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Geometric Properties for an Unified Class of Functions Characterized Using Fractional Ruscheweyh-Goyal Derivative Operator

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ABSTRACT

By means of the principle of subordination, we commence with a unified subclass of analytic functions involving the fractional Ruscheweyh-Goyal derivative operator introduced by Goyal and Goyal (2005). The properties like inclusion relationships, coefficient inequalities and distortion theorems for the above mentioned class have been analyzed. For analytic functions defined in open disk of unit radius, we have incorporated the differential sandwich theorem.

Keywords: Analytic functions; Convolution; Differential subordination; Fractional Ruscheweyh-Goyal derivative operator; Superordination

1. Fractional Ruscheweyh-Goyal Derivative

Let us assume an analytic and p-valent function denoted by $f(z)$ in an open disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

of unit radius and of the form

$$f(z) = z^p + \sum_{j=s}^{\infty} a_{p+j} z^{p+j}, \tag{1.1}$$

where $p, s \in \mathbb{N}$. The class of such function is denoted by A_p.

The fractional calculus is calculus of arbitrary order. Various authors are studying a number of applications of fractional

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calculus in various fields of science and engineering (see e.g. [1–3].

Goyal and Goyal [4] introduced the generalized Ruscheweyh derivative for p-valent functions (see also, [5–9]), involving the Saigo fractional differential operator $J_{0,z}^{\lambda,\kappa,\rho}$ (see, e.g. [10]) as follows:

Definition 1.1 (Fractional Ruscheweyh--Goyal derivative operator)**.** The generalized Ruscheweyh derivative for p-valent functions involving the Saigo fractional differential operator $J_{0,z}^{\lambda,\kappa,\rho}$ is defined by

$$\begin{aligned} J_p^{\lambda,\kappa} f(z) &:= \frac{\Gamma(\kappa-\lambda+\rho+2)}{\Gamma(\kappa+1)\Gamma(\rho+2)} z^p J_{0,z}^{\lambda,\kappa,\rho}(z^{\kappa-p} f(z)), \\ &= z^p + \sum_{r=n+p}^{\infty} a_k B_p^{\lambda,\kappa}(r) z^r, \end{aligned} \tag{1.2}$$

where,

$$B_p^{\lambda,\kappa}(r) := \frac{\Gamma(r-p+1+\kappa)}{\Gamma(r-p+1)} \times \frac{\Gamma(\rho+2+\kappa-\lambda)\Gamma(r+\rho-p+2)}{\Gamma(r+\rho-p+2+\kappa-\lambda)\Gamma(\rho+2)\Gamma(1+\kappa)},$$

$\lambda, \kappa, \rho \in \mathbb{R}$ and $0 \leq \lambda < 1$.

For $\kappa = \lambda$, fractional Ruscheweyh-Goyal derivative operator $J_p^{\lambda,\kappa}$ reduces to the Ruscheweyh derivative D^λ of order λ.

Eq. (1.2) can be expressed in the terms of convolution as:

$$J_p^{\lambda,\kappa} f(z) = z^p \cdot {}_2F_1(\kappa+1, \rho+2; \rho+2+\kappa-\lambda; z) * f(z).$$

For the fractional Ruscheweyh-Goyal derivative operator, recurrence relation is derived in [11] and is given below:

$$\begin{aligned} z\left[J_p^{\lambda,\kappa} f(z)\right]' &= (\kappa-\lambda+\rho+1) J_p^{\lambda+1,\kappa} f(z) \\ &\quad -(\kappa-\lambda+\rho+1-p) J_p^{\lambda,\kappa} f(z). \end{aligned} \tag{1.3}$$

The following definitions would be required in the current work: Let f and g are analytic functions defined in Δ. The function f is said to be subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$, $|w(z)| < 1$ for all $z \in \Delta$ such that

$$f(z) = g(w(z)) \text{ for all } z \in \Delta.$$

We denote this subordination by $f \prec g$ or $f(z) \prec g(z)$ for all $z \in \Delta$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ for all $z \in \Delta$.

Let $H = H(\Delta)$ denote the class of functions analytic in Δ. For a positive integer n and $a \in C$, let

$$H[a,n] = \{f \in H \mid f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$$

be denoted by $H_0 = H[0,1]$. We denote the set of all functions $f(z)$ analytic and injective on $\Delta \backslash E(f)$ by Q (see [12]), where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\}$$

such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \backslash E(f)$, where $\partial\Delta$ is a boundary of Δ.

