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Note on Recent Fixed Point Results in Graphical Rectangular b-Metric Spaces

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ABSTRACT

This paper aims to rectify the recent fixed point results on graphical rectangular b-metric spaces due to Mudasir Younis et al. (J. Fixed Point Theory Appl., doi:10.1007/s11784-019-0673-3, 2019). Moreover, we also give the answer of some open problem in the mentioned research related to the Kannan contraction mapping in the space described above with its fixed point theorems.

Keywords: Fixed point; Graph; Graphical rectangular b-metric; Kannan G'-contraction

1. Introduction

Throughout this paper, unless otherwise specified, let the diagonal of $X \times X$ be denoted by Δ for a nonempty set X. Furthermore, let $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ be a directed graph possessing no parallel edges, where $\mathfrak{U}(G)$ is the set of all vertices such that $\mathfrak{U}(G) \subseteq X$ and $\mathfrak{E}(G)$ is the set of all the edges of G containing all loops, that is, $\Delta \subseteq \mathfrak{E}(G)$. A path (or directed path) of length m between points $v, w \in \mathfrak{U}(G)$ is defined as a sequence $\{x_j\}_{j=0}^{m}$ of $(m+1)$ vertices with $v = x_0$, $w = x_m$ and $(x_{j-1}, x_j) \in \mathfrak{E}(G)$ for all $j = 1, 2, \ldots, m$. Consistent with Shukla [1], we denote

$$[u]_G^l = \{v \in X : \exists \text{ a path directing from } u \text{ } v \text{ having length } l\}.$$

In addition, a relation P on X is such that $(uPv)_G$ if there exists a path directing from u to v in G and the notion $w \in (uPv)G$ is used whenever w is contained in the path $(uPv)_G$. A sequence $\{x_n\}$ in X is called a G-termwise connected (briefly, G-TWC) if $(x_n P x_{n+1})_G$ for all $n \in \mathbb{N}$.

To avoid repetition, we assume the same terminology, notations and basic facts as having been utilized in [2]. For more details, one can also refer to [1,3–5]. The idea of a graphical rectangular b-metric space is a generalization of a rectangular b-metric space.

Definition 1.1 ([6]). *Let X be a non-empty*

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set and $d : X \times X \to [0, \infty)$ be a function. If d satisfies the following conditions:

(i) $d(x, y) = 0$ *iff* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) for each $x, y \in X$ and distinct points $u, v \in X \setminus \{x, y\}$, we have

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y),$$

then d is called a rectangular metric on X and (X, d) is called a rectangular metric space (briefly, a RMS).

Definition 1.2 ([7, 8]). *Let X be a non-empty set, $d : X \times X \to [0, \infty)$ be a function and $s \geq 1$. If d satisfies the following conditions:*

(i) $d(x, y) = 0$ *iff* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) for each $x, y \in X$ and distinct points $u, v \in X \setminus \{x, y\}$, we have

$$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)],$$

then d is called a rectangular b-metric on X and (X, d) is called a rectangular b-metric space (briefly, a R_bMS).

Definition 1.3 ([1]). *Let X be a non-empty set, G be a graph endowed with X, and $d_G : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

(i) $d_G(x, y) = 0$ *iff* $x = y$;

(ii) $d_G(x, y) = d_G(y, x)$ *for all* $x, y \in X$;

(iii) for each $x, y \in X$ with $(xPy)_G$ and $z \in (xPy)_G$, we have

$$d_G(x, y) \leq d_G(x, z) + d_G(z, y).$$

Then d_G is called a graphical metric on X and (X, d_G) is called a graphical metric space (briefly, a GMS).

Definition 1.4 ([2]). *Let X be a non-empty set, G be a graph endowed with X, $s \geq 1$, and $r_{G_b} : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

$(GR_bM - 1)$ $r_{G_b}(x, y) = 0$ *iff* $x = y$;

$(GR_bM - 2)$ $r_{G_b}(x, y) = r_{G_b}(y, x)$ *for all* $x, y \in X$;

$(GR_bM - 3)$ *for each $x, y \in X$ and distinct points $u, v \in X \setminus \{x, y\}$ with $(xPy)_G$ and $u, v \in (xPy)_G$, we have*

$$r_{G_b}(x, y) \leq s[r_{G_b}(x, u) + r_{G_b}(u, v) + r_{G_b}(v, y)].$$

Then r_{G_b} is called a graphical rectangular b-metric on X and (X, r_{G_b}) is called a graphical rectangular b-metric space (briefly, a GR_bMS).

Definition 1.5 ([2]). *If $s = 1$ in Definition 1.4, we call the resultant space a graphical rectangular metric space (briefly, $GRMS$) and denote it by (X, r_G), which is the graphical version of a rectangular metric space.*

Remark 1.6. It is easy to see that a GR_bMS is a $GRMS$ with $s = 1$.

Definition 1.7 ([2]). *Let (X, r_{G_b}) be a graphical rectangular b-metric space. A sequence $\{x_n\}$ in X is said to be*

i) a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$r_{G_b}(x_n, x_m) < \epsilon \text{ for all } n, m \geq n_0$$

i.e., $\lim_{n,m\to\infty} r_{G_b}(x_n, x_m) = 0$.

ii) converges to $x \in X$ if for given $\epsilon > 0$, there exist $m \in \mathbb{N}$ such that

$$r_{G_b}(x_n, x) < \epsilon \text{ for all } n \geq m$$

i.e., $\lim_{n\to\infty} r_{G_b}(x_n, x) = 0$.

