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Original research article

Green's Relations on a Ternary Monoid *Hyp*(2)

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ABSTRACT

The Green's relations on a monoid of hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and S. L. Wismath in [1]. Nevertheless, this study extends the similar concept to a ternary monoid of hypersubstitutions of type $\tau = (2)$ which is the set $Hyp(2)$ of all hypersubstitutions of type $\tau = (2)$ together with the ternary operation $[-,-,-]$ and identity element $\sigma_{id} = \sigma_{f(x_1,x_2)}$. Some of the algebraic-structural properties of this monoid were studied by the author in [5]. The Green's relations on a ternary semigroup were studied by Rabah Killil in [2]. In this paper, we apply the concept of Rabah Killil to a ternary monoid of hypersubstitutions of type $\tau = (2)$ and describe the classes of elements of this monoid under Green's relations $\mathcal{L}^T, \mathcal{R}^T, \mathcal{D}^T$ and some of the $\mathcal{I}^T-$class of elements.

Keywords: Green's relations; Hypersubstitutions; Ternary monoid

1. Introduction

In pure mathematics, the monoid of all hypersubstitutions were widely studied by many authors. Furthermore, they introduced the basic concepts and studied the algebraic-structural properties of this monoid. Moreover, they characterized the special elements and special classes of elements of this monoid.

In 1998, Denecke and Koppitz [3] studied the finite monoids of hypersubstitutions of type $\tau = (2)$. Also, they characterized the set of all finite submonoids of $Hyp(2)$ and studied some properties of all finite submonoids. In the same year, the monoid of hypersubstitutions of type $\tau = (2)$ was studied by Denecke and Wismath [1]. Furthermore, they constructed the monoid of hypersubstitutions of type $\tau = (2)$ and studied the semigroup properties of this monoid. Also, they characterized the idempotents and described Green's relations on this monoid. Afterward, in 2000, Wismath [4] extended the concept of [1]

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by studying the semigroup properties of the monoid of hypersubstitutions of type $\tau = (n), n \geq 2$. They characterized the projection, dual, and idempotent on this monoid, and also described the classes of these elements under Green's relations.

The ternary semigroup is the topic in pure mathematics which has been studied by many authors. In 2007, Kar and Maity [6] introduced the notion of congruence on ternary semigroup and studied the interested properties of it. They determined the cancellative congruence, group congruence, and Rees congruence on a ternary semigroup. In 2013, Kellil [2] studied Green's relations on a ternary semigroup in view of those obtained in binary semigroups. They studied the quality of idempotents with respect to the Green's relations and studied the particular case of the ternary inverse semigroup.

By using the concept of ternary semigroup, in 2019, Chansuriya and Leeratanavalee [5] constructed the ternary monoid of all hypersubstitutions of type $\tau = (2)$ and studied some algebraic-structural properties, idempotent and regular elements of this ternary monoid. Likewise, they determined the ideals of submonoid of this ternary monoid. In this study, we focus on the equivalence relations, called Green's relations, which are concerned with the mutual divisibility of elements. These relations are used in the classification of the elements of a ternary monoid. It is an important tool for study on divisibility in ternary monoids. In this paper, we describe the classes of elements of this ternary monoid $Hyp(2)$ under Green's relations $\mathcal{L}^T, \mathcal{R}^T, \mathcal{D}^T$ and some of the $\mathcal{I}^T$−class of elements. It is a tool for determining some structural properties of this ternary monoid.

2. Preliminaries

2.1 Hypersubstitutions

Let f_i be n_i-ary operation symbol of type τ which $n_i \in \mathbb{N}$ and $X_n = \{x_1, \cdots, x_n\}$ be a set of variables. The n-ary terms of type τ are inductively defined in the following way:

(i) Every variable $x_i \in X_n$ is an n-ary term.

(ii) If $t_1, \cdots, t_{n_i}$ are n-ary terms, then $f_i(t_1, \cdots, t_{n_i})$ is an n-ary term.

The set of all n-ary terms of type τ is denoted by $W_\tau(X_n)$ and we also have $W_\tau(X_n) := \cup_{n\in\mathbb{N}} W_\tau(X_n)$ as the set of all terms of type τ .

A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which preserves arities, i.e. $\sigma(f_i) \in W_\tau(X_{n_i})$. We denoted the set of all hypersubstitutions of type τ by $Hyp(\tau)$. To define the binary operation on $Hyp(\tau)$, we first give the concept of the superposition of terms as following.

