P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : https://tci-thaijo.org/index.php/SciTechAsia
Science & Technology Asia
Vol.25 No.1 January - March 2020
Page: [38-45]
Original research article

Certain New Integral Properties of a Product of Generalized Special Functions Associated with Feynman Integrals

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Received 9 October 2019; Received in revised form 16 January 2020
Accepted 17 January 2020; Available online 26 March 2020

ABSTRACT

In the present study we deal with certain integral properties pertaining to a general class of polynomials, Aleph function, $\overline{H}$-function, and the generalized M-Series. The main outcomes presented here are nature wise basic and unified and are possibly useful in many fields particularly statistical mechanics, probability theory, electrical networks, etc. The integrals involved here include several types of Feynman integrals, the precise division of Gaussian models in statistical mechanics, and many other special case functions. These results provide numerous corresponding remarkable results, such as simpler special functions and polynomials, which are special cases of expressions.

Keywords: Aleph-function; General class of polynomials; Generalized M-series; $\overline{H}$-function

1. Introduction

The integrals of Feynman path type are implemented for domains of quantum cosmology where traditional methods may fail to connect [1, 2]. These types of integrals are a re-establishment of quantum mechanics and are more basic than the traditional manner in operator form. Simple and multiple variables of Feynman path integrals are helpful for the study and expansion of hyper geometric series. In statistical mechanics, these integrals give more fruitful results.

The generalized M-series [3] is defined as

$$\begin{aligned} {}_pM_q^{\alpha,\beta}(z) &= (u_1,\ldots,u_p,v_1,\ldots,v_q;z) \\ &= \sum_{\xi=0}^{\infty}\frac{(u_1)_\xi\ldots(u_p)_\xi\, z^\xi}{(v_1)_\xi\ldots(v_q)_\xi\Gamma(\alpha\xi+\beta)}. \end{aligned} \tag{1.1}$$

Here $\alpha,\beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $(u_i)_\xi$

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$(i = 1, 2, \ldots, p)$ and $(v_i)_\xi$ $(i = 1, 2, \ldots, q)$ are Pochammer symbols. As usual, if the numerator parameter u_j of the series (1.1) is a negative integer or zero, it will terminate to a polynomial in terms of z. The series is defined when none of the denominator parameters v_j for all $j = 1, 2, \ldots, q$ is a zero or negative integer. The series is convergent for all z if $q \geq p$; it is convergent for $|z| < \delta = \alpha^\alpha$ if $p = q + 1$ and divergent, if $q + 1 < q$ in (1.1). Depending upon the conditions of the parameters, the series can be a convergent one when $p = q + 1$ and $|z| = \delta$.

Inayat-Hussain [4] proposed the $\overline{H}$-function and presented it in the subsequent manner:

$$\overline{H}^{Q,S}_{T,U}(x) = \overline{H}^{Q,S}_{T,U}\left[x \left| \begin{matrix} (c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T} \\ (d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\psi}\,(s)\, x^s\, ds, \qquad (1.2)$$

where

$$\overline{\psi}\,(s) = \frac{\prod_{k=1}^{R}\Gamma(d_k - \delta_k s)}{\prod_{k=Q+1}^{U}\{\Gamma(1 - d_k + \delta_k s)\}^{D_k}} \cdot \frac{\prod_{k=1}^{S}\{\Gamma(1 - c_k + \gamma_k s)\}^{C_k}}{\prod_{k=S+1}^{T}\Gamma(c_k - \gamma_k s)} \qquad (1.3)$$

More details of the $\overline{H}$-function and its convergence conditions are specified in the papers [4–6].

The Aleph ($\aleph$) function was introduced by Südland et al. [7] as the Mellin-Barnes type integral [7–9]:

$$\aleph[z] = \aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[z \left| \begin{matrix} (k_j,\beta_j)_{1,N},\ldots,[\tau_j(k_j,\beta_j)]_{N+1,P_i} \\ (c_j,\alpha_j)_{1,M},\ldots,[\tau_j(c_j,\alpha_j)]_{M+1,Q_i} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;\lambda}(s)\, z^{-s}\, ds. \qquad (1.4)$$

