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An Efficient Computation Approach for Abel's Integral Equations of the Second Kind

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ABSTRACT

This paper presents the application of a new powerful method named as q-homotopy analysis transform method (q-HATM). The q-HATM is a combination of the q-homotopy analysis scheme and the Laplace transform approach and more general than other existing techniques. Abel's integral equation of the second kind has been solved by using this method. We solve some examples and plot the graphs. The numerical solutions are shown in the form of graphs.

Keywords: Abel's integral equation of second kind; Laplace transform; q-homotopy analysis transform method

1. Introduction

Integral equations form in equations is a mathematical tool, useful in analysis of pure and applied sciences. There are so many problems in science and technology which usually are solved with the help of ordinary and partial differential equation. Such kind of problems can be solved more effectively with the help of integral equation methods by first converting differential equations in integral equations.

The term integral equation was introduced by Paul Du Bois-Raymond in 1888. Laplace used the integral transform to solve differential equation and difference equations in 1782. In 1826, Abel solved the integral equation having the form

$$f(\xi) = \int_0^{\xi} (\xi - t)^{-\alpha} g(t)dt, \qquad (1.1)$$

where f is a continuous function and $f(\alpha) = 0, 0 < \alpha < 1$. Abel's integral equation is solved by Huygens for $\alpha = \frac{1}{2}$.

In 1826, Poisson obtained the integral equation of type

$$g(\xi) = f(\xi) + \lambda \int_0^{\xi} K(\xi, t)g(t)dt, \qquad (1.2)$$

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where g is an unknown function and $K(\xi, t)$ is a kernel.

The kind of integral equation having infinite range of integration or discontinuous kernels are known as singular integral equations, and Abel's integral equation is one of the simplest form of singular integral equation. Abel's integral equation (AIE) is presented as

$$f(\xi) = \int_0^{\xi} \frac{g(t)}{(\xi - t)^{-\alpha}} dt, \ 0 < \alpha < 1, \tag{1.3}$$

where g is an unknown to be determined and the value of function f is known.

For more information, the references can be read about a first integral equation by Wazwaz [1]. In 1924, generalization of Abel's integral equation on a finite segment was studied by Zelion [2], the different type procedures and applications of Abel's integral equation used by Wazaz [3], Khan et al. [4], Agarwal et al. [5], numerical solution by Atkinson [6]. We can use fractional calculus for solving Chebyshev polynomials. This polynomial was used by Arazzadech et al. [7, 8]. Methods for Volterra integral equations of Abel type integral equation and procedures for nonlinear integral equation were presented by Wazwaz et al. [9–13] and solution of singular integral equation of Abel type by Panday et al. [14] and Huang et al. [15].

Abel's integral equation has been solved by several authors. The q-homotopy analysis transform method (q-HATM) [16–18] is stronger than other computational schemes. It is a combined form of q-HAM [19, 20] and Laplace transform technique. For more explanation, see references for fractional Fornbrg-Whitam equation via Laplace transform solved by Singh et al. [21], and a comparison of HATM and HPTM by Khan et al. [22]. Differential and integral equations have been solved using HAM by several authors [19, 23–26] and some differential and integral equations were solved by using q-HATM [16–18, 20, 27–29].

2. Preliminaries

This section takes us through some basic and useful definitions and properties of integral equations and Laplace transforms. These definitions and properties will be of use in the present paper.

Definition 2.1. A Laplace transform (LT) is defined as

$$L\left[f(\xi); s\right] = \overline{F}(s) = \int_0^{\infty} e^{-s\xi} f(\xi) d\xi, \ \xi > 0 \tag{2.1}$$

provided that the limit exists and is finite.