Definition 1.8 ([2]). *Let (X, r_{G_b}) be a graphical rectangular b-metric space endowed with a graph $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ and G' be a sub-graph of G with $\mathfrak{U}(G') = X$.*

i) X is said to be complete if every Cauchy sequence in X converges in X.

ii) X is said to be G'-complete if every G'-termwise connected Cauchy sequence in X converges in X.

Definition 1.9 ([2]). *Let A be a self-mapping on a graphical rectangular b-metric space (X, r_{G_b}) endowed with a graph G and the coefficient $s \geq 1$, and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Then A is called a (G, G')-contraction on X if it satisfies the following conditions:*

(GC-1) *for each $(x, y) \in \mathfrak{E}(G')$, we have $(Ax, Ay) \in \mathfrak{E}(G')$;*

(GC-2) *there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$r_{G_b}(Ax, Ay) \leq \lambda r_{G_b}(x, y)$$

for all $x, y \in X$ with $(x, y) \in \mathfrak{E}(G')$.

Definition 1.10 ([2]). *Let A be a self-mapping on a graphical rectangular b-metric space (X, r_{G_b}) endowed with a graph G and the coefficient $s \geq 1$, and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. A graph G' is said to satisfy the property $(\mathcal{P})$, if a G'-termwise connected A-Picard sequence $\{x_n\}$ converges in X, then there exist a limit $\xi \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $(x_n, \xi) \in \mathfrak{E}(G')$ or $(\xi, x_n) \in \mathfrak{E}(G')$ for all $n > n_0$.*

Theorem 1.11 ([2]). *Let (X, r_{G_b}) be a graphical rectangular b-metric space endowed with a graph G and the coefficient $s \geq 1$ and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Suppose that X is G'-complete, $A : X \to X$ is a (G, G')-contraction mapping and the following conditions hold:*

(I) G' satisfies the property $(\mathcal{P})$;

(II) there exist $x_0 \in X$ such that $Ax_0 \in [x_0]^l_{G'}$ for some $l \in \mathbb{N}$.

Then there exist $z^* \in X$ such that the A-Picard sequence $\{x_n\}$ with the initial value $x_0 \in X$ is G'-termwise connected and converges to both z^* and Az^*.

Definition 1.12 ([2]). *Let (X, r_{G_b}) be a graphical rectangular metric space and $A : X \to X$ be a (G, G')-contraction mapping. The quadruple (X, r_{G_b}, G', A) is said to have the property S^* if for each G'-termwise connected A-Picard sequence $\{x_n\}$ in X has the unique limit.*

In [2], authors also posed the following question.

- **Question:** Is it possible to establish analogous results of Edelstein [9], Hardy-Roger [10], Kannan [11] , Meir-Keeler [12], and Reich [13] type contractions in GR_bMS.

In this paper, we show that the conditions of Theorem 4.2 in [2] are not sufficient to prove the Cauchyness of the G'-termwise connected A-Picard sequence and hence it doesn't ensure the existence of fixed points in GR_bMS. In fact, we show that the inequality (4.4) of Theorem 4.2 in [2] (page 9-10) doesn't hold for even values $l \in \mathbb{N}$. To remedy this, we propose suitable conditions on the mentioned theorem (see condition (II) in Theorem 2.3 given below) and provide a corrected proof. Moreover, we provide a positive answer to the question of the existence of a fixed point for Kannan contraction mappings in the aforesaid space.

2. Main Results

We begin this section with the following example showing that the inequality (4.4) of Theorem 4.2 in [2] (page 9-10) doesn't hold for even values $l \in \mathbb{N}$.

Example 2.1. *Let $X = \{0\} \cup \{\frac{1}{3^n} : n \in \mathbb{N}\}$ and $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ be a graph associated with X, where $\mathfrak{U}(G) = X$ and $\mathfrak{E}(G) := \Delta \cup \{(\frac{1}{3^n}, \frac{1}{3^{n+1}}) \in X \times X : n \in \mathbb{N}\}$. Define a function $r_{G_b} : X \times X \to [0, \infty)$ by*

$$r_{G_b}(x, y) = 0 \text{ iff } x = y,$$

$$r_{G_b}\left(0, \frac{1}{3^n}\right) = r_{G_b}\left(\frac{1}{3^n}, 0\right) = \frac{1}{2} \text{ for all } n \in \mathbb{N},$$

$$r_{G_b}\left(\frac{1}{3^m}, \frac{1}{3^n}\right) = 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m \neq n \text{ and } 2 \text{ divides } |m - n|,$$

$$r_{G_b}\left(\frac{1}{3^m}, \frac{1}{3^n}\right) = \frac{1}{3^{n+m}} \text{ otherwise.}$$

Then (X, r_{G_b}) is a graphical rectangular metric space (i.e., GR_bMS with $s = 1$). Define a mapping $A : X \to X$ by

$$Ax = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{x}{3^4} & \text{otherwise.} \end{cases}$$

Then A is a (G, G')-contraction mapping on X with $\lambda = \frac{1}{3}$ and $G' = G$.