Here $z \neq 0$, $i = \sqrt{-1}$ and

$$\Omega^{M,N}_{P_i,Q_i,\tau_i;\lambda}(s) = \frac{\prod_{j=1}^{M}\Gamma(c_j+\alpha_j s)}{\sum_{i=1}^{\lambda}\tau_i \prod_{j=m+1}^{Q_i}\Gamma(1-c_{ji}-\alpha_{ji}s)} \cdot \frac{\prod_{j=1}^{N}\Gamma\left(1-k_j-\beta_j s\right)}{\prod_{j=n+1}^{P_i}\Gamma\left(k_{ji}+\beta_{ji}s\right)}. \qquad (1.5)$$

The path of integration $L = L_{i\gamma\infty}$, $(\gamma \in \mathbb{R})$ increases from $\gamma - i\infty$ to $\gamma + i\infty$, in such a way that poles of $\Gamma\left(1-k_j-\beta_j s\right)$ for all $j = 1, 2, \ldots, N$ do not match with the poles of $\Gamma\left(c_j+\alpha_j s\right)$ for all $j = 1, 2, \ldots, M$. The condition

$$0 \leq N \leq P_i, \ \ 1 \leq M \leq Q_i, \ \ \tau_i > 0$$

for $i = \overline{1, \lambda}$ is satisfactory for the parameters P_i,Q_i as non-negative integers. All the parameters k_j,c_j,k_{ji},c_{ji} $\in \mathbb{C}$ and β_j,α_j,β_{ji},$\alpha_{ji} > 0$. In (1.5) the empty product is taken as unity while the conditions below are defined for the integral (1.4):

$$\varphi_\ell > 0, \quad |arg(z)| < \frac{\pi}{2}\varphi_\ell, \ \ell = \overline{1,\lambda}; \qquad (1.6)$$

$$\varphi_\ell > 0, \quad |arg(z)| < \frac{\pi}{2}\varphi_\ell, \ \Re\{\zeta_\ell\}+1 < 0, \qquad (1.7)$$

where

$$\varphi_\ell = \sum_{j=1}^{N}A_j + \sum_{j=1}^{M}B_j - \tau_\ell\left(\sum_{j=N+1}^{P_\ell}A_{j\ell} + \sum_{j=M+1}^{Q_\ell}B_{j\ell}\right), \qquad (1.8)$$

$$\begin{aligned}\zeta_\ell &= \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j \\ &+ \tau_\ell \left(\sum_{j=M+1}^{Q_\ell} b_{j\ell} - \sum_{j=N+1}^{P_\ell} a_{j\ell} \right) \\ &+ \frac{1}{2}\left(P_\ell - Q_\ell\right) \qquad (1.9)\end{aligned}$$

for $\ell = \overline{1,\lambda}$. An elaborate description of $\aleph$-function is given in the papers by Saxena et al. [10, 11].

The series representation of Aleph ($\aleph$)-function is as below:

$$\begin{aligned}\aleph[z] &= \aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda} \left[z \left| \begin{matrix} (k_j,\beta_j)_{1,N},\ldots,[\tau_j(k_j,\beta_j)]_{N+1,P_i} \\ (c_j,\alpha_j)_{1,M},\ldots,[\tau_j(c_j,\alpha_j)]_{M+1,Q_i} \end{matrix} \right. \right] \\ &= \sum_{t=1}^{M} \sum_{w=0}^{\infty} \frac{(-1)^w \phi'(\eta_{t,w}) z^{-\eta_{t,w}}}{\alpha_t\, w!}, \qquad (1.10)\end{aligned}$$

where

$$\phi'(\eta_{t,w}) = \frac{\prod_{j=1}^{M} \Gamma\left(c_j+\alpha_j\eta_{t,w}\right)}{\sum_{i=1}^{\lambda} \tau_i \prod_{j=M+1}^{Q_i} \Gamma\left(1-c_{ji}-\alpha_{ji}\eta_{t,w}\right)} \frac{\prod_{j=1}^{N} \Gamma\left(1-k_j-\beta_j\eta_{t,w}\right)}{\prod_{j=N+1}^{P_i} \Gamma\left(k_{ji}+\beta_{ji}\eta_{t,w}\right)} \qquad (1.11)$$

and $\eta_{t,w} = \frac{c_t+w}{\alpha_t}$, $P_i > Q_i$, $|z| < 1$. Srivastava [12, Eq. (1)] proposed a polynomial given by (1.12) known as the general class of polynomials