Definition 2.2. The convolution of two functions is defined as

$$f * g = \int_0^{\xi} f(u) g(\xi - u)\, du. \tag{2.2}$$

The convolution theorem for the LT is presented as

$$\begin{aligned} L\left\{f * g\right\} &= L\left[\int_0^{\xi} f(u) g(\xi - u)\, du\right] \\ &= F(t) G(t). \end{aligned} \tag{2.3}$$

3. A Computational Technique

In this method, we study the basic theory and procedure of applied q-HATM for AIE of second kind. We take a general AIE of the form:

$$y(\xi) = f(\xi) - \int_a^{\xi} \frac{y(t)}{(\xi - t)^{\alpha}} dt, \ 0 < \alpha < 1. \tag{3.1}$$

First, taking LT operator on Eq. (3.1), we have

$$L[y(\xi)] = L[f(\xi)] - L\left[\int_a^{\xi} \frac{y(t)}{(\xi - t)^{\alpha}} dt\right]. \tag{3.2}$$

Now we define a nonlinear operator as

$$N\left[\phi(\xi;q)\right] = L\left[\int_a^{\xi}\frac{y(t)}{(\xi-t)^{\alpha}}dt\right] + L\left[\phi(\xi;q)\right] - L[f(\xi)], \quad (3.3)$$

where $q \in \left[0, \frac{1}{n}\right]$; $n \geq 1$ is an embedding parameter and $\phi(\xi;q)$ is a real function. Now, we take a homotopy

$$(1-nq)L\left[\phi(\xi;q) - y_0(\xi)\right] = \hbar qH(\xi)N\left[\phi(\xi;q\right], \quad (3.4)$$

where, L is the Laplace operator, $q \in \left[0, \frac{1}{n}\right]$; $n \geq 1$ is an embedding parameter, H is an auxiliary function which is nonzero, $\hbar$ is auxiliary parameter which is negative in almost all practical situations, $y_0(\xi)$ is an initial approximation of $y(\xi)$ and $\phi(\xi;q)$ is an unknown function. For embedding parameter $q = 0$ and $q = \frac{1}{n}$, the following conditions hold

$$\phi(\xi;0) = y_0(\xi) \text{ and } \phi\left(\xi;\frac{1}{n}\right) = y(\xi). \quad (3.5)$$

Consequently, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(\xi;q)$ transforms from an initial value of $y_0(\xi)$ to the final solution $y(\xi)$. With the help of Taylor's theorem about q, function $\phi(\xi;q)$ can also be written in series form as below. We have

$$\phi(\xi;q) = y_0(\xi) + \sum_{m=1}^{\infty} y_m(\xi)q^m, \quad (3.6)$$

where

$$y_m(\xi) = \frac{1}{\underline{|m}}\frac{\partial^m\phi(\xi;q)}{\partial q^m}\Big|_{q=0}. \quad (3.7)$$

If auxiliary parameter $\hbar$, the initial guess $y_0(\xi)$ and asymptotic parameter n are properly chosen, the series (3.7) converges at $q = \frac{1}{n}$. Then we have

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty}\left[y_m(\xi)\left(\frac{1}{n}\right)^m\right]. \quad (3.8)$$

Defining the vector

$$\overrightarrow{y}_m = \{y_0(\xi), y_1(\xi), ..., y_m(\xi)\} \quad (3.9)$$

and differentiating the deformation Eq. (3.3) m times with respect to q and then dividing by $\underline{|m}$ and finally setting $q = 0$, we can construct the m-th order deformation equation as given below

$$L\left[y_m(\xi) - k_m y_{m-1}(\xi)\right] = \hbar H(\xi)R_m\left[\overrightarrow{y}_{m-1}\right], \quad (3.10)$$

where

$$R_m\left[\overrightarrow{y}_{m-1}\right] = \frac{1}{\underline{|m-1}}\frac{\partial^{m-1}N\left[\phi(\xi;q)\right]}{\partial q^{m-1}}\Big|_{q=0} \quad (3.11)$$

and

$$k_m = \begin{cases} 0; & m \leq 1 \\ n; & m > 1 \end{cases}. \quad (3.12)$$

It is worth noting that in q-HATM we have freedom to take the initial guess $y_0(\xi)$, auxiliary parameter $\hbar$ which is nonzero and the asymptotic parameter n. Because of the existence of the factor $\left(\frac{1}{n}\right)^m$ in the series solution obtained in Eq.(3.8), the possibility exists for the method to be faster than what can be found from the standard HATM. It can be noted here that for $n = 1$ in Eq. (3.8), q-HATM converts into standard HATM. The auxiliary parameter $\hbar$ really plays a very significant role in controlling the convergence region and also convergence rate to the solution.