Now, we will prove that for any $x_0 \in X$ such that $Ax_0 \in [x_0]^l_{G'}$ for some $l \in \mathbb{N}$, the A-Picard sequence $\{x_n\}$ is not a Cauchy sequence. Note that the Property ($\mathcal{P}$) is not required to prove the Cauchyness of a a sequence $\{x_n\}$ (see the proof of Theorem 4.2 in [2]).

Case-I *If $x_0 = 0$, then $Ax_0 = \frac{1}{3}$. But there is no path from 0 to $\frac{1}{3}$. Then $Ax_0 \notin [x_0]^l_{G'}$ for all $l \in \mathbb{N}$. So we don't consider this case.*

Case-II *If $x_0 \in \{\frac{1}{3^n} : n \in \mathbb{N}\}$, then $Ax_0 \in [x_0]^4_{G'}$. Suppose that $x_0 = \frac{1}{3}$. Then $Ax_0 = x_1 = \frac{1}{3^5}$ and there exists a sequence $\{y_j\}^4_{j=0}$ such that $y_0 = x_0 = \frac{1}{3}$, $y_1 = \frac{1}{3^2}$, $y_2 = \frac{1}{3^3}$, $y_3 = \frac{1}{3^4}$, $y_4 = Ax_0 = \frac{1}{3^5}$ with $(y_{j-1}, y_j) \in \mathfrak{E}(G')$ for all $j = 1, 2, 3, 4$. This implies that $Ax_0 \in [x_0]^4_{G'}$. Since A is an edge preserving mapping, we can show that the sequence $\{x_n\}$ is a G'-termwise connected A-Picard sequence.*

Now, we will show that the inequality (4.4) of Theorem 4.2 in [2] (page 9-10) is not true for $m = 0$:

$$\begin{aligned} r_{G_b}(x_0, x_1) &= r_{G_b}(y_0, y_4) \\ &= r_{G_b}\left(\frac{1}{3}, \frac{1}{3^5}\right) \\ &= 1 \\ &\not\leq \frac{1}{3^3} + \frac{1}{3^5} + \frac{1}{3^7} + \frac{1}{3^9} \\ &= r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2) \\ &\quad + r_{G_b}(y_2, y_3) + r_{G_b}(y_3, y_4). \end{aligned}$$

Also, for any $n = 0, 1, 2, ...$, we have

$$r_{G_b}(x_n, x_{n+1}) = r_{G_b}\left(\frac{1}{3^{n+1}}, \frac{1}{3^{n+5}}\right) = 1.$$

This implies that $\{x_n\}$ is not a Cauchy sequence.

Remark 2.2. The above example demonstrates the technical difficulties in utilizing the path of even length between x_n and Ax_n.

To prove the next result, the following symbol is needed: for a graph $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ and $u \in \mathfrak{U}(G)$, we denote

$$[\blacktriangle u]^l_G = \{v \in X : \exists \text{ a path} \{x_j\}^l_{j=0} \text{from } u \text{ to } v \text{ with } x_{j-1} \neq x_j \ \forall j = 1, 2, \ldots, l\}.$$

Theorem 2.3. *Let (X, r_{G_b}) be a graphical rectangular b-metric space endowed with a graph G and the coefficient $s \geq 1$ and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Suppose that X is G'-complete, $A : X \to X$ is a one-to-one (G, G')-contraction mapping and the following conditions hold:*

(I) G' satisfies the property ($\mathcal{P}$);

(II) There exists $x_0 \in X$ such that $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, where l, m are odd positive integers.

Then there exist $z^* \in X$ such that the A-Picard sequence $\{x_n\}$ with the initial value $x_0 \in X$ is G'-termwise connected and converges to both z^* and Az^*.

Proof. Let $x_0 \in X$ be such that $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, where l, m are odd integers. Define an A-Picard sequence $\{x_n\}$ by $x_n = Ax_{n-1}$ for all $n \in \mathbb{N}$. Since $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, there exist a path $\{y_j\}^l_{j=0}$ such that $x_0 = y_0$, $Ax_0 = y_l$ and $(y_{j-1}, y_j) \in \mathfrak{E}(G')$ with $y_{j-1} \neq y_j$ for all $j = 1, 2, \ldots, l$ and a path $\{w_j\}^m_{j=0}$ such that $x_0 = w_0$, $A^2x_0 = w_m$ and $(w_{j-1}, w_j) \in \mathfrak{E}(G')$ with $w_{j-1} \neq w_j$ for all $j = 1, 2, \ldots, m$. Since A is a (G, G')-contraction mapping, by (GC-1), we have

$$(Ay_{j-1}, Ay_j) \in \mathfrak{E}(G') \text{ for all } j = 1, 2, \ldots, l.$$

Therefore, $\{Ay_j\}^l_{j=0}$ is a path from $Ay_0 = Ax_0 = x_1$ to $Ay_l = A^2x_0 = x_2$ of length l and $x_2 \in [x_1]^l_{G'}$. Continuing this process, for all $n \in \mathbb{N}$, we obtain $\{A^n y_j\}^l_{j=0}$ is a path from $A^n y_0 = A^n x_0 = x_n$ to $A^n y_l = A^n Ax_0 = x_{n+1}$ of length l and $x_{n+1} \in [x_n]^l_{G'}$. Thus, $\{x_n\}$ is a G'-termwise connected sequence.