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-N_1)_{UR}}{R!} A_{V,R}\, x^R, \quad (1.12)$$

for all $U = 0, 1, 2, 3, \ldots$, where the coefficients $A_{V,R}(V, R \geq 0)$ are arbitrarily taken as constants, real or complex. In recent years several special functions and their integral properties have been studied and their applications demonstrated in scientific problems [13–15]. In view of great significance and utilizations of special functions in scientific and engineering processes, we study some new integral properties of products of special functions. Several interesting and useful particular cases are also pointed out.

2. Main Results

Theorem 2.1. *If*

$$Re(1+\sigma+R\alpha_1+\alpha_2\frac{d_k}{\delta_k}+\xi\alpha_3-\alpha_4\eta_{t,w}) > 0,$$

$$Re(1+\rho+R\beta_1+\beta_2\frac{d_k}{\delta_k}+\xi\beta_3-\beta_4\eta_{t,w}) > 0,$$

$$|arg(a_4)| < \frac{\pi}{2}\varphi_\ell, \varphi_\ell > 0, |arg(a_1)| < \frac{1}{2}N\pi,$$

$N > 0$, where $k = 1, 2, \ldots, Q$, $j = 1, 2, \ldots, M$, $\sigma > 0$, $\rho > 0$, $\alpha_i, \beta_j > 0$ for all $i, j = 1, 2, 3, 4$, $a_1 > 0$, $|a_3| < 1$, $p \leq q$. Then the following integral relation holds

$$\begin{aligned}&\int_0^1\int_0^1 \left(\frac{1-a}{1-ay}y\right)^\sigma \left(\frac{1-y}{1-ay}\right)^\rho \left(\frac{1-ay}{(1-a)(1-y)}\right) \\ &\cdot \overline{H}^{Q,S}_{T,U}\left[a_2\left(\frac{1-a}{1-ay}y\right)^{\alpha_2}\left(\frac{1-y}{1-ay}\right)^{\beta_2}\right] \\ &\cdot {}_pM_q^{\alpha,\beta}\left[a_3\left(\frac{1-a}{1-ay}y\right)^{\alpha_3}\left(\frac{1-y}{1-ay}\right)^{\beta_3}\right] \\ &\cdot \aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4\left(\frac{1-a}{1-ay}y\right)^{\alpha_4}\left(\frac{1-y}{1-ay}\right)^{\beta_4}\right] \\ &\cdot S^{u'}_{v'}\left[a_1\left(\frac{1-a}{1-ay}y\right)^{\alpha_1}\left(\frac{1-y}{1-ay}\right)^{\beta_1}\right] \\ &\qquad dady \\ &= \sum_{R=0}^{[v'/u']} \sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M} \frac{-v'_{u'R}}{R!} A_{v',R}\end{aligned}$$

$$\cdot\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\Gamma(\alpha\xi+\beta)}\frac{(-1)^w}{\alpha_t w!}$$

$$\cdot\phi'(\eta_{t,w})z^{-\eta_{t,w}}{a_1}^R{a_3}^\xi{a_4}^{-\eta_{t,w}}\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\cdot\left[a_2\left|\begin{matrix}(1-\sigma-\alpha_1 R-\alpha_3\xi+\alpha_4\eta_{t,w},\alpha_2;1)\\ (d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-\beta_1 R-\beta_3\xi+\beta_4\eta_{t,w},\beta_2;1)\\(1-\sigma-\rho-(\alpha_1+\beta_1)R-(\alpha_3+\beta_3)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T}\\+(\alpha_4+\beta_4)\eta_{t,w},\alpha_2+\beta_2;1)\end{matrix}\right]. \quad (2.1)$$

Proof. To validate the result, we write the series elaboration for the generalized M-series, the Aleph function with the general class of polynomials given by (1.1), (1.10), (1.12), respectively. Then, after changing the arrangement of integration and summations that are acceptable due to the absolute convergence involved in the procedure, integrating with respect to both the variables x and y between 0 and 1 and using a well-known result [16, p.145], we obtained the desired outcome (2.1) following some simplification. □