Let $W_\tau(X_n)$ be the set of all n-ary terms of type $\tau = (n_i)_{i\in I}$ where $, n \in \mathbb{N}$. The superposition operation $S^n : W_\tau(X_n) \times W_\tau(X_n)^n \to W_\tau(X_n)$ (for terms) is inductively defined by the following steps:

(i) $S^n(x_i, t_1, \ldots, t_n) := t_i$ where $x_i \in X_n, t_1, \ldots, t_n \in W_\tau(X_n)$

(ii) $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$ where $f_i(s_1, \ldots, s_{n_i}) \in W_\tau(X_n)$.

For any $\sigma \in Hyp(\tau)$, the extension $\hat{\sigma}$ of σ is a mapping $\hat{\sigma} : W_\tau(X_n) \to W_\tau(X_n)$ where $t \in W_\tau(X)$ defined inductively by

(i) $\hat{\sigma}[x] := x$, for any variable $x \in X$ and

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_{n_i})] := S^{n_i}(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])$ where $\hat{\sigma}[f_j]; 1 \leq j \leq n_i$ are already defined.

Then the binary operation $\circ_h$ on $Hyp(\tau)$ is defined by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma_1} \circ \sigma_2$ for any$\sigma_1, \sigma_2 \in Hyp(\tau)$ and $\circ$ is the usual composition of mappings.

Proposition 2.1 ([7]). *If $\hat{\sigma}$ is the extension of a hypersubstitution σ, then for $n \in \mathbf{N}$,* $\hat{\sigma}[S_n^n(t,t_1,\ldots,t_n)] = S_n^n(\hat{\sigma}[t],\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$.

Proposition 2.2 ([7]). *Let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then $\hat{\sigma}_1 \circ \sigma_2$ is a hypersubstitution, and*

$$(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Let σ_{id} be a hypersubstitution which is defined by $\sigma_{id}(f_i) = f_i(x_1,\ldots,x_{n_i})$ for all $i \in I$. Then the set $\underline{Hyp(\tau)} := (Hyp(\tau), \circ_h, \sigma_{id})$ forms a monoid. For more details on hypersubstitutions and the monoid of hypersubstitutions of type τ see [7].

2.2 The Ternary monoid $Hyp(2)$

We first give the concept of the ternary monoid $Hyp(2)$ as follows.

Define a ternary operation $[-,-,-] : Hyp(2)\times Hyp(2)\times Hyp(2) \to Hyp(2)$ by

$$[\sigma_1,\sigma_2,\sigma_3] := \sigma_1 \circ_h \sigma_2 \circ_h \sigma_3$$

for each $\sigma_1,\sigma_2,\sigma_3 \in Hyp(2)$. Then we obtain the following propositions.

Proposition 2.3 ([5]). *$\underline{Hyp(2)} := (Hyp(2), [-,-,-])$ is a ternary semigroup.*

Let $\sigma_t, \sigma_{id} \in Hyp(2)$ where $t \in W(\{x_1,x_2\})$ and $\sigma_{id} = \sigma_{f(x_1,x_2)}$. Then we have

$$\begin{aligned}[\sigma_{id},\sigma_{id},\sigma_t](f) &= (\sigma_{id} \circ_h \sigma_{id} \circ_h \sigma_t)(f)\\ &= (\sigma_{id} \circ_h \sigma_t)(f)\\ &= \sigma_t(f).\end{aligned}$$

Similarly, we have $[\sigma_{id},\sigma_t,\sigma_{id}](f) = \sigma_t(f) = [\sigma_t,\sigma_{id},\sigma_{id}](f)$.

Proposition 2.4 ([5]). *$\underline{Hyp(2)} := (Hyp(2), [-,-,-], \sigma_{id})$ is a ternary monoid.*

For more details on ternary monoid $Hyp(2)$ see [5].