2. The class $S_p^{\lambda,\kappa}(\alpha, \beta; \chi)$

Inspired by the present efforts in [28], and by using the operator $J_p^{\lambda,\kappa}$, we define and study a novel unified subclass of class A_p, which is introduced using the principle of subordination and is given underneath:

Definition 2.1. If the following subordination constraint for $\alpha \in C$, $\lambda, \kappa, \rho \in \mathbb{R}$, $\Re(\alpha)$, $\Re(\beta) > 0$, $0 \leq \lambda < 1$,

$$(1-\alpha)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta + \alpha\left(\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta \prec \chi(z), \tag{2.1}$$

is satisfied, then the functions $f(z) \in A_p$ are said to be in the class $S_p^{\lambda,\kappa}(\alpha, \beta; \chi)$.

A number of classes follow as special cases of the above defined class.

Special Cases:

1. If $\chi(z) = \frac{1+Az}{1+Bz}$, $1 \geq B > A \geq -1$, then we denote the class $S_p^{\lambda,\kappa}(\alpha, \beta; \chi)$ by $S_p^{\lambda,\kappa}(\alpha, \beta; A, B)$. Hence, $f(z)$ belongs to the class $S_p^{\lambda,\kappa}(\alpha, \beta; A, B)$, if it fulfills the relation

$$\left| \frac{\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta + \frac{z}{\Lambda}\left\{\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta\right\}' - 1}{A - B\left[\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta + \frac{z}{\Lambda}\left\{\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta\right\}'\right]} \right| \leq 1, \quad (2.2)$$

$\Lambda = \frac{\beta(\kappa-\lambda+p+1)}{\alpha}$.

2. If $\chi(z) = \alpha z q'(z) + q(z)$, then the above mentioned class is represented by $S_p^{\lambda,\kappa}(\alpha, \beta; q)$.

3. If $\lambda = \kappa = 1; \beta = 1; p = 1;$ and $A = 2\gamma - 1; B = 1$, then

$$\begin{aligned} S_p^{\lambda,\kappa}(\alpha, \beta; A, B) &= S_1^{1,1}(\alpha, 1; 2\gamma - 1, 1) \\ &= R(\alpha, \gamma) \end{aligned}$$

for all $0 \leq \gamma < 1$. This class was studied by Altintas [13].

4. If $\lambda = \kappa = 1;\ \beta = 1;\ p = 1;\ \alpha = 0$ and $A = 2\gamma - 1;\ B = 1$, then

$$\begin{aligned} S_p^{\lambda,\kappa}(\alpha, \beta; A, B) &= S_1^{1,1}(0, 1; 2\gamma - 1, 1) \\ &= T^{**}(\gamma) \end{aligned}$$

for all $0 \leq \gamma < 1$. This class was introduced and investigated by Sarangi and Uralegaddi [14] and Al-Amiri [15].

5. If $\lambda = \kappa = 1; \beta = 1; p = 1; \alpha = 0$ and $A = \{(1+\varepsilon)\gamma - 1\}\delta; B = \varepsilon\delta$, then

$$\begin{aligned} &S_p^{\lambda,\kappa}(\alpha, \beta; A, B) \\ &= S_1^{1,1}(0, 1; \{(1+\varepsilon)\gamma - 1\}\delta, \varepsilon\delta) \\ &= P^*(\gamma, \varepsilon) \end{aligned}$$

for all $0 \leq \gamma < 1, 0 < \delta \leq 1, 0 \leq \varepsilon < 1$. This class was studied by Owa and Aouf [16].

6. If $\lambda = \kappa = 1;\ \beta = 1;\ p = 1;\ \alpha = 0$ and $A = (2\gamma - 1)\delta;\ B = \delta$, then

$$\begin{aligned} &S_p^{\lambda,\kappa}(\alpha, \beta; A, B) \\ &= S_1^{1,1}(0, 1; (2\gamma - 1)\delta, \delta) \\ &= P^*(\gamma, \delta) \end{aligned}$$

for all $0 \leq \gamma < 1$ and $0 < \delta \leq 1$. This class was studied by Gupta and Jain [17].

7. If $\lambda = \kappa; \beta = 1$ and $p = 1$, then

$$S_1^{\lambda,\lambda}(\alpha, \beta; A, B) = S^\lambda(\alpha, 1; A, B),$$

which is studied by Chen [18].

8. If $\lambda = \kappa = 0;\ \beta = 1;\ p = 1$ and $A = 2\gamma - 1;\ B = 1$ with $0 \leq \gamma < 1$, then

$$S_p^{\lambda,\kappa}(\alpha, \beta; A, B) = S(\alpha, 1; 2\gamma - 1, 1).$$

This class has been considered by Bhoosnurmath and Swamy [19].