Theorem 2.2. *If*

$$Re(1+\rho+R\beta_1+\beta_2\frac{d_k}{\delta_k}+\xi\beta_3-\beta_4\eta_{t,w})>0,$$

$$Re(1+\sigma+R\alpha_1+\alpha_2\frac{(c_k-1)}{\gamma_k}+\xi\alpha_3-\alpha_4\eta_{t,w})<0,$$

$$Re(1+\rho+R\beta_1+\beta_2\frac{(c_k-1)}{\gamma_k}+\xi\beta_3-\beta_4\eta_{t,w})<0,$$

$$|arg(a_4)|<\frac{\pi}{2}\varphi_\ell,\varphi_\ell>0,|arg(a_1)|<\frac{1}{2}N\pi,$$

$N>0$, where $k=1,2,\ldots,Q$, $j=1,2,\ldots,M$, $\sigma>0$, $\rho>0$, $\alpha_i,\beta_j>0$ for all $i,j=1,2,3,4$, $a_1>0$, $|a_3|<1$, $p\le q$. Then the following integral relation exists

$$\int_0^\infty\int_0^\infty\psi(u+v)u^{\sigma-1}v^{\rho-1}s^{u'}_{v'}\left[a_1u^{\alpha_1}v^{\beta_1}\right]$$

$$\cdot\overline{H}^{Q,S}_{T,U}\left[a_2u^{\alpha_2}v^{\beta_2}\right]{}_p\overset{\alpha,\beta}{M}{}_q\left[a_3u^{\alpha_3}v^{\beta_3}\right]$$

$$\cdot\aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4u^{\alpha_4}v^{\beta_4}\right]dudv$$

$$=\sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\cdot\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\Gamma(\alpha\xi+\beta)}\frac{(-1)^w}{\alpha_t w!}$$

$$\cdot\phi'(\eta_{t,w})z^{-\eta_{t,w}}{a_1}^R{a_3}^\xi{a_4}^{-\eta_{t,w}}$$

$$\int_0^\infty\psi(z)z^{\sigma+\rho+(\alpha_1+\beta_1)R+(\alpha_3+\beta_3)\xi}$$

$$z^{-(\alpha_4+\beta_4)\eta_{t,w}-1}dz\cdot\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\cdot\left[a_2z^{\alpha_2+\beta_2}\left|\begin{matrix}(1-\sigma-\alpha_1R-\alpha_3\xi+\alpha_4\eta_{t,w};\alpha_2;1)\\(d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-\beta_1R-\beta_3\xi+\beta_4\eta_{t,w};\beta_2;1)\\ [1-\sigma-\rho-(\alpha_1+\beta_1)R-(\alpha_3+\beta_3)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T}\\+(\alpha_4+\beta_4)\eta_{t,w]};\alpha_2+\beta_2;1)\end{matrix}\right]. \quad (2.2)$$

Proof. To establish the result (2.2), we used the equations (1.1), (1.10) and (1.12) then changed the order of integration and sums that are acceptable due to the absolute convergence involved within the procedure. Then, doing the integration with respect to both the variables u and v between the limits 0 and ∞ and using a well-known result [16, p.177], we arrived at the desired outcomes. □

Theorem 2.3. *If*

$$Re(1+\sigma+R\alpha_1+\alpha_2\frac{d_k}{\delta_k}+\xi\alpha_3-\alpha_4\eta_{t,w})>0,$$

$$Re(1+\rho+R\beta_1+\beta_2\frac{d_k}{\delta_k}+\xi\beta_3-\beta_4\eta_{t,w})>0,$$

$$Re(1+\sigma+R\alpha_1+\alpha_2\frac{d_k}{\delta_k}+\xi\alpha_3-\alpha_4\eta_{t,w})>0,$$