2.3 Green's relations on a ternary semigroup

Let T be a ternary semigroup and $T^1 := T \cup \{1\}$ where 1 is the identity for the ternary operation. Define five equivalence relations $\mathcal{L}^T, \mathcal{R}^T, \mathcal{I}^T, \mathcal{H}^T, \mathcal{D}^T$ on T by:

- $a\mathcal{L}^T b$ if and only if for all $a, b \in T^1$, $a = xyb$ and $b = uva$ for some $x, y, u, v \in T^1$.
- $a\mathcal{R}^T b$ if and only if for all $a, b \in T^1$, $a = bxy$ and $b = auv$ for some $x, y, u, v \in T^1$.
- $a\mathcal{I}^T b$ if and only if for all $a, b \in T^1$, $a = xby$ and $b = uav$ for some $x, y, u, v \in T^1$.
- For all $a, b \in T^1$, $a\mathcal{H}^T b$ if and only if $a\mathcal{L}^T b$ and $a\mathcal{R}^T b$.
- $\mathcal{D}^T = \mathcal{L}^T \circ \mathcal{R}^T$.

The equivalence relations $\mathcal{L}^T, \mathcal{R}^T, \mathcal{I}^T, \mathcal{H}^T, \mathcal{D}^T$ are called the *Green's relations on T*. For any $a \in T^1$, we denoted the equivalence classes $\mathcal{L}^T, \mathcal{R}^T, \mathcal{I}^T, \mathcal{H}^T$, and $\mathcal{D}^T$ containing a by $L_a^T, R_a^T, I_a^T, H_a^T$, and D_a^T, respectively. For more details on Green's relations on ternary semigroups see [2, 8].

Throughout this present paper, we denote:

σ_t := the hypersubstitution σ of type τ which maps f to the term t,

$var(t)$:= the set of all variables occurring in the term t,

$vb(t)$:= the total number of variables occuring in the term t,

$W_{(2)}(\{x_1, x_2\}) := \{t \in W_{(2)}(X_2) \mid x_1, x_2 \in var(t)\}$,

$W_{(2)}(\{x_1\}) := \{t \in W_{(2)}(X_2) \mid x_1 \in var(t), x_2 \notin var(t)\}$,

$W_{(2)}(\{x_2\}) := \{t \in W_{(2)}(X_2) \mid x_1 \notin var(t), x_2 \in var(t)\}$,

$W(\{x_1\}) := W_{(2)}(\{x_1\}) \setminus \{x_1\}$,

$W(\{x_2\}) := W_{(2)}(\{x_2\}) \setminus \{x_2\}$,

$P(2) := \{\sigma \in Hyp(2) \mid \sigma(f)$ is a variable$\}$,

$E_{x_1} := \{\sigma_{f(x_1,u)} | u \in W_{(2)}(\{x_1\})\}$,

$E_{x_2} := \{\sigma_{f(v,x_2)} | v \in W_{(2)}(\{x_2\})\}$,

$\overline{f(c,d)}$:= the term obtained from $f(c,d)$ by interchanging all occurrences of the letters x_1 and x_2, i.e. $\overline{f(c,d)} = s^2(\underline{f(c,d)}, x_2, x_1)$ and $f(c,d) = s^2(\overline{f(c,d)}, x_2, x_1)$,

$f(c,d)'$:= the term defined inductively by $x_i' = x_i$ and $f(c,d)' = f(d',c')$,

$\overline{S} := \{\bar{s} \mid s \in S\}$,

$S' := \{s' \mid s \in S\}$.

3. Green's relations on a ternary monoid Hyp(2)

Lemma 3.1. *Let $\sigma_r, \sigma_s, \sigma_t \in Hyp(2)$. If $[\sigma_s, \sigma_t, \sigma_r] = \sigma_{id}$, then the following statements hold:*

(i) $\sigma_s = \sigma_t = \sigma_r = \sigma_{id}$,

(ii) $\sigma_s = \sigma_{id}$ *and* $\sigma_t = \sigma_r = \sigma_{f(x_2,x_1)}$,

(iii) $\sigma_t = \sigma_{id}$ *and* $\sigma_s = \sigma_r = \sigma_{f(x_2,x_1)}$,

(iv) $\sigma_r = \sigma_{id}$ *and* $\sigma_s = \sigma_t = \sigma_{f(x_2,x_1)}$.

Proof. Assume that $[\sigma_s, \sigma_t, \sigma_r] = \sigma_{id}$. Since $f(x_1, x_2) \notin X_2$, then $s, t, r \notin X_2$. Thus $s = f(a,b), t = f(c,d)$ and $r = f(m,n)$ for some $a,b,c,d,m,n \in W_{(2)}(X_2)$. Since $[\sigma_s, \sigma_t, \sigma_r](f) = (\sigma_s \circ_h \sigma_t \circ_h \sigma_r)(f) = \sigma_{id}(f)$, then we have $\hat{\sigma}_{f(a,b)}[S^2(f(c,d), \hat{\sigma}_t[m], \hat{\sigma}_t[n])]$
$= \hat{\sigma}_{f(a,b)}[f(S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n]), S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n]))]$
$= f(x_1, x_2)$.