4. Numerical examples

Here some examples are presented to verify and validate the present work and some graphs have been plotted between exact and approximate solution for different values of $\hbar$. Examples are given as follows:

Example 4.1. By considering AIE of second kind in the following manner,

$$y(\xi) = \xi + \frac{4}{3}\xi^{\frac{3}{2}} - \int_a^{\xi}\frac{y(t)}{\sqrt{\xi-t}}dt, \ 0 \leq \xi \leq 1. \quad (4.1)$$

An exact solution of the given AIE may be taken as $y(\xi) = \xi$.

Now, by taking Laplace transform on Eq. (4.1), it leads to the following result:

$$L[y(\xi)] = L\left[\xi + \frac{4}{3}\xi^{\frac{3}{2}}\right] - \sqrt{\frac{\pi}{s}}L[y(\xi)]. \tag{4.2}$$

We assume the nonlinear operator as

$$N[\phi(\xi;q)] = \sqrt{\frac{\pi}{s}}L[\phi(\xi;q)] + L[\phi(\xi;q)] - L\left[\xi + \frac{4}{3}\xi^{\frac{3}{2}}\right]. \tag{4.3}$$

Using the above technique of projected numerical approach, the m-th order deformation equation for $H(\xi) = 1$ can be constructed as

$$L\left[y_m(\xi) - k_m y_{m-1}(\xi)\right] = \hbar R_m\left[\overrightarrow{y}_{m-1}\right], \tag{4.4}$$

where

$$R_m\left[\overrightarrow{y}_{m-1}\right] = \sqrt{\frac{\pi}{s}}L[y_{m-1}(\xi)] + L[y_{m-1}(\xi)] - \left(1 - \frac{k_m}{n}\right)L\left[\xi + \frac{4}{3}\xi^{\frac{3}{2}}\right] \tag{4.5}$$

Now, taking inverse LT on Eq. (4.4), we have

$$y_m(\xi) = k_m y_{m-1}(\xi) + \hbar L^{-1}\left[R_m\left\{\overrightarrow{y}_{m-1}\right\}\right]. \tag{4.6}$$

Considering the initial approximation

$$y_0(\xi) = \xi + \frac{4}{3}\xi^{\frac{3}{2}}$$

and iterative formula (4.6), we obtain

$$y_0(\xi) = \xi + \frac{4}{3}\xi^{\frac{3}{2}}, \tag{4.7}$$

$$y_1(\xi) = \hbar\left(\frac{4}{3}\xi^{\frac{3}{2}} + \frac{\pi}{2}\xi^2\right), \tag{4.8}$$

$$y_2(\xi) = \frac{4}{3}\hbar(n+\hbar)\xi^{\frac{3}{2}} + \frac{\pi}{2}\hbar(n+2\hbar)\xi^2 + \frac{8\pi}{15}\hbar^2\xi^{\frac{5}{2}}, \tag{4.9}$$

$$y_3(\xi) = \frac{4}{3}\hbar(n+\hbar)^2\xi^{\frac{3}{2}} + \frac{\pi}{2}\hbar(n+\hbar)(n+2\hbar)\xi^2 + \frac{4\pi}{15}\hbar^2(7n+12\hbar)\xi^{\frac{5}{2}} + \frac{\pi^2}{2}\hbar^3\xi^3. \tag{4.10}$$

Proceeding in the same manner, we can also compute the rest of the components of $y_m(\xi)$ for $m \geq 4$. The solution of the given AIE by using q-HATM is expressed in series form as follows:

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty}\left[y_m(\xi)\left(\frac{1}{n}\right)^m\right]$$

. Taking, $n = 1$ and $\hbar = -1$, we obtain

$$y(\xi) = \xi + \frac{4}{5}\xi^3 + \ldots;\ 0 \leq \xi \leq 1. \tag{4.11}$$

In the standard case, $n = 1$ and $\hbar = -1$, the series solution converge to the exact solution $y(\xi) = \xi$ as $m \to \infty$. The comparative analysis between q-HATM solution at $n = 1$ and and exact solution is demonstrated in Fig. 1. It can be easily noticed from Fig. 1 that the q-HATM solution is in agreement with the exact solution of AIE.