$|arg(a_4)| < \frac{\pi}{2}\varphi_\ell, \varphi_\ell > 0, |arg(a_1)| < \frac{1}{2}N\pi,$ $N > 0$, *where* $k = 1, 2, \ldots, Q$, $j = 1, 2, \ldots, M$, $\sigma > 0$, $\rho > 0$, $\alpha_i, \beta_j > 0$ *for all* $i, j = 1, 2, 3, 4$, $a_1 > 0$, $|a_3| < 1$, $p \leq q$. *Then the following integral relation holds*

$$\int_0^1\int_0^1 \phi(\gamma\delta)(1-\gamma)^{\sigma-1}(1-\delta)^{\rho-1}v^{\sigma} \cdot \overline{H}^{Q,S}_{T,U}\left[a_2(1-\gamma)^{\alpha_2}(1-\delta)^{\beta_2}v^{\alpha_2}\right] \cdot {}_pM_q^{\alpha,\beta}\left[a_3(1-\gamma)^{\alpha_3}(1-\delta)^{\beta_3}\right] \cdot \aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4(1-\gamma)^{\alpha_4}(1-\delta)^{\beta_4}v^{\alpha_4}\right] \cdot s^{u'}_{v'}\left[a_1(1-\gamma)^{\alpha_1}(1-\delta)^{\beta_1}v^{\alpha_1}\right] d\gamma d\delta$$

$$= \sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R} \cdot \frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\Gamma(\alpha\xi+\beta)}\frac{(-1)^w}{\alpha_t w!} \cdot \phi'(\eta_{t,w})z^{-\eta_{t,w}}a_1{}^R a_3{}^\xi a_4{}^{-\eta_{t,w}} \cdot \int_0^1 \phi(z)(1-z)^{\sigma+\rho+(\alpha_1+\beta_1)R+(\alpha_3+\beta_3)\xi} (1-z)^{-(\alpha_4+\beta_4)\eta_{t,w}-1}dz \cdot \overline{H}^{Q,S+2}_{T+2,U+1} \left[a_2(1-z)^{\alpha_2+\beta_2}\left|\begin{matrix}(1-\sigma-\alpha_1R-\alpha_3\xi+\alpha_4\eta_{t,w};\alpha_2;1) \\ (d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right. \begin{matrix}(1-\rho-\beta_1R-\beta_3\xi+\beta_4\eta_{t,w};\beta_2;1) \\ {[1-\sigma-\rho-(\alpha_1+\beta_1)R-(\alpha_3+\beta_3)\xi+(\alpha_4+\beta_4)\eta_{t,w};\alpha_2+\beta_2;1]}\end{matrix} \begin{matrix} c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T} \end{matrix}\right]. \quad (2.3)$$

Proof. Using (1.1), (1.10), (1.11) and (1.12), changing the sequence of integrations and sums that are permissible due to the absolute convergence involved within the method, and doing the integration with respect to both the variables γ and δ between the limits 0 and ∞, and looking at a known result [16, p.243], establishes result (2.3). □

Theorem 2.4. *If*

$$Re(1+\alpha+\sigma+R\alpha_1+\alpha_2\frac{d_k}{\delta_k}+\xi\alpha_3-\alpha_4\eta_{t,w}) > 0,$$

$$Re(1+\rho+\beta+R\beta_1+\beta_2\frac{d_k}{\delta_k}+\xi\beta_3-\beta_4\eta_{t,w}) > 0,$$

$|arg(a_4)| < \frac{\pi}{2}\varphi_\ell, \varphi_\ell > 0, |arg(a_1)| < \frac{1}{2}N\pi,$ $N > 0$, *where* $k = 1, 2, \ldots, Q$, $j = 1, 2, \ldots, M$, $\sigma > 0$, $\rho > 0$, $\alpha_i, \beta_j > 0$ *for all* $i, j = 1, 2, 3, 4$, $a_1 > 0$, $|a_3| < 1$, $p \leq q$. *Then the following integral relation exists*

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\sigma+\alpha}\left(\frac{1-y}{1-xy}\right)^{\beta}\frac{1}{(1-x)} \cdot P_n^{\alpha,\beta}\left[1-2a_1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\right] \cdot \overline{H}^{Q,S}_{T,U}\left[a_3\left(\frac{1-x}{1-xy}y\right)^{\alpha_3}\left(\frac{1-y}{1-xy}\right)^{\beta_3}\right] \cdot {}_pM_q^{\alpha,\beta}\left[a_4\left(\frac{1-x}{1-xy}y\right)^{\alpha_4}\left(\frac{1-y}{1-xy}\right)^{\beta_4}\right] \cdot \aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_5\left(\frac{1-x}{1-xy}y\right)^{\alpha_5}\left(\frac{1-y}{1-xy}\right)^{\beta_5}\right] \cdot s^{u'}_{v'}\left[a_2\left(\frac{1-x}{1-xy}y\right)^{\alpha_2}\left(\frac{1-y}{1-xy}\right)^{\beta_2}\right]dxdy$$