That is

$$S^2(a, \hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])], \hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])]) = x_1 \quad (3.1)$$

$$S^2(b, \hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])], \hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])]) = x_2. \quad (3.2)$$

If $a = x_1$, then, by (3.1), we obtain that $\hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])] = x_1$. So $c = m = x_1$ or $c = x_2, n = x_1$.

If $a = x_2$, then, by (3.1), we obtain that $\hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])] = x_1$. So $d = m = x_1$ or $d = x_2, n = x_1$.

If $b = x_1$, then, by (3.2), we obtain that $\hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])] = x_2$. So $c = n = x_2$ or $c = x_1, m = x_2$.

If $b = x_2$, then, by (3.2), we obtain that $\hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])] = x_2$. So $d = n = x_2$ or $d = x_1, m = x_2$. □

Lemma 3.2. *Let $\sigma_r, \sigma_s, \sigma_t \in Hyp(2)$. If $[\sigma_s, \sigma_t, \sigma_r] = \sigma_{f(x_2,x_1)}$, then the following statements hold:*

(i) $\sigma_s = \sigma_t = \sigma_r = \sigma_{f(x_2,x_1)}$,

(ii) $\sigma_s = \sigma_{f(x_2,x_1)}$ *and* $\sigma_t = \sigma_r = \sigma_{id}$,

(iii) $\sigma_t = \sigma_{f(x_2,x_1)}$ *and* $\sigma_s = \sigma_r = \sigma_{id}$,

(iv) $\sigma_r = \sigma_{f(x_2,x_1)}$ *and* $\sigma_s = \sigma_t = \sigma_{id}$.

Proof. The proof is similar to Lemma 3.1 by considering the following equations:

$$S^2(a, \hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])], \hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])]) = x_2 \quad (3.3)$$

$$S^2(b, \hat{\sigma}_s[S^2(c, \hat{\sigma}_t[m], \hat{\sigma}_t[n])], \hat{\sigma}_s[S^2(d, \hat{\sigma}_t[m], \hat{\sigma}_t[n])]) = x_1. \quad (3.4)$$

□

Proposition 3.3. *For any* $\sigma_t \in Hyp(2) \setminus P(2)$*, we have* $\sigma_t \mathcal{R}^T \sigma_{\bar{t}}$*,* $\sigma_t \mathcal{L}^T \sigma_{t'}$ *and* $\sigma_t \mathcal{D}^T \sigma_{t'} \mathcal{D}^T \sigma_{\bar{t}} \mathcal{D}^T \sigma_{\overline{t'}}$.

Proof. Let $\sigma_t \in Hyp(2) \setminus P(2)$. Then we have $\sigma_t \circ_h \sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)} = \sigma_{\bar{t}}$, $\sigma_{\bar{t}} \circ_h \sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)} = \sigma_t$, $\sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)} \circ_h \sigma_{t'} = \sigma_t$ and $\sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)} \circ_h \sigma_t = \sigma_{t'}$. So $\sigma_t \mathcal{L}^T \sigma_{t'}$ and $\sigma_t \mathcal{R}^T \sigma_{\bar{t}}$. Therefore, $\sigma_t \mathcal{D}^T \sigma_{t'} \mathcal{D}^T \sigma_{\bar{t}} \mathcal{D}^T \sigma_{\overline{t'}}$. □

Proposition 3.4. *All of* $\mathcal{R}^T-, \mathcal{L}^T-$ *and* $\mathcal{D}^T-$*class of* σ_{id} *are equal to* $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$*. Moreover, the* $\mathcal{I}^T-$*class of* σ_{id} *is equal to the* $\mathcal{D}^T-$*class.*