Since $(A^n y_{j-1}, A^n y_j) \in \mathfrak{E}(G')$ for $j = 1, 2, \ldots, l$ and $n \in \mathbb{N}$. By (GC-2), for each $j = 1, 2, \ldots, l$, we have

$$\begin{aligned} r_{G_b}(A^n y_{j-1}, A^n y_j) &\leq \lambda r_{G_b}(A^{n-1} y_{j-1}, A^{n-1} y_j) \\ &\;\;\vdots \\ &\leq \lambda^n r_{G_b}(y_{j-1}, y_j). \end{aligned} \tag{2.1}$$

Similarly, we can show that $\{A^n w_j\}^m_{j=0}$ is a path from $A^n w_0 = A^n x_0 = x_n$ to $A^n w_m = A^n A^2 x_0 = x_{n+2}$ of length m and $x_{n+2} \in [x_n]^m_{G'}$ for all $n \in \mathbb{N}$.

Since $(A^n w_{j-1}, A^n w_j) \in \mathfrak{E}(G')$ for $j = 1, 2, \ldots, m$ and $n \in \mathbb{N}$. By (GC-2), for each $j = 1, 2, \ldots, m$, we have

$$\begin{aligned} r_{G_b}(A^n w_{j-1}, A^n w_j) &\leq \lambda r_{G_b}(A^{n-1} w_{j-1}, A^{n-1} w_j) \\ &\;\;\vdots \\ &\leq \lambda^n r_{G_b}(w_{j-1}, w_j). \end{aligned} \tag{2.2}$$

Now, we obtain

$$\begin{aligned} r_{G_b}(x_0, x_1) &\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2) \\ &\quad + r_{G_b}(y_2, y_l)] \\ &\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2)] \\ &\quad + s^2[r_{G_b}(y_2, y_3) + r_{G_b}(y_3, y_4) \\ &\quad + r_{G_b}(y_4, y_l)] \\ &\;\;\vdots \\ &\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2)] \\ &\quad + s^2[r_{G_b}(y_2, y_3) + r_{G_b}(y_3, y_4)] \\ &\quad + \cdots + s^{\frac{l-1}{2}}[r_{G_b}(y_{l-3}, y_{l-2}) \\ &\quad + r_{G_b}(y_{l-2}, y_{l-1}) + r_{G_b}(y_{l-1}, y_l)] \\ &=: D_l \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} r_{G_b}(x_0, x_2) &\leq s[r_{G_b}(w_0, w_1) + r_{G_b}(w_1, w_2) \\ &\quad + r_{G_b}(w_2, w_m)] \\ &\leq s[r_{G_b}(w_0, w_1) + r_{G_b}(w_1, w_2)] \\ &\quad + s^2[r_{G_b}(w_2, w_3) + r_{G_b}(w_3, w_4) \\ &\quad + r_{G_b}(w_4, w_m)] \\ &\;\;\vdots \end{aligned}$$

$$\begin{aligned}
&\leq s[r_{G_b}(w_0,w_1)+r_{G_b}(w_1,w_2)]\\
&\quad + s^2[r_{G_b}(w_2,w_3)+r_{G_b}(w_3,w_4)]\\
&\quad + \cdots + s^{\frac{m-1}{2}}[r_{G_b}(w_{m-3},w_{m-2})\\
&\quad + r_{G_b}(w_{m-2},w_{m-1})\\
&\quad + r_{G_b}(w_{m-1},w_m)]\\
&=: D_m. \qquad (2.4)
\end{aligned}$$

By using (GR_bM-3) and (GC-1) and inequalities (2.1) and (2.3), we have

$$\begin{aligned}
r_{G_b}(x_n,x_{n+1}) &= r_{G_b}(A^nx_0,A^nx_1)\\
&= r_{G_b}(A^ny_0,A^ny_l)\\
&\leq s[r_{G_b}(A^ny_0,A^ny_1)\\
&\quad + r_{G_b}(A^ny_1,A^ny_2)\\
&\quad + r_{G_b}(A^ny_2,A^ny_l)]\\
&\leq s[r_{G_b}(A^ny_0,A^ny_1)\\
&\quad + r_{G_b}(A^ny_1,A^ny_2)]\\
&\quad + s^2[r_{G_b}(A^ny_2,A^ny_3)\\
&\quad + r_{G_b}(A^ny_3,A^ny_4)\\
&\quad + r_{G_b}(A^ny_4,A^ny_l)]\\
&\ \ \vdots\\
&\leq s[r_{G_b}(A^ny_0,A^ny_1)\\
&\quad + r_{G_b}(A^ny_1,A^ny_2)]\\
&\quad + s^2[r_{G_b}(A^ny_2,A^ny_3)\\
&\quad + r_{G_b}(A^ny_3,A^ny_4)] + \cdots\\
&\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^ny_{l-3},A^ny_{l-2})\\
&\quad + r_{G_b}(A^ny_{l-2},A^ny_{l-1})\\
&\quad + r_{G_b}(A^ny_{l-1},A^ny_l)]\\
&\leq \lambda^nD_l. \qquad (2.5)
\end{aligned}$$

