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Evolution of Modulational Instability in Non-linear Hirota Types Equation

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ABSTRACT

The Time Dependent Variational Approach (TDVA) is applied to estimate the modulational instability (MI) for Non-Linear Hirota type equations. This approach is new to obtain the nonlinear dispersion relation (NDR) for such equations. The classical modulational instability criterion is nascent and it establishes many prospective for the MI domain because of the generalized dispersion relation. Analyzing the Non Linear Partial Differential Equations, we rederive the classical modulational instability criterion.

Keywords: Modulational instability (MI); Non-linear hirota types equation; Time dependent variational approach (TDVA)

1. Introduction

Modulational instability (MI) exists in many different branches of physics and biology. Many phenomena can be explained by the interlinking of physical and biological systems on the basis of modulational instabilities (MI). MI is obtainable due to the established relation between the nonlinearity and diffraction/dispersion special effects. It has been applied to produce train soliton-like pulses, which are a predecessor to solitons configuration [1–3]. MI has been foreseen theoretically for propagation of electromagnetic and plasma waves in different media [4–9].

Consider a one dimensional non-linear partial differential (NLPD) equation as

$$i\frac{\partial E}{\partial t} + \beta_1\frac{\partial^2 E}{\partial x^2} + i\gamma_1\frac{\partial^3 E}{\partial x^3} + \delta_1|E|^2E + 3i\alpha_1|E|^2\frac{\partial E}{\partial x} = 0. \quad (1.1)$$

Here $E(x,t)$ is a high-frequency field and α_1,β_1, γ_1 and δ_1 are real constants with the relation

$$\alpha_1\beta_1 - \delta_1\gamma_1 = \text{constant}.$$

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If $\alpha_1 = \gamma_1 = 0$ then Eq.(1.1) represents a non-linear Schrödinger equation [10].

If $\beta_1 = \delta_1 = 0$ the equation reduces to the modified Korteweg-de Vries equation.

We consider $\beta_1 = \gamma_1 = 1$, $\delta_1 = \alpha_1 = -\frac{1}{3}$ Eq. (1.1) reduces to Hirota equation

$$i\left[E_t + E_{xxx} - |E|^2 E_x\right] + E_{xx} - \tfrac{1}{3}|E|^2 E = 0. \tag{1.2}$$

2. Modulational Instability for Modal Equation

The augmentation of modulation instability of an uninterrupted wave is discussed in two steps. In the first step, the equilibrium situation for simple and precise monochromatic wave solutions is considered. In the second step, a small perturbation on the equilibrium situation in amplitude propagates a set of equations from which one draws a conclusion about the nonlinear dispersion relation. By setting for Eq. (1.2)

$$E(x,t) = E_0 e^{i(kx+wt)}$$

we get

$$w = k^3 - k^2 + E_0^2 k - \tfrac{1}{3}E_0^2 \tag{2.1}$$

a small perturbation in amplitude

$$E = E_0\left(1 + \Im\psi\right) e^{i(kx+wt)}. \tag{2.2}$$

The perturbation $\psi(x,t)$, is put in Eq. (1.2) then we may get

$$i\frac{\partial\psi}{\partial t} + i\left(E_0^2 + 2k - 3k^2\right)\frac{\partial\psi}{\partial x} + \left(E_0 - 3k\right)\frac{\partial^2\psi}{\partial x^2} + i\frac{\partial^3\psi}{\partial x^3} + \left(2k - \tfrac{1}{3}\right)E = 0. \tag{2.3}$$

For the solution let

$$\psi(x,t) = Me^{i(Qx-\Omega t)} + N^* e^{-i(Qx-\Omega^* t)}, \tag{2.4}$$

where the wave number Q and the frequency Ω are perturbations, respectively (* indicates complex conjugation).

Using the dispersion relation for evolution of MI, one obtains Im $\Omega > 0$

$$\left[\Omega - kQ\left(2-3k\right) + Q^3\right] = i\,|Q|\sqrt{\left(E_0^2\left(2k - \tfrac{1}{3}\right) - Q^2\right)}. \tag{2.5}$$

This is an instability region of model equation (1.2) for perturbation. The value of $\Omega(k)$ is quadratic in terms of the wave numbers $Q > 0$ and the plane wave parameters. Their imaginary roots are in linearly unstable mode, with extension rate $|\mathrm{Im}\,(\Omega(k))|$.