$$= \sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\cdot\frac{(u_1)_\xi \ldots (u_p)_\xi}{(v_1)_\xi \ldots (v_q)_\xi}\frac{1}{\Gamma(\alpha\xi+\beta)}\frac{(-1)^w}{\alpha_t w!}$$

$$\phi'(\eta_{t,w})z^{-\eta_{t,w}}a_2{}^R a_4{}^\xi a_5{}^{-\eta_{t,w}}\frac{(1+\alpha)_n}{n}$$

$$_2F_1\left[-n,1+\alpha+\beta+n;(1+\alpha);a_1\right]\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\cdot\left[a_3\left|\begin{matrix}(1-\alpha-\sigma-\alpha_1 k_1-\alpha_2 R-\alpha_4\xi+\alpha_5\eta_{t,w},\alpha_3;1)\\(d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right.\right.$$

$$\begin{matrix}(-\beta-\beta_1 K_1-\beta_2 R-\beta_4\xi+\beta_5\eta_{t,w},\beta_3;1)\\ [-\alpha-\beta-\sigma-(\alpha_1+\beta_1)k_1-(\alpha_2+\beta_2)R-(\alpha_4+\beta_4)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T}\\+(\alpha_5+\beta_5)\eta_{t,w},(\alpha_3+\beta_3);1]\end{matrix}\right]. \qquad (2.4)$$

Proof. Using (1.1), (1.10), (1.11) and (1.12), interchanging the sequence of integration and sums that are permissible due to the absolute convergence involved in the process, and doing integration within the limits 0 and 1 with respect to the variables x and y, we obtain the desired outcome. □

3. Special Cases

In the particular case for generalized hypergeometric sereies [3] when $\alpha = 1, \beta = 1$ we have

$$_p\overset{1,1}{M}_q(z)=\sum_{\xi=0}^{\infty}\frac{(u_1)_\xi\ldots(u_p)_\xi\, z^\xi}{(v_1)_\xi\ldots(v_q)_\xi\xi!}={}_pF_q(z).$$

All cases are shown below.

(3A)

$$\int_0^1\int_0^1\left(\frac{1-a}{1-ay}y\right)^\sigma\left(\frac{1-y}{1-ay}\right)^\rho\left(\frac{1-ay}{(1-a)(1-y)}\right)$$

$$\cdot\overline{H}^{Q,S}_{T,U}\left[a_2\left(\frac{1-a}{1-ay}y\right)^{\alpha_2}\left(\frac{1-y}{1-ay}\right)^{\beta_2}\right]$$

$$\cdot{}_pF_q\left[a_3\left(\frac{1-a}{1-ay}y\right)^{\alpha_3}\left(\frac{1-y}{1-ay}\right)^{\beta_3}\right]$$

$$\cdot\aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4\left(\frac{1-a}{1-ay}y\right)^{\alpha_4}\left(\frac{1-y}{1-ay}\right)^{\beta_4}\right]$$

$$\cdot s^{u'}_{v'}\left[a_1\left(\frac{1-a}{1-ay}y\right)^{\alpha_1}\left(\frac{1-y}{1-ay}\right)^{\beta_1}\right]dady$$

$$=\sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M'}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\xi!}\frac{(-1)^w\phi'(\eta_{t,w})}{w!B'_t}$$

$$\cdot a_1{}^R a_3{}^\xi a_4{}^{-\eta_{t,w}}\cdot\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\left[a_2\left|\begin{matrix}(1-\sigma-\alpha_1 R-\alpha_3\xi+\alpha_4\eta_{t,w},\alpha_2;1)\\(d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-\beta_1 R-\beta_3\xi+\beta_4\eta_{t,w},\beta_2;1)\\(1-\sigma-\rho-(\alpha_1+\beta_1)R-(\alpha_3+\beta_3)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;C_k)_{1,S},(c_k,\gamma_k)_{S+1,T}\\+(\alpha_4+\beta_4)\eta_{t,w},\ \alpha_2+\beta_2;1)\end{matrix}\right]$$

exists under conditions required for (2.1).