Example 4.2. In this case, let us consider AIE of second kind as follows:

$$y(\xi) = \xi^2 + \frac{16}{15}\xi^{\frac{5}{2}} - \int_a^x \frac{y(t)}{\sqrt{\xi - t}}dt,\ 0 \leq \xi \leq 1. \tag{4.12}$$

An exact solution of the given AIE is $y(\xi) = \xi^2$. Taking the Laplace transform of both sides of Eq. (4.12), gives the following result:

$$L\left[y(\xi)\right] = L\left[\xi^2 + \frac{16}{15}\xi^{\frac{5}{2}}\right] - \sqrt{\frac{\pi}{s}}L[y_m(\xi)]. \tag{4.13}$$

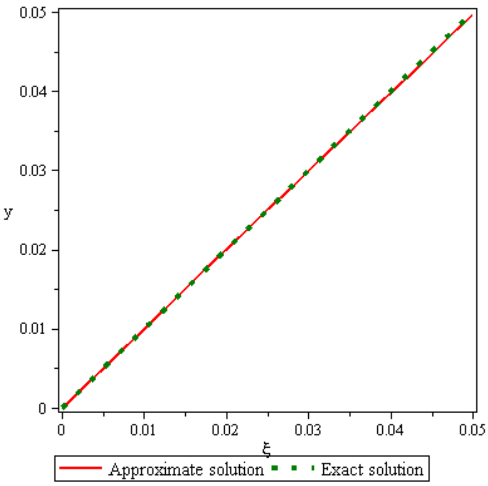

Fig. 1. Comparison between q-HATM solution and exact solution.

We assume the nonlinear operator as

$$N\left[\phi\left(\xi;q\right)\right]=\sqrt{\frac{\pi}{s}}L[\phi\left(\xi;q\right)]+L[\phi\left(\xi;q\right)]-L\left[\xi^{2}+\frac{16}{15}\xi^{\frac{5}{2}}\right]. \quad (4.14)$$

By using the above technique of suggested numerical scheme, the m-th order deformation equation for $H(\xi)=1$ can be constructed in the following manner:

$$L\left[y_m\left(\xi\right)-k_m y_{m-1}\left(\xi\right)\right]=\hbar R_m\left(\vec{y}_{m-1}\right), \quad (4.15)$$

where

$$R_m\left[\overrightarrow{y}_{m-1}\right]=\sqrt{\frac{\pi}{s}}L[y_{m-1}(\xi)]+L[y_{m-1}(\xi)]-\left(1-\frac{k_m}{n}\right)L\left[\xi^{2}+\frac{16}{15}\xi^{\frac{5}{2}}\right]. \quad (4.16)$$

Now, taking inverse Laplace transform on Eq. (4.4), we have

$$y_m\left(\xi\right)=k_m y_{m-1}\left(\xi\right)+\hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \quad (4.17)$$

Considering the initial approximation $y_0(\xi)=\xi^2+\frac{16}{15}\xi^{\frac{5}{2}}$ and iterative formula (4.17), we obtain

$$y_0(\xi)=\xi^2+\frac{16}{15}\xi^{\frac{5}{2}}, \quad (4.18)$$

$$y_1(\xi)=\hbar\left(\frac{16}{15}\xi^{\frac{5}{2}}+\frac{\pi}{3}\xi^{3}\right), \quad (4.19)$$

$$y_2(\xi)=\frac{16}{15}\hbar(n+\hbar)\xi^{\frac{5}{2}}+\frac{\pi}{3}\hbar(n+2\hbar)\xi^{3}+\frac{32\pi}{105}\hbar^{2}\xi^{\frac{7}{2}}, \quad (4.20)$$

$$y_3(\xi)=\frac{16}{15}\hbar(n+\hbar)^{2}\xi^{\frac{5}{2}}+\frac{\pi}{3}\hbar(n+\hbar)(n+3\hbar)\xi^{3}+\frac{16\pi}{105}\hbar^{2}(9n+11\hbar)\xi^{\frac{7}{2}}+\frac{\pi^{2}}{12}\hbar^{3}\xi^{4}. \quad (4.21)$$