We now apply TDVA [11] to recognize the period of unstable wave numbers. Lagrangian for the Hirota equation (1.2)

$$L = \int_{-\infty}^{+\infty}\left[\tfrac{i}{2}(E^* E_t - E E_t^*) - |E_x|^2 - \tfrac{i}{2}|E|^2 E^* E_x - iE_x^* E_{xx} - \tfrac{1}{6}(E^* E)^2\right]dx \tag{2.6}$$

and consider the plane wave solution

$$E(x,t) = e^{i(kx+wt)}\left[E_0 + a(t)e^{(i\Phi_a(t)+qx)} + b(t)\mathrm{e}^{(i(\Phi_b(t)-qx))}\right]. \tag{2.7}$$

Here we consider the intervallic boundary situation on the $E(x,t)$ with integration limits $x \in [0, 2\pi]$ after substitution in Eq. (2.5), then Lagrangian for the Hirota equation

$$L=\pi\left\{\begin{array}{l} 2q^2\left((a+b)^2-2ab\right) \\ -2\left(a^2\Phi_a+b^2\Phi_b\right) \\ -4kq\left(a^2-b^2\right)+q\left(E_0^3\right. \\ +E_0^2 ab\sin\left(\Phi_a+\Phi_b\right) \\ \left.+a^4-b^4\right)+\frac{1}{2}\left(\frac{1}{3}-k\right)\left(E_0^4\right. \\ \left.-2E_0^2-4E_0^2 ab\cos\left(\Phi_a+\Phi_b\right)\right) \\ -\frac{1}{2}\left(\frac{1}{3}-k\right)\left(\left(a^2+b^2\right)^2\right. \\ \left.-2ab\left(ab+2\right)\right) \\ -k^3\left(E_0^2+a^2+b^2\right) \\ -k^2 q\left(3a^2-3b^2\right) \\ -kq^2\left(3a^2-b^2\right) \end{array}\right\}. \tag{2.8}$$

It's dependent on a, b, Φ_a and Φ_b. Equation of motion for $a(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right)-\frac{\partial L}{\partial a}=0$$

$$\begin{aligned}\Rightarrow a\Phi_a = \; & C_1 b+C_2 b\Big[\sin\left(\Phi_a+\Phi_b\right) \\ & -\cos\left(\Phi_a+\Phi_b\right) \\ & +\tfrac{b}{2}\left(\tfrac{1}{3}-k\right)\left(\left(4q+1\right)a^2+2b^2\right)\Big].\end{aligned} \tag{2.9}$$

Equation of motion for $\Phi_a(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}_a}\right)=\frac{\partial L}{\partial \Phi_a}$$

$$\Rightarrow \dot{a}=C_2 b\left[\sin\left(\Phi_a+\Phi_b\right)-\cos\left(\Phi_a+\Phi_b\right)\right]. \tag{2.10}$$

Equation of motion for $b(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right)=\frac{\partial L}{\partial b}$$

$$\begin{aligned}\Rightarrow b\dot{\Phi}_b = \; & C_3 b+C_2 a\Big[\sin\left(\Phi_a+\Phi_b\right) \\ & -\cos\left(\Phi_a+\Phi_b\right)+\tfrac{a}{2}\left(\tfrac{1}{3}-k\right)\Big((4q \\ & +1)b^2+a^2\Big)\Big].\end{aligned} \tag{2.11}$$

Equation of motion for $\Phi_b(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}_b}\right)=\frac{\partial L}{\partial \Phi_b}$$

$$\Rightarrow \dot{b}=C_2 a\left[\sin\left(\Phi_a+\Phi_b\right)-\cos\left(\Phi_a+\Phi_b\right)\right], \tag{2.12}$$

where

$$\begin{aligned} C_1 = \; & \tfrac{1}{2}\left(\tfrac{1}{3}-k\right)E_0^2-q^2-2kq-\tfrac{1}{2}qE_0^2 \\ & -\tfrac{1}{2}k^3+\tfrac{3}{2}k^2 q+\tfrac{3}{2}kq^2, \\ C_2 = \; & -\left(\tfrac{1}{3}+k\right)E_0^2, \\ C_3 = \; & \tfrac{1}{2}\left(\tfrac{1}{3}-k\right)E_0^2-q^2+2kq \\ & -\tfrac{1}{2}qE_0^2-\tfrac{1}{2}k^3+\tfrac{3}{2}k^2 q+\tfrac{3}{2}kq^2,\end{aligned}$$

when $a=b, \Phi=\Phi_a+\Phi_b$, then

$$\dot{a}=\frac{da}{dt}=C_2 a\left(\sin\Phi-\cos\Phi\right), \tag{2.13}$$

$$\dot{\Phi}=\frac{d\Phi}{dt}=\left(C_1+C_3\right)+2C_2\sin\Phi-2C_2\cos\Phi. \tag{2.14}$$