(3B)

$$\int_0^\infty\int_0^\infty\psi(u+v)\,u^{\sigma-1}\,v^{\rho-1}s^{u'}_{v'}\left[a_1u^{\alpha_1}v^{\beta_1}\right]$$

$$\cdot\overline{H}^{Q,S}_{T,U}\left[a_2u^{\alpha_2}v^{\beta_2}\right]{}_pF_q\left[a_3u^{\alpha_3}v^{\beta_3}\right]$$

$$\cdot\aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4u^{\alpha_4}v^{\beta_4}\right]dudv$$

$$=\sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\cdot\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\xi!}$$

$$\cdot\frac{(-1)^w\phi'(\eta_{t,w})z^{-\eta_{t,w}}}{\alpha_t\, w!}a_1{}^R a_3{}^\xi a_4{}^{-\eta_{t,w}}$$

$$\int_0^\infty\psi(z)z^{\sigma+\rho+(\alpha_1+\beta_1)R+(\alpha_3+\beta_3)\xi}$$

$$z^{-(\alpha_4+\beta_4)\eta_{t,w}-1}\,dz\cdot\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\left[a_2z^{\alpha_2+\beta_2}\left|\begin{matrix}(1-\sigma-\alpha_1R-\alpha_3\xi+\alpha_4\eta_{t,w};\alpha_2;1)\\(d_k,\delta_k)_{1,Q},\ (d_k,\delta_k;\ D_k)_{Q+1,U}\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-\beta_1R-\beta_3\xi+\beta_4\eta_{t,w};\beta_2;1)\\ [1-\sigma-\rho-(\alpha_1+\beta_1)R-(\alpha_3+\beta_3)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;\,C_k)_{1,S},\,(c_k,\gamma_k)_{S+1,T}\\ +(\alpha_4{+}\beta_4)\eta_{t,w}\alpha_2{+}\beta_2;\,1]\end{matrix}\right]$$

valid beneath the equivalent conditions necessary for (2.2).

(3C)

$$\int_0^1\int_0^1 \phi(\gamma\delta)(1-\gamma)^{\sigma-1}(1-\delta)^{\rho-1}v^{\sigma}$$

$$\cdot{}_pF_q\left[a_3(1-\gamma)^{\alpha_3}(1-\delta)^{\beta_3}\right]$$

$$\cdot\overline{H}^{Q,S}_{T,U}\left[a_2(1-\gamma)^{\alpha_2}(1-\delta)^{\beta_2}v^{\alpha_2}\right]$$

$$\cdot\aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_4(1-\gamma)^{\alpha_4}(1-\delta)^{\beta_4}v^{\alpha_4}\right]$$

$$\cdot s^{u'}_{v'}\left[a_1(1-\gamma)^{\alpha_1}(1-\delta)^{\beta_1}v^{\alpha_1}\right]\,d\gamma d\delta$$

$$=\sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\cdot\frac{1}{\xi!}\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{(-1)^w}{\alpha_t\,w!}$$

$$\cdot\phi'(\eta_{t,w})a_1{}^R a_3{}^\xi a_4{}^{-\eta_{t,w}}$$

$$\cdot\int_0^1\phi(z)(1-z)^{\sigma+\rho+(\alpha_1+\beta_1)R+(\alpha_3+\beta_3)\xi}$$

$$(1-z)^{-(\alpha_4+\beta_4)\eta_{t,w}-1}dz\cdot\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\left[a_2(1-z)^{\alpha_2+\beta_2}\middle|\begin{matrix}(1{-}\sigma{-}\alpha_1R{-}\alpha_3\xi{+}\alpha_4\eta_{t,w};\alpha_2;\,1)\\(d_k,\delta_k)_{1,Q},\,(d_k,\delta_k;\,D_k)_{Q+1,U}\end{matrix}\right.$$

$$\begin{matrix}(1{-}\rho{-}\beta_1R{-}\beta_3\xi{+}\beta_4\eta_{t,w};\beta_2;\,1)\\ [1{-}\sigma{-}\rho{-}(\alpha_1{+}\beta_1)R-(\alpha_3{+}\beta_3)\xi\end{matrix}$$

$$\left.\begin{matrix}(c_k,\gamma_k;\,C_k)_{1,S},\,(c_k,\gamma_k)_{S+1,T}\\ +(\alpha_4{+}\beta_4)\eta_{t,w};\,\alpha_2{+}\beta_2;\,1]\end{matrix}\right]$$

justifiable under the same conditions as surrounding (2.3);