Similarly, by using (GR_bM-3) and (GC-1) and inequalities (2.2) and (2.4), we have

$$\begin{aligned}
r_{G_b}(x_n,x_{n+2}) &= r_{G_b}(A^nx_0,A^nx_2)\\
&= r_{G_b}(A^nw_0,A^nw_m)\\
&\leq s[r_{G_b}(A^nw_0,A^nw_1)\\
&\quad + r_{G_b}(A^nw_1,A^nw_2)\\
&\quad + r_{G_b}(A^nw_2,A^nw_m)]\\
&\leq s[r_{G_b}(A^nw_0,A^nw_1)\\
&\quad + r_{G_b}(A^nw_1,A^nw_2)]\\
&\quad + s^2[r_{G_b}(A^nw_2,A^nw_3)\\
&\quad + r_{G_b}(A^nw_3,A^nw_4)\\
&\quad + r_{G_b}(A^nw_4,A^nw_m)]\\
&\ \ \vdots\\
&\leq s[r_{G_b}(A^nw_0,A^nw_1)\\
&\quad + r_{G_b}(A^nw_1,A^nw_2)]\\
&\quad + s^2[r_{G_b}(A^nw_2,A^nw_3)\\
&\quad + r_{G_b}(A^nw_3,A^nw_4)] + \cdots\\
&\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^nw_{m-3},A^nw_{m-2})\\
&\quad + r_{G_b}(A^nw_{m-2},A^nw_{m-1})\\
&\quad + r_{G_b}(A^nw_{m-1},A^nw_m)]\\
&= \lambda^nD_m. \qquad (2.6)
\end{aligned}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence i.e., for all $p \geq 1$, $r_{G_b}(x_n,x_{n+p}) \to 0$ as $n \to \infty$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}\cup\{0\}$, then $r_{G_b}(x_n,x_{n+p}) \to 0$ as $n \to \infty$. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$.

Case-I: If p is odd integer, then

$$\begin{aligned}
r_{G_b}(x_n,x_{n+p}) &\leq s[r_{G_b}(x_n,x_{n+1})\\
&\quad + r_{G_b}(x_{n+1},x_{n+2})]\\
&\quad + s^2[r_{G_b}(x_{n+2},x_{n+3})\\
&\quad + r_{G_b}(x_{n+3},x_{n+4})] + \cdots\\
&\quad + s^{\frac{p-1}{2}}[r_{G_b}(x_{n+p-3},x_{n+p-2})\\
&\quad + r_{G_b}(x_{n+p-2},x_{n+p-1})\\
&\quad + r_{G_b}(x_{n+p-1},y_{n+p})].
\end{aligned}$$

By using inequality (2.5), we have

$$\begin{aligned}
r_{G_b}(x_n,x_{n+p}) &\leq s[\lambda^nD_l + \lambda^{n+1}D_l]\\
&\quad + s^2[\lambda^{n+2}D_l + \lambda^{n+3}D_l] + \cdots\\
&\quad + s^{\frac{p-1}{2}}[\lambda^{n+p-3}D_l\\
&\quad + \lambda^{n+p-2}D_l + \lambda^{n+p-1}D_l]\\
&\leq s^{\frac{p-1}{2}}\left(\frac{\lambda^n}{1-\lambda}\right)D_l
\end{aligned}$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty.$$

Case-II: If p is even integer, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \leq\ & s[r_{G_b}(x_n, x_{n+1}) \\ & + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) \\ & + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-2}{2}}[r_{G_b}(x_{n+p-4}, x_{n+p-3}) \\ & + r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\ & + r_{G_b}(x_{n+p-2}, y_{n+p})]. \end{aligned}$$

By using inequality (2.5) and (2.6), we have

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \leq\ & s[\lambda^n D_l + \lambda^{n+1} D_l] \\ & + s^2[\lambda^{n+2} D_l + \lambda^{n+3} D_l] + \cdots \\ & + s^{\frac{p-2}{2}}[\lambda^{n+p-4} D_l \\ & + \lambda^{n+p-3} D_l + \lambda^{n+p-2} D_m] \\ \leq\ & s^{\frac{p-2}{2}} \left(\frac{\lambda^n}{1-\lambda} \right) (D_l + D_m) \\ & \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

From Case-I and Case-II, we can say that $\{x_n\}$ is a Cauchy sequence. Since X is G'-complete, $\{x_n\}$ is a convergent sequence. By our assumption, there exist $z^* \in X$ and $n_0 \in \mathbb{N}$ such that $x_n \rightarrow z^*$ as $n \rightarrow \infty$ and $(x_n, z^*) \in \mathfrak{E}(G')$ or $(z^*, x_n) \in \mathfrak{E}(G')$ for all $n > n_0$. Suppose that $(x_n, z^*) \in \mathfrak{E}(G')$ for all $n > n_0$. By (GC-2), we have

$$r_{G_b}(Ax_n, Az^*) \leq \lambda r_{G_b}(x_n, z^*)$$

for all $n > n_0$. This implies that

$$r_{G_b}(Ax_n, Az^*) \rightarrow 0 \text{ as } n \rightarrow \infty$$

i.e., $x_{n+1} \rightarrow Az^*$. So, Az^* is also a limit of $\{x_n\}$.

