_m - \ell \geq \varepsilon\}| = 0.$$

Also, we write, $stat_E \lim_{n\to\infty} x_n = \ell$.

Next, we suppose that (a_n) and (b_n) are sequences of non-negative integers such that $a_n < b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = \infty$. The above conditions are known as the regularity conditions for the deferred Euler mean (see Agnew [16]). Next, we define the deferred Euler mean $D(E, q)$, as

$$\phi_n = \frac{1}{(1+q)^{b_n}} \sum_{m=a_n+1}^{b_n} \binom{b_n}{m} q^{b_n-m} x_m$$

for all $n \in \mathbb{N}$.

Remark 2.2. *If $a_n = 0$ and $b_n = n$ for all $n \in \mathbb{N}$, then the ϕ_n-mean reduces to the Euler (E, q)-mean. Moreover, if $a_n = 0$, $b_n = n$ for all $n \in \mathbb{N}$ and $q = 1$, then the ϕ_n-mean coincides with the Euler mean of order one, that is, the $(E, 1)-$mean.*

Definition 2.3. A sequence (x_n) is said to be deferred Euler $D(E, q)$ statistically convergent to ℓ if, for each $\varepsilon > 0$,

$$K_\varepsilon = \left\{m : m \leq (1+q)^{b_n} \text{ and } q^{b_n-m} x_m - \ell| \geq \varepsilon\right\}$$

has zero natural density, that is,

$$\lim_{n\to\infty} \frac{|K_\varepsilon|}{(1+q)^{b_n}} = 0.$$

We write it as $stat_{D(E)} x_n = \ell$.

Definition 2.4. A sequence (x_n) is said to be statistically deferred Euler $D(E, q)$-summable to ℓ if, for each $\varepsilon > 0$,

$$d_{D(E)}(F_\varepsilon) = 0,$$

where

$$F_\varepsilon = \{m : m \in \mathbb{N}\text{and}|\phi_n - \ell| \geq \varepsilon\}.$$

We write it as $DE(stat)\lim_{n\to\infty} x_n = \ell$.

Theorem 2.5. *Suppose that*

$$q^{b_n - m}|x_n - \ell| \leq M$$

for all $n \in \mathbb{N}$. If a sequence (x_n) is deferred Euler statistically convergent to ℓ, then it is statistically deferred Euler $D(E, q)$-summable to ℓ, but the converse is not true.

Proof. Suppose (x_n) is deferred Euler statistically convergent to ℓ, this implies that

$$\lim_{n\to\infty} \frac{|K_\varepsilon|}{(1+q)^{b_n}} = 0.$$

We then obtain,

$$\begin{aligned} &|\phi_n - \ell| \\ &= \left| \frac{1}{(1+q)^{b_n}} \sum_{m=a_n+1}^{b_n} q^{b_n - m}(x_m - \ell) \right| \\ &\leq \frac{1}{(1+q)^{b_n}} \sum_{\substack{m=a_n+1 \\ m\in K_\epsilon}}^{b_n} q^{b_n - m} \left|(x_m - \ell)\right| \\ &\quad + \frac{1}{(1+q)^{b_n}} \sum_{\substack{m=a_n+1 \\ m\notin K_\epsilon}}^{b_n} q^{b_n - m} \left|(x_m - \ell)\right| \\ &\leq \frac{M}{(1+q)^{b_n}} |K_\varepsilon| + \varepsilon \to \varepsilon. \end{aligned}$$

Clearly, $\phi_n \to \ell$ as $n \to \infty$. Thus, the sequence (x_n) is statistically deferred Euler, $D(E, q)$- summable to ℓ. □

Now, we consider the following counter example to see that it is not true conversely.

Example 2.6. Consider the sequences $(a_n) = (2n)$, $(b_n) = (4n)$ and suppose that (x_n) is a sequence given by,

$$x_n = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ -\frac{n}{2}, & n \text{ is even.} \end{cases}$$

It is trivially observed that (x_n) is neither classically convergent nor statistically convergent. Also, it is not Euler statistically convergent, but it is deferred Euler statistically summable to 0, that is, $D(E, q)$-summable to 0.

3. A Korovkin-type Theorem

In this section, for a sequence of positive linear operators on the class of continuous functions, $C[a, b]$, we prove a Korovkin type approximation theorem by using the notion of our proposed statistical deferred Euler summability mean and then considering the Bernstein operator. We demonstrate that our proposed method is stronger than classical and statistical versions of Korovokin-type approximation theorems in [1, 2, 17].

It is well-known that $C[a, b]$ is a Banach space with a norm defined by

$$\|g\| = \sup\{g(t) : a \leq t \leq b\}$$

for all $g \in C[a, b]$.

Suppose that $A : C[a, b] \to C[a, b]$ is a positive linear operator, that is, A satisfies the condition that $g \geq 0$ implies $A(g) \geq 0$. Also, the value of $A(g)$ at a point t is denoted by $A(g; t)$. Now, we establish the following theorem.

Theorem 3.1. *Suppose that $A : C[a, b] \to C[a, b]$ is a positive linear operator. Then*

$$DE(stat)\lim_{n\to\infty} \|A_n(g; t) - g(t)\| = 0 \tag{3.1}$$

if and only if

$$DE\ (stat) \lim_{n\to\infty} \left\| A_n(g_j;t) - g_j(x) \right\| = 0, \tag{3.2}$$

where $g_0(t) = 1$, $g_1(t) = t$ *and* $g_2(t) = t^2$.