(3D)

$$\int_0^1\int_0^1\left(\frac{y(1-x)}{1-xy}\right)^{\sigma+\alpha}\left(\frac{1-y}{1-xy}\right)^{\beta}\frac{1}{(1-x)}$$

$$\cdot P_n^{\alpha,\beta}\left[1-2a_1\left(\frac{y(1-x)}{1-xy}\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\right]$$

$$\cdot\overline{H}^{Q,S}_{T,U}\left[a_3\left(\frac{y(1-x)}{1-xy}\right)^{\alpha_3}\left(\frac{1-y}{1-xy}\right)^{\beta_3}\right]$$

$$\cdot{}_pF_q\left[a_4\left(\frac{y(1-x)}{1-xy}\right)^{\alpha_4}\left(\frac{1-y}{1-xy}\right)^{\beta_4}\right]$$

$$\cdot\aleph^{M,N}_{P_i,Q_i,\tau_i;\lambda}\left[a_5\left(\frac{y(1-x)}{1-xy}\right)^{\alpha_5}\left(\frac{1-y}{1-xy}\right)^{\beta_5}\right]$$

$$\cdot s^{u'}_{v'}\left[a_2\left(\frac{y(1-x)}{1-xy}\right)^{\alpha_2}\left(\frac{1-y}{1-xy}\right)^{\beta_2}\right]dxdy$$

$$=\sum_{R=0}^{[v'/u']}\sum_{\xi=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=1}^{M}\frac{(-v')_{u'R}}{R!}A_{v',R}$$

$$\cdot\frac{(u_1)_\xi\ldots(u_p)_\xi}{(v_1)_\xi\ldots(v_q)_\xi}\frac{1}{\xi!}\frac{(-1)^w\phi'(\eta_{t,w})}{\alpha_t\,w!}$$

$$\cdot a_2{}^R a_4{}^\xi a_5{}^{-\eta_{t,w}}.\frac{(1+\alpha)_n}{\lfloor n}$$

$$\cdot{}_2F_1\left[-n,1{+}\alpha{+}\beta{+}n;(1{+}\alpha);a_1\right]\cdot\overline{H}^{Q,S+2}_{T+2,U+1}$$

$$\left[a_3\middle|\begin{matrix}(1{-}\alpha{-}\sigma{-}\alpha_1k_1{-}\alpha_2R{-}\alpha_4\xi{+}\alpha_5\eta_{t,w},\alpha_3;1)\\(d_k,\delta_k)_{1,Q},(d_k,\delta_k;D_k)_{Q+1,U}\end{matrix}\right.$$

$$\left.\begin{matrix}(-\beta{-}\beta_1K_1{-}\beta_2R{-}\beta_4\xi{+}\beta_5\eta_{t,w},\beta_3;\,1)\\ [-\alpha{-}\beta{-}\sigma{-}(\alpha_1{+}\beta_1)k_1{-}(\alpha_2{+}\beta_2)R\end{matrix}\right]$$

$$\left.\begin{matrix}(c_k,\gamma_k;\,C_k)_{1,S},\,(c_k,\gamma_k)_{S+1,T}\\ -(\alpha_4{+}\beta_4)\xi{+}(\alpha_5{+}\beta_5)\eta_{t,w},(\alpha_3{+}\beta_3);\,1]\end{matrix}\right]$$

valid in adjoining conditions (2.4).

(3E) Substituting $\beta = 1$; $\tau_i = 1$; $\lambda = 1$ in (2.1), we acquire the known outcome recently obtained by Chaurasia and Singh [17].

4. Conclusion

Due to the generality of the Aleph-function, it can be expressed in terms of various very useful functions, viz the H function, I and G function. Also, by suitably specializing various parameters of the M-series and general class of polynomials, our results yield numerous special cases which may help to solve problems occurring in science, engineering, mathematical physics, etc.

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