Similarly, we can prove this for the case $(z^*, x_n) \in \mathfrak{E}(G')$ for all $n > n_0$. This completes the proof. □

Theorem 2.4. *Assume that all hypotheses of Theorem 2.3 hold and further suppose that the quadruple (X, r_{G_b}, G', A) has the property S^*. Then A has a fixed point in X.*

Proof. From the proof of Theorem 2.3 and Property S^*, we get this result. □

Theorem 2.5. *Assume that all hypotheses of Theorem 2.4 hold and further suppose that $(z^*, w^*) \in \mathfrak{E}(G')$ for all $z^*, w^* \in Fix(A)$, where $Fix(A)$ is the set of all fixed points of A. Then A has the unique fixed point.*

Proof. From Theorem 2.4, A has a fixed point. Now, we will show the uniqueness of a fixed point. Assume that z^* and w^* are two distinct fixed points of A. By the assumption, we obtain $(z^*, w^*) \in \mathfrak{E}(G')$. By (GC-1), we have $(Az^*, Aw^*) \in \mathfrak{E}(G')$. Now, by (GC-2), we have

$$\begin{aligned} & r_{G_b}(Az^*, Aw^*) \leq \lambda r_{G_b}(z^*, w^*) \\ & \Rightarrow r_{G_b}(z^*, w^*) \leq \lambda r_{G_b}(z^*, w^*) \\ & \Rightarrow \lambda \geq 1, \end{aligned}$$

which is a contradiction to $\lambda < 1$. Hence, A has the unique fixed point. □

Now, in order to provide a positive answer to the question of the existence of fixed points for Kannan contraction mappings in GR_bMS, we first define the following definition:

Definition 2.6. Let A be a self-mapping on a graphical rectangular b-metric space (X, r_{G_b}) endowed with a graph G and the coefficient $s \geq 1$, and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Then A is called a Kannan G'-contraction on X if it satisfies the following conditions:

(KGC-1) for each $(x, y) \in \mathfrak{E}(G')$, we have $(Ax, Ay) \in \mathfrak{E}(G')$.

(KGC-2) there exists $\lambda \in [0, \frac{1}{s+1})$ such that

$$r_{G_b}(Ax, Ay) \leq \lambda[r_{G_b}(x, Ax) + r_{G_b}(y, Ay)]$$

for all $x, y \in X$ with $(x, y) \in \mathfrak{E}(G')$.

Lemma 2.7. *Let (X, r_{G_b}) be a graphical rectangular b-metric space endowed with a graph G and the coefficient $s \geq 1$ and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Suppose that $A : X \to X$ is a Kannan G'-contraction mapping. If $(x, Ax) \in \mathfrak{E}(G')$ for every $x \in X$, then*

$$r_{G_b}(A^n x, A^n y) \leq \frac{\lambda^n}{1-\lambda}[r_{G_b}(x, Ax) + r_{G_b}(y, Ay)]$$

for all $n \in \mathbb{N}$ whenever $(x, y) \in \mathfrak{E}(G')$.

Proof. Let $(x, y) \in \mathfrak{E}(G')$. By (KGC-1), we have

$$(A^n x, A^n y) \in \mathfrak{E}(G') \tag{2.7}$$

for all $n \in \mathbb{N}$. Define $\psi(x, y) := r_{G_b}(x, Ax) + r_{G_b}(y, Ay)$ for each $(x, y) \in \mathfrak{E}(G')$. Then

$$\begin{aligned}\psi(Ax, Ay) &= r_{G_b}(Ax, A^2x) + r_{G_b}(Ay, A^2y)\\ &\leq \lambda[r_{G_b}(x, Ax) + r_{G_b}(Ax, A^2x)]\\ &\quad + \lambda[r_{G_b}(y, Ay) + r_{G_b}(Ay, A^2y)]\\ &= \lambda[\psi(x, y) + \psi(Ax, Ay)].\end{aligned}$$

This implies that

$$\psi(Ax, Ay) \leq \frac{\lambda}{1-\lambda}\psi(x, y). \tag{2.8}$$

By repeating this process, we have

$$\psi(A^n x, A^n y) \leq \frac{\lambda^n}{1-\lambda}\psi(x, y) \tag{2.9}$$

for all $n \in \mathbb{N}$. By the Kannan G'-contractive condition, we get

$$\begin{aligned}r_{G_b}(Ax, Ay) &\leq \lambda[r_{G_b}(x, Ax) + r_{G_b}(y, Ay)]\\ &= \lambda\psi(x, y).\end{aligned} \tag{2.10}$$

Now, we obtain

$$\begin{aligned}r_{G_b}(A^2x, A^2y) &\leq \lambda[r_{G_b}(Ax, A^2x) + r_{G_b}(Ay, A^2y)]\\ &\leq \lambda\{\lambda[r_{G_b}(x, Ax) + r_{G_b}(Ax, A^2x)]\\ &\quad + \lambda[r_{G_b}(y, Ay) + r_{G_b}(Ay, A^2y)]\}\\ &= \lambda^2[\psi(x, y) + \psi(Ax, Ay)]\\ &\leq \lambda^2\left[\psi(x, y) + \frac{\lambda}{1-\lambda}\psi(x, y)\right]\\ &= \frac{\lambda^2}{1-\lambda}\psi(x, y).\end{aligned}$$

In the same way, one can show that

$$r_{G_b}(A^n x, A^n y) \leq \frac{\lambda^n}{1-\lambda}\psi(x, y), \tag{2.11}$$

that is,

$$r_{G_b}(A^n x, A^n y) \leq \frac{\lambda^n}{1-\lambda}[r_{G_b}(x, Ax) + r_{G_b}(y, Ay)]. \tag{2.12}$$

This completes the proof. □

The following theorem ensures the existence of fixed points for Kannan contraction mappings in GR_bMS.