9. If $\lambda = \kappa;\ \beta = 1;\ p = 1$ and $\frac{D^\lambda f(z)}{z}$ is replaced by $(D^\lambda f(z))'$, then the class $S_p^{\lambda,\kappa}(\alpha, \beta; A, B)$ reduces to class $Q(\lambda, \alpha; A, B)$ which was studied by Attiya and Aouf [20].

3. Preliminaries

The following lemmas are needed to prove our results:

Lemma 3.1 ([21]). *The function h, which is analytic in Δ given as*

$$h(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \ldots,$$

and let there be another function φ with $\varphi(0) = 1$ be analytic and convex(univalent) in Δ, and if

$$h(z)+\frac{zh'(z)}{\zeta} \prec \varphi(z)\,(\Re(\zeta) > 0; \zeta \neq 0; z \in \Delta). \tag{3.1}$$

Then,

$$\begin{aligned} h(z) &\prec \xi(z) \\ &= \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1}\varphi(t)dt \\ &\prec \varphi(z) \end{aligned}$$

for all $z \in \Delta$ and the best dominant of (3.1) is $\xi(z)$.

Lemma 3.2 ([22]). *If $|k|$ reaches its highest value inside the circle $r = |z| < 1$ at z_0, where k is an analytic function in Δ which is non-constant with $k(0) = 0$, then*

$$z_0 k'(z_0) = \beta k(z_0),$$

where $\beta \in \mathbb{R}$ with $\beta \geq 1$.

Lemma 3.3 ([23]). *Suppose G is analytic and a convex function in open unit disk Δ. If $s, t \prec G$, where $s, t \in A$, then*

$$\lambda s + (1 - \lambda t) \prec G$$

for $1 \geq \lambda \geq 0$.

Lemma 3.4 ([24]). *Suppose that*

$$\Re\left\{1 + \frac{zs''(z)}{s'(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\eta}\right)\right\},$$

where $s(z)$ is univalent in Δ, and η is a non-zero complex number, and if

$$h(z) + \gamma z h'(z) \prec s(z) + \gamma z s'(z),$$

then $h(z) \prec s(z)$, where $h(z)$ is analytic in Δ. Further the best dominant is $s(z)$.

Lemma 3.5 ([12]). *Assuming $h(0) = a$, $h(z)$ is convex in Δ, let $\eta \in C, \Re(\eta) > 0$. If $g \in H[a, 1]$ and $g(z) + \eta z g'(z)$ is univalent in an open disk Δ of radius unity, then*

$$h(z) + \gamma z h'(z) \prec g(z) + \gamma z g'(z),$$

here $h(z) \prec g(z)$ and the best subordinant is $h(z)$.

Lemma 3.6 ([25]). *Taking $h(z)$ to be analytic in an open disk of radius unity which is defined in the following manner*

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,$$

and $p(z)$ is convex and analytic in Δ defined as

$$p(z) = 1 + \sum_{k=1}^{\infty} d_k z^k,$$

If $h \prec p$, then

$$|c_k| \leq |d_1|$$

for all $k \in \mathbb{N}$.

4. Integrals Means

We begin with integral means results below by using Lemma 1.

Theorem 4.1. *For $\alpha \in C$ and $f \in S_p^{\lambda,\kappa}(\alpha, \beta; \chi)$ with $\Re(\alpha, \beta) > 0$, then*

$$\begin{aligned} \left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta} &\prec \frac{\Lambda}{n}\int_0^1 \left(\frac{1 + Azu}{1 + Bzu}\right) u^{\frac{\Lambda}{n}-1} du \\ &\prec \frac{1 + Az}{1 + Bz}, \end{aligned} \tag{4.1}$$

$$z \in \Delta,\ \Lambda = \frac{(\kappa - \lambda + \rho + 1)\beta}{\alpha}.$$

Proof. Here the function p_1 is defined as

$$p_1(z) = \left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta}, (z \in \Delta), \tag{4.2}$$

then p_1 is analytic in Δ with $p_1(0) = 1$. On taking the derivative of (Eq. (4.2)) of both sides and by applying (1.3), we get

$$\left\{\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right\}\left\{\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right\}^{\beta}\alpha + \left\{\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right\}^{\beta}(1-\alpha) = \left\{p_1(z) + p_1{}'(z).\frac{z}{\Lambda}\right\} \prec \frac{1+Az}{1+Bz}, (z \in \Delta). \tag{4.3}$$