Proof. Since $1, t$ and $t^2 \in C[a,b]$, the implication (3.1) $\Longrightarrow$(3.2) is trivially true. In order to prove Theorem 3.1, we first assume that (3.2) is true. Suppose that $g \in C[a,b]$ for all $t \in [a,b]$ As g is bounded in $[a,b]$, then there exists a constant $\lambda > 0$ such that

$$|g(t)| \leq \lambda$$

for all $t \in [a,b]$. This implies that

$$|g(y) - g(t)| \leq 2\lambda$$

for all $t, y \in [a,b]$. Clearly, g is continuous in $[a,b]$, that is, for given $\varepsilon \in> 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|g(y) - g(t)| < \varepsilon \text{ whenever } |y-t| < \delta. \tag{3.3}$$

Let us choose $\phi = (y-t)^2$. Then if $|y-t| \geq \delta$ for all $y, t \in [a,b]$, we obtain

$$|f(y) - f(t)| < \frac{2\lambda}{\delta^2}\phi(y,t). \tag{3.4}$$

From (3.3) and (3.4), we have

$$|g(y) - g(t)| < \varepsilon + \frac{2\lambda}{\delta^2}\phi,$$

where $\phi = (y-t)^2$. Now, we operate $A_n(1;t)$ to this inequality, since $A_n(g;t)$ is linear and monotone. Therefore,

$$\begin{aligned} & A_n(g;t) - g(t) \\ &= \varepsilon \left[A_n(1;t) - 1\right] + \varepsilon \\ &\quad + \left\{\left[A_n(y^2;t) - t^2\right] + 2t\left[A_n(y;t) - t\right]\right\} \\ &\quad + \frac{2\lambda}{\delta^2} t^2 \left[A_k(1;t) - 1\right] \\ &\quad + g(x)\left[A_n(1;t) - 1\right]. \end{aligned}$$

This implies that

$$\begin{aligned} & |A_n(g;t) - g(t)| \\ &= \varepsilon + \left(\varepsilon + \frac{2\lambda b^2}{\delta^2} + \lambda\right) |A_n(1;t) - 1| \\ &\quad + \frac{2\lambda}{\delta^2} \left|A_n(y^2;t) - x^2\right| \\ &\quad + \frac{4\lambda b}{\delta^2} |A_n(y;t) - t|, \end{aligned}$$

where $b = \max |x|$.

Next, taking supremum over $x \in [a,b]$, we obtain

$$\begin{aligned} & \|A_n(g;t) - g(t)\|_{C[a,b]} \\ &\leq \varepsilon + \mu \sum_{i=0}^{2} \left\| A_n(g_j;t) - g_j(t) \right\|, \end{aligned} \tag{3.5}$$

where

$$\mu = \left\{\varepsilon + \frac{2\lambda b^2}{\delta^2} + \lambda, \frac{2\lambda}{\delta^2}, \frac{4\lambda b}{\delta^2}\right\}.$$

Now on the basis of our proposed mean, we replace $A_n(g;\ t)$ by

$$L_n(g;t) = \frac{1}{(1+q)^{b_n}} \sum_{m=a_n+1}^{b_n} q^{b_n - m} A_m(g;\ t)$$

in (3.5). Next, for a given $\alpha > 0$, we consider $\varepsilon^1 > 0$ such that $0 < \varepsilon < \alpha$, and we define the sets

$$\begin{aligned} H &= \{n \in \mathbb{N} : n \leq N \\ &\qquad \text{and } |L_n(g;t) - g(t)| \geq \alpha\}, \\ H_j &= \Big\{n \in \mathbb{N} : n \leq N \\ &\qquad \text{and } \left|L_n(g_j;t) - g_j(t)\right| \geq \frac{\alpha - \varepsilon^1}{2\mu}\Big\}. \end{aligned}$$

We, easily find from equation (3.5) that,

$$H \leq \sum_{j=0}^{2} H_j.$$

Hence,

$$\frac{\|H\|}{n} \leq \sum_{j=0}^{2} \frac{\|H_j\|}{n}. \quad (3.6)$$

Finally, by using the above assumption for the implication in (3.2) and Definition 2.4, the term in the right hand side of (3.6) tends to zero as $n \to \infty$. Subsequently, we get

$$DE\,(stat) \lim_{n\to\infty} \|A_n(g;\ t) - g(t)\|_{C[a,b]} = 0.$$

Hence, the implication (3.1) holds true. Thus, the proof of Theorem 3.1 is completed. □

We now present below an example to illustrate the results of Theorem 3.1 by help of the Bernstein operator.

Example 3.2. Let g be a function defined on $C\,[0,1]$. Consider the operators,

$$B_n(g;\ t) = \sum_{m=0}^{n} g\left(\frac{m}{n}\right)\frac{n}{m}x^m(1-x)^{n-m}$$

for all $t \in [0,1]$. Here, we also have

$$\begin{aligned} B(1;\ t) &= 1, \\ B(y;t) &= t, \\ B(y^2;\ t) &= t^2 + \frac{(t-t^2)}{n}. \end{aligned}$$

Let $A : C\,[0,1] \to C\,[0,1]$ be a positive linear operator defined by

$$A_n(g;\ t) = (1 + x_n)B_n(g;\ t), \quad (3.7)$$

where (x_n) is a sequence same as in Example 2.6. It is trivial that the sequence (A_n) holds for our assertion (3.2) of Theorem 3.1. Thus, we obtain

$$DE\,(stat) \lim_{n\to\infty} \left\|A_n(g_j;\ t) - g_j(t)\right\| = 0,$$

Therefore, by Theorem 3.1

$$DE\,(stat) \lim_{n\to\infty} \|A_n(g;\ t) - g(t)\| = 0.$$

Moreover, (x_n) is neither statistically nor classically convergent. Thus, here the classical and statistical versions of Korovkin-type approximation theorem does not work for the operators defined in (3.7). So, this application clearly indicates that our Theorem 2 is a non-trivial generalization of the classical and statistical version of Korovkin-type approximation theorem (see [1, 17]).

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