Let $A=C_1+C_3$, $B=C_2$. Then Eq. (2.14)

$$\frac{2(A-2B)}{\sqrt{(A^2-8B^2)}}\tan^{-1}\left[\frac{(A-2B)\tan\left(\frac{\Phi}{2}\right)+2B}{\sqrt{(A^2-8B^2)}}\right]=t,$$

$$\begin{aligned}\Phi = \; & 2\tan^{-1}\left\{\tfrac{1}{(A-2B)}\Big[\sqrt{A^2-8B^2}\right. \\ & \left.\tan\left(t\tfrac{\sqrt{(A^2-8B^2)}}{2(A-2B)}\right)-2B\Big]\right\},\end{aligned}$$

$$\begin{aligned}\tan\tfrac{\Phi}{2} = \; & \sqrt{\tfrac{(A^2-8B^2)}{(A-2B)^2}}\tan\left[\tfrac{t}{2}\sqrt{\tfrac{(A^2-8B^2)}{(A-2B)^2}}\right] \\ & -\tfrac{2B}{(A-2B)}.\end{aligned} \tag{2.15}$$

If $A^2 - 8B^2 > 0 \Rightarrow (C_1 + C_3)^2 - 8{C_2}^2 > 0$

$$\Rightarrow \left(E_2^0(\tfrac{1}{3} - k - q) + q^2(k - 2 - k^3 + 3k^2 q)\right)^2 - 8\left(-(\tfrac{1}{3+k}E_0^2)\right)^2 > 0, \qquad (2.16)$$

then by Eq. (2.13)

$$\log a = \frac{(A^2-8B^2)}{2B(A-3B)} \log\left[\frac{1+\tan^2\frac{\Phi}{2}}{A^2+\tan^2\frac{\Phi}{2}}\right] - \frac{(A-2B)^2}{(A^2-8B^2)}\tan^{-1}\left[\frac{(A-2B)^2}{(A^2-8B^2)}\tan\frac{\Phi}{2}\right] + \frac{(A^2-8B^2)}{(A-3B)}\left[\frac{\Phi}{2} + \frac{(A-2B)^2}{(A^2-8B^2)} \tan^{-1}\left[\frac{(A-2B)^2}{(A^2-8B^2)}\tan\frac{\Phi}{2}\right]\right]. \qquad (2.17)$$

Eq. (2.17) shows when $A - 2B < 0$ is stable and when $A - 2B0$ is unstable. Three and two dimension simulation for Eq. (2.7) whose parameters are derived in Eqs. (2.8)-(2.17) are represented in Fig. 1 and Fig. 2.

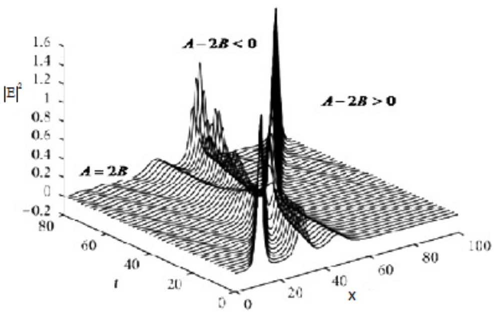

Fig. 1. Three dimension simulation for Eq. (2.7) with time measure $\Delta t = {}^1/_{50}$, $0 \le t \le 80$ and Space measure $\Delta x = {}^1/_5$, $0 \le x \le 100$

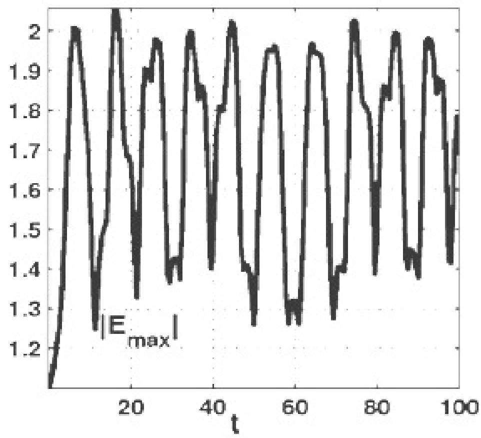

Fig. 2. Two dimension time step $\Delta t = 0.02$, $0 \le t \le 100$ for Eq. (2.17) and $x = 0$

3. Result and Discussion

In this paper, modulational instabilities of the Hirota equation are discussed in the form of nonlinear dispersion relation (NDR) by a time dependent variational approach. We obtain a Euler-Lagrange equation of motion in favour of the time-dependent variables and scrutinize their strength for different amplitudes (wave number and frequency) of perturbation of wave solution. It is found that the analogous solidity circumstance obtaining a new ordinary differential equation, whose exceptional situations communicate to stability/instability criteria, is recognized. To the best of our knowledge, the time dependent variational approach (TDVA) has not been implemented for constructing the modulational instability and solutions of this model.

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