An application of Lemma 3.1 to (4.3) yields

$$\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta} \prec \frac{\Lambda}{n}\int_0^z \left(\frac{1+At}{1+Bt}\right)\left(\frac{t}{z}\right)^{\frac{\Lambda}{n}-1} dt = \frac{\Lambda}{n}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du \prec \frac{1+Az}{1+Bz}, (z \in \Delta). \tag{4.4}$$

Thus the proof is completed. □

Theorem 4.2. *Taking $\alpha \in \mathbb{C}$ along with $f \in S_p^{\lambda,\kappa}(\alpha,\beta; A, B)$, $1 \geq B > A \geq -1$, $\Re(\alpha,\beta) > 0$, then we have*

$$\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1} du < \Re\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta} < \frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1} du. \tag{4.5}$$

The extremal function of (4.5) is given by

$$J_p^{\lambda,\kappa} f(z) F_{\alpha,\beta,A,B}(z) = z^p\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du\right)^{\frac{1}{\beta}}, \tag{4.6}$$

where $\Lambda = \frac{\beta(\kappa-\lambda+\rho+1)}{\alpha}$.

Proof. By taking $f \in S_p^{\lambda,\kappa}(\alpha,\beta; A, B)$ with $\Re(\alpha,\beta) > 0$ and using Theorem 4.1, it can be concluded that Eq. (4.1) holds, which implies that

$$\Re\left[\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta}\right] < \sup_{z\in\Delta} \Re\left[\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du,\right] \leq \frac{\Lambda}{n}\int_0^1 \sup_{z\in\Delta} \Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du, < \frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1} du. \tag{4.7}$$

and

$$\Re\left[\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta}\right] > \inf_{z\in\Delta} \Re\left[\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du,\right] \geq \frac{\Lambda}{n}\int_0^1 \inf_{z\in\Delta} \Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\Lambda}{n}-1} du, > \frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1} du. \tag{4.8}$$

Combining Eqs. (4.7) and (4.8), we get

$$\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1} du < \Re\left[\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta}\right] < \frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1} du. \tag{4.9}$$

□

Corollary 4.3. *For $\alpha \in$ and $f \in S_p^{\lambda,\kappa}(\alpha,\beta; A, B)$ with $1 \geq B > A \geq -1$, and $\Re(\alpha,\beta) > 0$, then*

$$\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1} du < \Re\left[\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^{\beta}\right] \tag{4.10}$$

$$< \frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1}du. \tag{4.11}$$

The extremal function is given by (4.6) for the function given in (4.10).

Proof. This corollary can be proved in the similar manner as done in Theorem 4.2. □

Using Theorem 4.2 and Corollary 4.3, the following distortion theorems are derived for the class $S_p^{\lambda,\kappa}(\alpha,\beta;A,B)$.

Corollary 4.4. *Let* $\alpha \in C$ *and* $f \in S_p^{\lambda,\kappa}(\alpha,\beta;A,B)$ *with* $1 \geq B > A \geq -1$ *and* $\Re(\alpha,\beta) > 0$. *Then*

$$r^p\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Aur}{1-Bur}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{\beta}} < |J_p^{\lambda,\kappa}f(z)| < r^p\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Aur}{1+Bur}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{\beta}}, \tag{4.12}$$

$r = |z| < 1$.

The extremal function for the above inequality is mentioned in (4.6).

Corollary 4.5. *For* $\alpha \in C$ *and* $f \in S_p^{\lambda,\kappa}(\alpha,\ \beta;\ A,\ B)$, $1 \geq B > A \geq -1$ *with* $\Re(\alpha,\beta) > 0$. *Then*

$$r^p\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Aur}{1+Bur}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{\beta}} < |J_p^{\lambda,\kappa}f(z)| < r^p\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Aur}{1-Bur}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{\beta}}, \tag{4.13}$$

$r = |z| < 1$.

The extremal function for the above inequality is mentioned in (4.6).

By keeping in mind that,

$(\Re(v))^{\frac{1}{2}} \leq \Re(v^{\frac{1}{2}}) \leq |v|^{\frac{1}{2}}(\Re(v) \geq 0; v \in C)$.