Proceeding in the same manner, we can also compute the rest of the components of $y_m(\xi)$ for $m\geq 4$. The solution of given AIE by using q-HATM can be expressed in series form as follows:

$$y(\xi)=y_0(\xi)+\sum_{m=1}^{\infty}\left[y_m(\xi)\left(\frac{1}{n}\right)^{m}\right].$$

Taking, $n=1$ and $\hbar=-1$, we obtain

$$y(\xi)=\xi^{2}-\frac{\pi^{2}}{12}\xi^{4}+\ldots;0\leq\xi\leq 1. \quad (4.22)$$

In the standard case, $n=1$ and $\hbar=-1$, the series solution converge to the exact solution $y(\xi)=\xi^2$ as $m\rightarrow\infty$. The comparative examination between q-HATM solution at $n=1$ and $\hbar=-1$ exact solution is displayed in Fig. 2. It can be easily seen from Fig. 2 that the q-HATM solution is in agreement with the exact solution.

Example 4.3. Here we would consider the same AIE which is of second kind in the following way:

$$y(\xi)=\frac{1}{\sqrt{\xi}}+\pi-\int_0^{\xi}\frac{y(t)}{\sqrt{\xi-t}}dt,\ \ 0\leq\xi\leq 1. \quad (4.23)$$

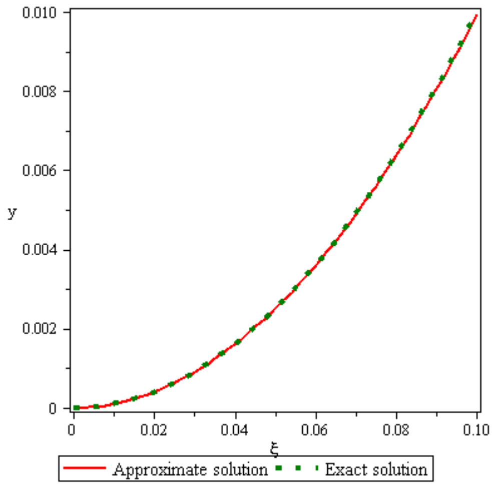

Fig. 2. Comparison between q-HATM solution and exact solution.

And an exact solution of the considered integral equation is $y(\xi) = \frac{1}{\sqrt{\xi}}$. Taking LT of both sides of Eq. (4.3.1), gives the following result:

$$L\left[y(\xi)\right] = L\left[\frac{1}{\sqrt{\xi}} + \pi\right] - \sqrt{\frac{\pi}{s}} L\left[y(\xi)\right]. \tag{4.24}$$

We assume the nonlinear operator as

$$N\left[\phi\left(\xi; q\right)\right] = \sqrt{\frac{\pi}{s}} L[\phi\left(\xi; q\right)] + L[\phi\left(\xi; q\right)] - L\left[\frac{1}{\sqrt{\xi}} + \pi\right]. \tag{4.25}$$

Using the above technique of projected numerical approach, the m-th order deformation equation for $H(\xi) = 1$ can be constructed as

$$L\left[y_m\left(\xi\right) - k_m y_{m-1}\left(\xi\right)\right] = \hbar R_m\left(\vec{y}_{m-1}\right), \tag{4.26}$$

where

$$R_m\left[\overrightarrow{y}_{m-1}\right] = \sqrt{\frac{\pi}{s}} L[y_{m-1}(\xi)] + L[y_{m-1}(\xi)] - \left(1 - \frac{k_m}{n}\right) L\left[\frac{1}{\sqrt{\xi}} + \pi\right]. \tag{4.27}$$

Now, taking inverse LT on Eq. (4.26), we have

$$y_m\left(\xi\right) = k_m y_{m-1}\left(\xi\right) + \hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \tag{4.28}$$

Considering the initial approximation $y_0(\xi) = \frac{1}{\sqrt{\xi}} + \pi$ and iterative formula (4.28), we obtain

$$y_0(\xi) = \frac{1}{\sqrt{\xi}} + \pi, \tag{4.29}$$

$$y_1(\xi) = \pi\hbar(1 + 2\xi^{\frac{1}{2}}), \tag{4.30}$$

$$y_2(\xi) = \pi\hbar(n+\hbar) + 2\pi\hbar(n+2\hbar)\xi^{\frac{1}{2}} + \pi^2\hbar^2\xi, \tag{4.31}$$

$$y_3(\xi) = \pi\hbar(n+\hbar)^2 + 2\pi\hbar(n+\hbar)(n+3\hbar)\xi^{\frac{1}{2}} + \pi^2\hbar^2(2n+3\hbar)\xi + \frac{4\pi^2}{3}\hbar^3\xi^{\frac{3}{2}}. \tag{4.32}$$