Theorem 2.8. *Let (X, r_{G_b}) be a graphical rectangular b-metric space endowed with a graph G and the coefficient $s \geq 1$ and G' be a subgraph of G with $\Delta \subseteq \mathfrak{E}(G')$. Suppose that X is G'-complete, $A : X \to X$ is a one-to-one Kannan G'-contraction mapping and the following conditions hold:*

(I) there exist $x_0 \in X$ such that $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, where l, m are odd positive integers;

(II) $(x, Ax) \in \mathfrak{E}(G')$ for every $x \in X$;

(III) A is sequentially continuous, i.e., if $\{x_n\}$ is a sequence in X and $z \in X$ with $r_{G_b}(x_n, z) \to 0$, then $r_{G_b}(Ax_n, Az) \to 0$.

Then there exist $z^* \in X$ such that the A-Picard sequence $\{x_n\}$ with the initial value $x_0 \in X$ is G'-termwise connected and converges to both z^* and Az^*.

Proof. Let $x_0 \in X$ be such that $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, where l, m are odd integers. Define an A-Picard sequence $\{x_n\}$ by $x_n = Ax_{n-1}$ for all $n \in \mathbb{N}$. Since $Ax_0 \in [\blacktriangle x_0]^l_{G'}$ and $A^2x_0 \in [\blacktriangle x_0]^m_{G'}$, there exist a path $\{y_j\}^l_{j=0}$ such that $x_0 = y_0$, $Ax_0 = y_l$ and $(y_{j-1}, y_j) \in \mathfrak{E}(G')$ with $y_{j-1} \neq y_j$ for all $j = 1, 2, \ldots, l$ and a path $\{w_j\}^m_{j=0}$ such that $x_0 = w_0$, $A^2x_0 = w_m$ and $(w_{j-1}, w_j) \in \mathfrak{E}(G')$ with $w_{j-1} \neq w_j$ for all $j = 1, 2, \ldots, m$. Since A is a Kannan G'-contraction mapping, by (KGC-1), we get

$$(Ay_{j-1}, Ay_j) \in \mathfrak{E}(G') \text{ for } j = 1, 2, \ldots, l.$$

Therefore, $\{Ay_j\}^l_{j=0}$ is a path from $Ay_0 = Ax_0 = x_1$ to $Ay_l = A^2x_0 = x_2$ of length l and $x_2 \in [x_1]^l_{G'}$. Continuing this process, we obtain $\{A^ny_j\}^l_{j=0}$ is a path from $A^ny_0 = A^nx_0 = x_n$ to $A^ny_l = A^nAx_0 = x_{n+1}$ of length l and $x_{n+1} \in [x_n]^l_{G'}$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a G'-termwise connected sequence. Now, we have

$$\begin{aligned}
r_{G_b}(x_n, x_{n+1}) &= r_{G_b}(A^ny_0, A^ny_l)\\
&\leq s[r_{G_b}(A^ny_0, A^ny_1)\\
&\quad + r_{G_b}(A^ny_1, A^ny_2)\\
&\quad + r_{G_b}(A^ny_2, A^ny_l)]\\
&\leq s[r_{G_b}(A^ny_0, A^ny_1)\\
&\quad + r_{G_b}(A^ny_1, A^ny_2)]\\
&\quad + s^2[r_{G_b}(A^ny_2, A^ny_3)\\
&\quad + r_{G_b}(A^ny_3, A^ny_4)\\
&\quad + r_{G_b}(A^ny_4, A^ny_l)]\\
&\vdots\\
&\leq s[r_{G_b}(A^ny_0, A^ny_1)\\
&\quad + r_{G_b}(A^ny_1, A^ny_2)]\\
&\quad + s^2[r_{G_b}(A^ny_2, A^ny_3)\\
&\quad + r_{G_b}(A^ny_3, A^ny_4)] + \cdots\\
&\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^ny_{l-3}, A^ny_{l-2})\\
&\quad + r_{G_b}(A^ny_{l-2}, A^ny_{l-1})\\
&\quad + r_{G_b}(A^ny_{l-1}, A^ny_l)].
\end{aligned} \tag{2.13}$$

Since $(y_{j-1}, y_j) \in \mathfrak{E}(G')$, we have $(A^ny_{j-1}, A^ny_j) \in \mathfrak{E}(G')$ for all $j = 1, 2, \ldots, l$ and for all $n \in \mathbb{N}$. By using Lemma 2.7, the inequality (2.13) becomes

$$\begin{aligned}
r_{G_b}(x_n, x_{n+1}) \leq \frac{\lambda^n}{1-\lambda}\{&s[\psi(y_0, y_1) + \psi(y_1, y_2)]\\
&+ s^2[\psi(y_2, y_3) + \psi(y_3, y_4)] + \cdots\\
&+ s^{\frac{l-1}{2}}[\psi(y_{l-3}, y_{l-2})\\
&+ \psi(y_{l-2}, y_{l-1}) + \psi(y_{l-1}, y_l)]\}.
\end{aligned} \tag{2.14}$$