Corollary 4.6. *For* $\alpha \in C$ *and* $f \in S_p^{\lambda,\kappa}(\alpha,\ \beta;\ A,\ B)$, $1 \geq B > A \geq -1$ *with* $\Re(\alpha,\beta) > 0$. *Then*

$$\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{2}} < \Re\left[\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\frac{\beta}{2}}\right] < \left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{2}}. \tag{4.14}$$

Corollary 4.7. *For* $\alpha \in C$ *and* $f \in S_p^{\lambda,\kappa}(\alpha,\beta;A,B)$, $1 \geq B > A \geq -1$ *with* $\Re(\alpha,\beta) > 0$. *Then*

$$\left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1+Au}{1+Bu}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{2}} < \Re\left[\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\frac{\beta}{2}}\right] < \left(\frac{\Lambda}{n}\int_0^1 \left(\frac{1-Au}{1-Bu}\right) u^{\frac{\Lambda}{n}-1}du\right)^{\frac{1}{2}}. \tag{4.15}$$

5. Subordination and Superordination

In this section we prove multiple theorems for the class $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ to show the subordination and superordination results.

Theorem 5.1. *Let* $q(0) = 1$ *and* $q(z)$ *be univalent in* Δ*and* $\Re(\alpha,\beta) > 0$, $\alpha \in C$. *Suppose that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{-\Re(\Lambda), 0\}, \tag{5.1}$$

$\Lambda = \frac{\beta(\kappa-\lambda+1+\rho)}{\alpha}$. *If* $f(z) \in A_p$ *fulfills the subordination,*

$$(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} +$$

$$\alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} \prec q(z)+\frac{\alpha z}{\beta(\kappa-\lambda+\rho+1)}q'(z), \quad (5.2)$$

then,

$$\left\{\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right\}^{\beta} \prec q(z),$$

and the best dominant is $q(z)$.

Proof. Suppose that

$$p(z)=\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}, \quad (5.3)$$

performing the differentiation of the above equation with respect to the variable z and using (1.3), we obtain

$$zp'(z)=\alpha\,\Lambda\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\left\{\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}-1\right\}.$$

Hence, we get

$$p(z)+\frac{z}{\Lambda}p'(z)=(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}+\alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}.$$

By the relation (5.2), we obtain

$$p(z)+\frac{z}{\Lambda}p'(z)\prec q(z)+\frac{z}{\Lambda}q'(z).$$

According to Lemma 3.4,

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\prec q(z).$$

Thus the proof is completed. □

The following corollary is obtained by taking the convex function $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 5.1.

Corollary 5.2. *Let $\Re(\alpha,\beta)>0$, $\alpha\in C$ and $1\geq B>A\geq -1$. When the following subordination*

$$S_p^{\lambda,\kappa}(\alpha,\beta;\chi)\prec\frac{1+Az}{1+Bz}+\frac{(A-B)\alpha z}{\beta(\kappa-\lambda+\rho+1)(1+Bz)^2}, \quad (5.4)$$

where, $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ defined in (2.1), is fulfilled by $f(z)\in A_p$, then

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 5.3. *When $q(z)$ with $q(0)=1$ is convex in Δ and $\Re(\alpha,\beta)>0$, $\alpha\in C$. If*

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\in Q\cap H(q(0),1),\ f(z)\in A_p,$$

and the following superordination

$$q(z)+q'(z)\frac{\alpha z}{\beta(\kappa-\lambda+1+\rho)}\prec S_p^{\lambda,\kappa}(\alpha,\beta;\chi), \quad (5.5)$$

is fulfilled by $q(z)\in S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ is univalent in Δ where, $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ is defined in (4), then

$$q(z)\prec\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta},$$

and the best subordinant is $q(z)$.

Proof. Proceeding similarly as in Theorem 5.1 and letting $p(z)$ as given by (5.3), we redraft subordination (5.5) in the form

$$q(z)+\frac{\alpha z}{\beta(\kappa-\lambda+\rho+1)}q'(z)\prec p(z)+\frac{\alpha z}{\beta(\kappa-\lambda+\rho+1)}p'(z).$$

The above theorem is derived by applying Lemma 3.5. □

Corollary 5.4. *Suppose that* $\Re(\alpha,\beta) > 0$, $\alpha \in C$ *and* $1 \geq B > A \geq -1$. *If*

$$\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta \in Q \cap H(q(0),\, 1),\; f(z) \in A_p,$$

and the superordination

$$\frac{1+Az}{1+Bz} + \frac{\alpha(A-B)z}{\beta(\kappa-\lambda+\rho+1)(1+Bz)^2} \prec S_p^{\lambda,\kappa}(\alpha,\beta;\chi),$$

is satisfied by $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ *which is univalent in* Δ *then,*

$$\frac{1+Az}{1+Bz} \prec \left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta,$$

and the best subordinant is the function $\dfrac{1+Az}{1+Bz}$.