Proceeding in this manner, we can also compute the rest of the components of $y_m(\xi)$ for $m \geq 4$. The solution of the considered integral equation by q-HATM is expressed in the following series form as

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty}\left[y_m(\xi)\left(\frac{1}{n}\right)^m\right].$$

Taking, $n = 1$ and $\hbar = -1$, we obtain,

$$y(\xi) = \frac{1}{\sqrt{\xi}} - \frac{4\pi^2}{3}\xi^3 + \ldots;\quad 0 \leq \xi \leq 1. \tag{4.33}$$

In standard case $n = 1$ and $\hbar = -1$, the series solution converge to the exact solution $y(\xi) = \frac{1}{\sqrt{\xi}}$ as $m \to \infty$. The comparative analysis between the q-HATM solution at $n = 1$ and $\hbar = -1$ and the exact solution is presented in Fig. 3. It can be easily noticed from Fig. 3 that the q-HATM solution is in agreement with exact solution.

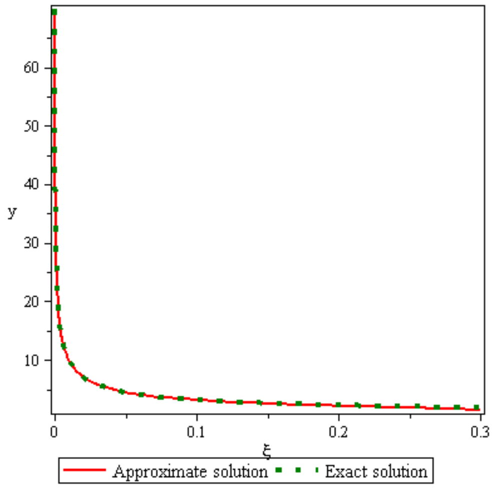

Fig. 3. Comparison between q-HATM solution and exact solution.

Example 4.4. Again consider the following AIE of second kind,

$$y(\xi) = 2\sqrt{\xi} - \int_0^{\xi} \frac{y(t)}{\sqrt{\xi - t}} dt, \quad 0 \leq \xi \leq 1. \tag{4.34}$$

We get to an exact solution of the equation as $y(\xi) = 1 - e^{\pi\xi} erfc\left(\sqrt{\pi\xi}\right)$. Taking LT of both sides of Eq. (4.34), produces the following result:

$$L\left[y(\xi)\right] = L\left[2\sqrt{\xi}\right] - \sqrt{\frac{\pi}{s}} L\left[y(\xi)\right]. \tag{4.35}$$

We assume the nonlinear operator as

$$N\left[\phi\left(\xi;q\right)\right] = \sqrt{\frac{\pi}{s}} L[\phi\left(\xi;q\right)] + L[\phi\left(\xi;q\right)] - L\left[2\sqrt{\xi}\right]. \tag{4.36}$$

Using the projected numerical technique given above, the m-th order deformation equation for $H(\xi) = 1$ can be constructed as

$$L\left[y_0(\xi) - k_m y_{m-1}(\xi)\right] = \hbar R_m\left(\overrightarrow{y}_{m-1}\right), \tag{4.37}$$

where

$$R_m\left[\overrightarrow{y}_{m-1}\right] = \sqrt{\frac{\pi}{s}} L[y_{m-1}(\xi)] + L[y_{m-1}(\xi)] - \left(1 - \frac{k_m}{n}\right) L[2\sqrt{\xi}]. \tag{4.38}$$

Now, taking the inverse Laplace transform for Eq. (4.37), we have

$$y_m\left(\xi\right) = k_m y_{m-1}\left(\xi\right) + \hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \tag{4.39}$$