Similarly, we have

$$\begin{aligned}
r_{G_b}(x_n, x_{n+2}) &= r_{G_b}(A^nw_0, A^nw_m)\\
&\leq s[r_{G_b}(A^nw_0, A^nw_1)\\
&\quad + r_{G_b}(A^nw_1, A^nw_2)]\\
&\quad + s^2[r_{G_b}(A^nw_2, A^nw_3)\\
&\quad + r_{G_b}(A^nw_3, A^nw_4)] + \cdots\\
&\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^nw_{m-3}, A^nw_{m-2})\\
&\quad + r_{G_b}(A^nw_{m-2}, A^nw_{m-1})\\
&\quad + r_{G_b}(A^nw_{m-1}, A^ny_m)].
\end{aligned} \tag{2.15}$$

Since $(w_{j-1}, w_j) \in \mathfrak{E}(G')$, we have $(A^nw_{j-1}, A^nw_j) \in \mathfrak{E}(G')$ for all $j = 1, 2, \ldots, m$ and for all $n \in \mathbb{N}$. By using Lemma 2.7, the inequality (2.15) becomes

$$\begin{aligned}
r_{G_b}(x_n, x_{n+2}) \leq \frac{\lambda^n}{1-\lambda}\{&s[\psi(w_0, w_1)\\
&+ \psi(w_1, w_2)]\\
&+ s^2[\psi(w_2, w_3)\\
&+ \psi(w_3, w_4)] + \cdots\\
&+ s^{\frac{m-1}{2}}[\psi(w_{m-3}, w_{m-2})
\end{aligned}$$

$$+ \psi(w_{m-2}, w_{m-1}) + \psi(w_{m-1}, w_m)]. \quad (2.16)$$

Now, we will show that the G'-termwise connected A-Picard sequence $\{x_n\}$ is a Cauchy sequence i.e., for $p \geq 1$, $r_{G_b}(x_n, x_{n+p}) \to 0$ as $n \to \infty$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $r_{G_b}(x_n, x_{n+p}) \to 0$ as $n \to \infty$. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Case-I. If p is an odd integer, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \leq\ & s[r_{G_b}(x_n, x_{n+1}) \\ & + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) \\ & + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-1}{2}}[r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\ & + r_{G_b}(x_{n+p-2}, x_{n+p-1}) \\ & + r_{G_b}(x_{n+p-1}, x_{n+p})]. \end{aligned}$$

From inequality (2.14), we can say that

$$r_{G_b}(x_n, x_{n+p}) \to 0 \text{ as } n \to \infty. \quad (2.17)$$

Case-II. If p is an even integer, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \leq\ & s[r_{G_b}(x_n, x_{n+1}) \\ & + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) \\ & + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-2}{2}}[r_{G_b}(x_{n+p-4}, x_{n+p-3}) \\ & + r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\ & + r_{G_b}(x_{n+p-2}, x_{n+p})]. \end{aligned}$$

From inequality (2.14) and (2.16), we have

$$r_{G_b}(x_n, x_{n+p}) \to 0 \text{ as } n \to \infty. \quad (2.18)$$

From Case-(I) and Case-(II), one can say that $\{x_n\}$ is a Cauchy sequence. Since X is G'-complete, there exist $z^* \in X$ such that $x_n \to z^*$ as $n \to \infty$. Since A is sequentially continuous, we obtain $x_{n+1} = Ax_n \to Az^*$ as $n \to \infty$. This completes the proof. □

Theorem 2.9. *Assume that all hypotheses of Theorem 2.8 hold and further suppose that the quadruple (X, r_{G_b}, G', A) has the property S^*. Then A has a fixed point in X.*

Proof. From the proof of Theorem 2.8 and Property S^*, we get this result. □

Theorem 2.10. *Assume that all hypotheses of Theorem 2.9 hold and further suppose that $(z^*, w^*) \in \mathfrak{E}(G')$ for all $z^*, w^* \in Fix(A)$, where $Fix(A)$ is the set of all fixed points of A. Then A has the unique fixed point.*

Proof. From Theorem 2.9, A has a fixed point. Now, we will show the uniqueness of a fixed point. Assume that z^* and w^* are two fixed points of A. By the assumption, we obtain $(z^*, w^*) \in \mathfrak{E}(G')$. By (KGC-1), we have $(Az^*, Aw^*) \in \mathfrak{E}(G')$. Now, by (KGC-2), we have

$$\begin{aligned} r_{G_b}(z^*, w^*) &= r_{G_b}(Az^*, Aw^*) \\ &\leq \lambda[r_{G_b}(z^*, Az^*) \\ &\quad + r_{G_b}(w^*, Aw^*)] = 0. \end{aligned}$$

This implies that $z^* = w^*$. Hence, A has the unique fixed point. □

3. Conclusion

In this work we presented an example of a graphical rectangular b-metric space in which a A-Picard sequence in the sense of Mudasir Younis et.al.[2] is not Cauchy. To overcome this drawback, we formulated suitable conditions and made appropriate corrections to Theorem 4.2 given in [2]. Moreover, we provided a positive answer to the question of the existence of a fixed point for Kannan contraction mappings in the aforesaid space.

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