The following sandwich-type theorem is obtained by combining Theorem 5.1 and Theorem 5.3.

Theorem 5.5. *Suppose that* $q_1(z)$ *with* $q_1(0) = 1$ *and* $q_2(z)$ *with* $q_2(0) = 1$ *are convex functions in* Δ, *and x satisfy (20),* $\Re(\alpha,\beta) > 0, \alpha \in C$. *If*

$$\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta \in Q \cap H(q(0),\, 1),\; f(z) \in A_p,$$

and the relation

$$q_1(z) + \frac{z}{\Lambda}q'_1(z) \prec S_p^{\lambda,\kappa}(\alpha,\beta;\chi) \prec q_2(z) + \frac{z}{\Lambda}q'_2(z),$$

is satisfied by $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ *which is univalent in* Δ, *where* $S_p^{\lambda,\kappa}(\alpha,\beta;\chi)$ *is given by (2.1), then*

$$q_1(z) \prec \left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta \prec q_2(z).$$

the best dominant is $q_2(z)$ *and best subordinant is* $q_1(z)$.

Remark 5.6. *The sandwich results for the operator*

$$\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta$$

are derived by using Corollaries 5.2 and 5.4.

Theorem 5.7. *Suppose that*

$$\phi(z) = \frac{z\left[\left(\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta - 1\right]'}{\left[\left(\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta - 1\right]},$$

$f(z) \in A_p$, $z \in \Delta$. *If* ϕ *fulfills any conditions given below:*

$$Re(\phi(z)) \begin{cases} < \frac{1}{|\zeta|^2}\Re(\zeta) & (\Re(\zeta) > 0), \\ \neq 0 & (\Re(\zeta) = 0), \\ > \frac{1}{|\zeta|^2}\Re(\zeta) & (\Re(\zeta) < 0), \end{cases} \tag{5.6}$$

or,

$$Im(\phi(z)) \begin{cases} > -\frac{1}{|\zeta|^2}\Im(\zeta) & (\Im(\zeta) > 0), \\ \neq 0 & (\Im(\zeta) = 0), \\ < -\frac{1}{|\zeta|^2}\Im(\zeta) & (\Im(\zeta) < 0), \end{cases} \tag{5.7}$$

where $\zeta \in C\backslash\{0\}$, *then*

$$\left|\left[\left(\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta - 1\right]^\zeta\right| < 1-\gamma, 1 \geq \gamma \geq 0$$

Proof. Let the following function χ be defined as,

$$\left[\left(\frac{J_p^{\lambda+1,\kappa} f(z)}{J_p^{\lambda,\kappa} f(z)}\right)\left(\frac{J_p^{\lambda,\kappa} f(z)}{z^p}\right)^\beta - 1\right]^\zeta = (1-\gamma)\chi(z), \tag{5.8}$$

It is simple to understand that the function $\chi(z)$ with $\chi(0) = 0$ is analytic in Δ.

Differentiating logarithmically both sides of (5.8) w.r.t. z, we get

$$z\frac{\chi'(z)}{\chi(z)} = \zeta\frac{z\left[\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} - 1\right]'}{\left[\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} - 1\right]}, \tag{5.9}$$

($z \in \Delta; \zeta \in \mathbb{C}\backslash\{0\}$). Now, let us define the function ϕ by

$$\varphi = \frac{\bar{\zeta}}{|\zeta|^2}\frac{z\chi'(z)}{\chi(z)}, (z \in \Delta; \zeta \in C\backslash\{0\}). \tag{5.10}$$

Assuming that a point $z_0 \in \Delta$ exists such that

$$\max_{|z|\leq|z_0|}|\chi(z)| = 1 = |\chi(z_0)|.$$

We know by Lemma 3.2 that

$$z\chi'(z_0) = k\chi(z_0), (1 \leq k). \tag{5.11}$$

Eqs. (5.10) and (5.11) results in

$$\Re(\phi(z_0)) = \Re\left(\frac{\bar{\zeta}}{|\zeta|^2}\frac{z_0\chi'(z_0)}{\chi(z_0)}\right) = \Re\left(\frac{\bar{\zeta}}{|\zeta|^2}k\right)$$

$$= \frac{k}{|\zeta|^2}\Re(\zeta)\begin{cases} \geq \frac{1}{|\zeta|^2}\Re(\zeta) & (\Re(\zeta) > 0), \\ = 0 & (\Re(\zeta) = 0), \\ \leq \frac{1}{|\zeta|^2}\Re(\zeta) & (\Re(\zeta) < 0), \end{cases} \tag{5.12}$$