Consider the initial approximation $y_0(\xi) = 2\sqrt{\xi}$ and iterative formula (4.39), we obtain

$$y_0(\xi) = 2\sqrt{\xi}, \tag{4.40}$$

$$y_1(\xi) = \pi\hbar\xi, \tag{4.41}$$

$$y_2(\xi) = \pi\hbar(n+\hbar)\xi + \frac{4\pi}{3}\hbar^2\xi^{\frac{3}{2}}, \tag{4.42}$$

$$y_3(\xi) = \pi\hbar(n+\hbar)^2\xi + \frac{8\pi}{3}\hbar^2(n+\hbar)\xi^{\frac{3}{2}} + \frac{\pi^2}{2}\hbar^3\xi^2, \tag{4.43}$$

$$y_4(\xi) = \pi\hbar(n+\hbar)^3\xi + 4\pi\hbar^2(n+\hbar)^2\xi^{\frac{3}{2}} + \frac{3\pi^2}{2}\hbar^3(n+\hbar)\xi^2 + \frac{8\pi^2}{15}\hbar^5\xi^{\frac{5}{2}}. \tag{4.44}$$

Proceeding in this manner, we can also compute the rest of the components of $y_m\left(\xi\right)$ for $m \geq 5$. The solution of given AIE by q-HATM is expressed in the subsequent series form as

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty}\left[y_m(\xi)\left(\frac{1}{n}\right)^m\right].$$

Taking, $n = 1$ and $\hbar = -1$, we obtain,

$$y(\xi) = 2\sqrt{\xi} - \pi\xi + \frac{4\pi}{3}\xi^{\frac{3}{2}} - \frac{\pi^2}{2}\xi^2$$

$$-\frac{8\pi^2}{15}\xi^{\frac{5}{2}}+\ldots;\ \ 0\le\xi\le 1. \tag{4.45}$$

Or

$$y(\xi)=\sum_{m=1}^{\infty}\frac{(-1)^{n-1}(\pi\xi)^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}=1-E_{\frac{1}{2}}\left(-\sqrt{\pi\xi}\right),$$

where

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}$$

is a well-known Mittag-Leffler function. So,

$$y(\xi)=1-e^{\pi\xi}erfc\left(\sqrt{\pi\xi}\right). \tag{4.46}$$

In the standard case, $n = 1$ and $\hbar = -1$, the series solution converges to the exact solution $y(\xi) = 1 - e^{\pi\xi}erfc\left(\sqrt{\pi\xi}\right)$ as $m \to \infty$. The comparative study between the q-HATM solution at $n = 1$ and $\hbar = -1$ and the exact solution is exhibited in Fig. 4. It can be easily seen from 4 that the q-HATM solution is in agreement with the exact solution.

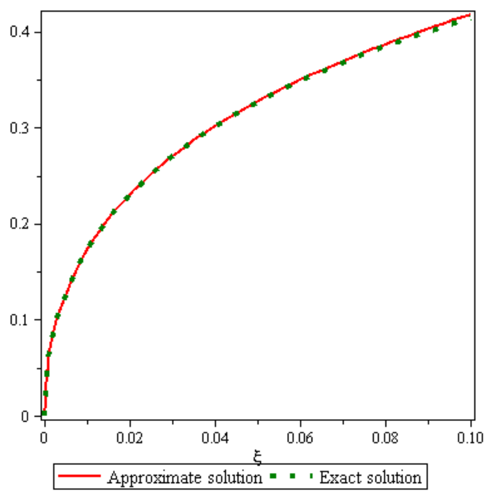

Fig. 4. Comparison between q-HATM solution and exact solution.

5. Conclusions

In the analytical view of this paper, the q-HATM is successfully applied on the second kind of Abel's integral equation. The q-HATM provides a large convergence region with great efficiency and positiveness. We obtain the analytical result of the AIEs in the form of series solution which is computed very easily. In the fourth section, four examples have been given to investigate and demonstrate the versatility of the newly obtained approaches. By comparison with other methods, q-HATM is very powerful and a stronger method than others and we see more advantages to this method. In this method we can choose the value of auxiliary linear operator L, auxiliary function $H(\xi)$ and initial function $y_0(\xi)$ with freedom. Specialization of this method lies in the fact that the obtained solutions having auxiliary parameter and asymptotic parameter n provide an easy way to adjust and control the convergence region and the rate of convergence of the derived series solution.

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