and,

$$\Im(\phi(z_0)) = \Im\left(\frac{\bar{\zeta}}{|\zeta|^2}\frac{z_0\chi'(z_0)}{\chi(z_0)}\right) = \Im\left(\frac{\bar{\zeta}}{|\zeta|^2}k\right)$$

$$= -\frac{k}{|\zeta|^2}\Im(\zeta)\begin{cases} \leq -\frac{1}{|\zeta|^2}\Im(\zeta) & (\Im(\zeta) > 0), \\ = 0 & (\Im(\zeta) = 0), \\ \geq -\frac{1}{|\zeta|^2}\Im(\zeta) & (\Im(\zeta) < 0). \end{cases} \tag{5.13}$$

But the inequalities in (5.12) and (5.13) contradict the inequalities in (5.6) and (5.7), respectively.

Thus, we reach the conclusion that $|\chi(z)| < 1, (z \in \Delta)$, which implies that,

$$\left|\left[\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} - 1\right]^{\zeta}\right| = (1-\gamma)|\chi(z)| < 1-\gamma.$$

Thus the proof is completed. □

Theorem 5.8. *Suppose that* $\Re(\alpha) > 0$, *and* $f \in S_p^{\lambda,\kappa}(0, \beta; 1-2\delta, -1)$ $(0 \leq \delta < 1)$, *then* $f \in S_p^{\lambda,\kappa}(\alpha, \beta; 1-2\delta, -1)$ *for* $K(\alpha, \beta, \lambda, \rho) > |z|$, *where*

$$K(\alpha,\beta,\lambda,\rho) = \frac{-\alpha + \sqrt{\left\{\alpha^2 + \beta^2(\kappa-\lambda+\rho+1)^2\right\}}}{\beta(\kappa-\lambda+\rho+1)}. \tag{5.14}$$

The best possible bound is $K(\alpha,\beta,\lambda,\rho)$.

Proof. Assuming that,

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} = \delta + (1-\delta)h(z),$$

$$(z \in \Delta;\ 0 \leq \delta < 1),$$

where h has a positive real part in Δ and h is analytic in Δ. Differentiating both sides and applying the recurrence relation (3), we obtain

$$(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} = p(z) + \frac{\alpha z}{\beta(\kappa-\lambda+\rho+1)}p'(z),$$

i.e.

$$(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} = \delta+(1-\delta)h(z) + \frac{\alpha z}{\beta(\kappa-\lambda+\rho+1)}(1-\delta)h'(z),$$

$$\Re\left[(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}-\delta\right] = (1-\delta)\Re\left(h(z)+\frac{z}{\Lambda}h'(z)\right) \geq (1-\delta)\Re\left(h(z)-\frac{1}{\Lambda}|zh'(z)|\right). \tag{5.15}$$

By applying the well-known estimate given in [26] as:

$$|zh'(z)| \leq \frac{2r}{1-r^2}\Re(h(z)), (r=|z|<1),$$

in (5.15), we get

$$\Re\left[(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}-\delta\right] \geq (1-\delta)\left[1-\frac{2\alpha r}{\beta(\kappa-\lambda+\rho+1)(1-r^2)}\right],$$

$\Re(h(z))>0$ for $K(\alpha,\beta,\lambda,\rho)>r$. Writing

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} = \delta+(1-\delta)\left(\frac{1-z}{1+z}\right),$$
$$(f\in A_p,\ z\in\Delta),$$

to prove the bound $K(\alpha,\beta,\lambda,\rho)$ is the best possible. By keeping in mind that,

$$\Re\left[(1-\alpha)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}-\delta\right] = (1-\delta)\Re\left(\frac{1-z}{1+z}-\frac{2\alpha z}{\beta(\kappa-\lambda+\rho+1)(1+z)^2}\right) = 0$$

for $z = K(\alpha,\beta,\lambda,\rho)$, it can be concluded that the bound is the best possible. Therefore Theorem 5.8 is proved. □

6. Inclusion Relation

Theorem 6.1. *Let* $\Re(\alpha_2)\geq\Re(\alpha_1)\geq 0$ *and* $1\geq A_1\geq A_2>B_2\geq B_1\geq -1$. *Then*

$$S_p^{\lambda,\kappa}(\alpha_2,\beta;A_2,B_2)\subseteq S_p^{\lambda,\kappa}(\alpha_1,\beta;A_1,B_1). \tag{6.1}$$

Proof. Let us consider that $f\in S_p^{\lambda,\kappa}(\alpha_2,\beta;A_2,B_2)$, we have

$$(1-\alpha_2)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} + \alpha_2\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta} \prec \frac{1+A_2z}{1+B_2z}.$$

Since, $1\geq A_1\geq A_2>B_2\geq B_1\geq -1$ we easily determine

$$(1-\alpha_2)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$

$$+\alpha_2\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$\prec \frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \qquad (6.2)$$

i.e. $f\in S_p^{\lambda,\kappa}(\alpha_1,\beta;A_1,B_1)$. Thus, statement of Theorem 6.1 holds for $\alpha_2=\alpha_1\geq 0$. If $\alpha_2>\alpha_1\geq 0$, using Theorem 4.1 and (6.2), we say that $f\in S_p^{\lambda,\kappa}(0,\beta;A_1,B_1)$, i.e.

$$\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\prec\frac{1+A_1z}{1+B_1z}. \qquad (6.3)$$

At the same time, we have

$$(1-\alpha_1)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$+\alpha_1\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$=\left(1-\frac{\alpha_1}{\alpha_2}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$+\frac{\alpha_1}{\alpha_2}\left[(1-\alpha_2)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\right.$$
$$\left.+\alpha_2\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\right]. \qquad (6.4)$$

Moreover, since $0\leq\frac{\alpha_1}{\alpha_2}<1$ and

$$h_1(z)=\frac{1+A_1z}{1+B_1z},(z\in\Delta),$$

is convex and analytic in Δ. Using (6.2), (6.3) and (6.4) and Lemma 3.3, we conclude that

$$(1-\alpha_1)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$+\alpha_1\left(\frac{J_p^{\lambda+1,\kappa}f(z)}{J_p^{\lambda,\kappa}f(z)}\right)\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}$$
$$\prec\frac{1+A_1z}{1+B_1z},$$

i.e. $f\in S_p^{\lambda,\kappa}(\alpha_1,\beta;A_1,B_1)$, which means that the statement (6.1) of Theorem 6.1 holds. □

7. Coefficient Inequalities

For the class $S_p^{\lambda,\kappa}(\alpha,\beta;A,B)$, the coefficient inequalities results are given below:

Theorem 7.1. *If the function $f(z)\in A_p$ satisfies*

$$\sum_{r=p+n}^{\infty}\left(1+\frac{n}{\Lambda}\right)B_p^{\lambda,\kappa}(r)|a_r|\leq\frac{|A-B|}{1+|B|}, \qquad (7.1)$$

where, $\lambda,\kappa,\rho\in R$, $\Re(\alpha,\beta)>0,\alpha\in J$ and $1\geq\lambda>0$, then $f(z)\in S_p^{\lambda,\kappa}(\alpha,\beta;A,B)$. For the function $f(z)$, where

$$f(z)=z^p+\frac{|A-B|}{(1+|B|)\left(1+\frac{2}{\Lambda}\right)a_{p+2}B_p^{\lambda,\kappa}(p+2)}z^{p+2}, \qquad (7.2)$$

the result (7.1) is sharp.

Proof. For $|z|=1$, we have

$$\left|\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}+\frac{z}{\Lambda}\left\{\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\right\}-1\right|$$
$$-\left|A-B\left[\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}+\frac{z}{\Lambda}\left\{\left(\frac{J_p^{\lambda,\kappa}f(z)}{z^p}\right)^{\beta}\right\}\right]\right|$$
$$=\left|\sum_{r=p+n}^{\infty}\left(1+\frac{n}{\Lambda}\right)B_p^{\lambda,\kappa}(r)a_rz^r\right|$$
$$-\left|(A-B)-B\sum_{r=p+n}^{\infty}\left(1+\frac{n}{\Lambda}\right)B_p^{\lambda,\kappa}(r)a_rz^r\right|$$
$$\leq\sum_{r=p+n}^{\infty}\left[1+\frac{n}{\Lambda}\right]B_p^{\lambda,\kappa}(r)|a_r|-|A-B|$$

$$+ |B| \sum_{r=p+n}^{\infty} \left[1 + \frac{n}{\Lambda}\right] B_p^{\lambda,\kappa}(r)|a_r|$$

$$= (1 + |B|) \sum_{r=p+n}^{\infty} \left[1 + \frac{n}{\Lambda}\right] B_p^{\lambda,\kappa}(r)|a_r|$$

$$- |A - B|,$$

$$\leq 0. \quad \text{(byhypothesis)}.$$

Hence, with the help of the maximum modulus theorem, $f \in S_p^{\lambda,\kappa}(\alpha, \beta; A, B)$. □

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