Proof. By Lemma 3.1 and Lemma 3.2, we obtain that σ_{id} and $\sigma_{f(x_2,x_1)}$ are $\mathcal{R}^T-, \mathcal{L}^T-, \mathcal{I}^T-$ and $\mathcal{D}^T-$related. Thus we have the $\mathcal{R}^T-, \mathcal{L}^T-, \mathcal{I}^T-$ and $\mathcal{D}^T-$class of σ_{id} is contain at least $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Let $\sigma_s \in Hyp(2)$ where $\sigma_s \mathcal{D}^T \sigma_{id}$. Then $\sigma_s \mathcal{L}^T \sigma_t$ and $\sigma_t \mathcal{R}^T \sigma_{id}$ for some $\sigma_t \in Hyp(2)$. So there exist $\sigma_u, \sigma_v, \sigma_r, \sigma_l, \sigma_p, \sigma_q, \sigma_m, \sigma_n \in Hyp(2)$ such that $\sigma_s = \sigma_u \circ_h \sigma_v \circ_h \sigma_t$, $\sigma_t = \sigma_r \circ_h \sigma_l \circ_h \sigma_s$ and $\sigma_t = \sigma_{id} \circ_h \sigma_p \circ_h \sigma_q$, $\sigma_{id} = \sigma_t \circ_h \sigma_m \circ_h \sigma_n$.

Now, we consider $\sigma_{id} = \sigma_t \circ_h \sigma_m \circ_h \sigma_n$. By Lemma 3.1, we have $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$. Since $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$ and $\sigma_s = \sigma_u \circ_h \sigma_v \circ_h \sigma_t$, again by Lemma 3.1, we get $\sigma_s = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$. So the $\mathcal{D}^T-$class of σ_{id} is equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

From $\mathcal{L}^T \subseteq \mathcal{D}^T$ and $\mathcal{R}^T \subseteq \mathcal{D}^T$, we have the $\mathcal{L}^T-$ and $\mathcal{R}^T-$class of σ_{id} are equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Let $\sigma_t \in Hyp(2)$ where $\sigma_s \mathcal{I}^T \sigma_{id}$. Then there exist $\sigma_u, \sigma_v, \sigma_m, \sigma_n \in Hyp(2)$ such that $\sigma_t = \sigma_u \circ_h \sigma_{id} \circ_h \sigma_v$ and $\sigma_{id} = \sigma_m \circ_h \sigma_t \circ_h \sigma_n$.

From $\sigma_{id} = \sigma_m \circ_h \sigma_t \circ_h \sigma_n$, by Lemma 3.1, we get $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$. So, the $\mathcal{I}^T-$class of σ_{id} is equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. □

Lemma 3.5. *The element* $\sigma_{x_i} \in P(2)$ *is* $\mathcal{L}^T-$*related only to itself and* $\mathcal{R}^T-$*related,* $\mathcal{I}^T-$*related and* $\mathcal{D}^T-$*related to all elements in* $P(2)$*. Thus the set* $P(2)$ *forms the* $\mathcal{R}^T-$*,* $\mathcal{I}^T-$ *and* $\mathcal{D}^T-$*class of* σ_{x_i}.

Proof. Since $[\sigma, \sigma, \sigma_{x_i}](f) = (\sigma \circ_h \sigma \circ_h \sigma_{x_i})(f) = \sigma_{x_i}(f)$ for all $\sigma \in Hyp(2)$, then we have that any $\sigma_{x_i} \in P(2)$ can be $\mathcal{L}^T-$related only to itself. Since $(\sigma_{x_j} \circ_h \sigma \circ_h \sigma_{x_i})(f) = \sigma_{x_i}(f)$ for all $\sigma \in Hyp(2)$, so σ_{x_i} is $\mathcal{R}^T-$related to all elements in $P(2)$. In the same way, we also have $(\sigma \circ_h \sigma_{x_j} \circ_h \sigma_{x_i})(f) = \sigma_{x_i}(f)$ for all $\sigma \in Hyp(2)$, thus σ_{x_i} is $\mathcal{I}^T-$related to all elements in $P(2)$. Since $\mathcal{R}^T \subseteq \mathcal{D}^T$, then σ_{x_i} is $\mathcal{D}^T-$related.

Let $\sigma_s \in Hyp(2)$ such that $\sigma_s \mathcal{I}^T \sigma_{x_i}$. Then $\sigma_{x_i} = \sigma_t \circ_h \sigma_s \circ_h \sigma_r$ and $\sigma_s = \sigma_p \circ_h \sigma_{x_i} \circ_h \sigma_q$ for some $\sigma_p, \sigma_q, \sigma_r, \sigma_t \in Hyp(2)$. Because of $\sigma_p \circ_h \sigma_{x_i} \circ_h \sigma_q \in P(2)$, this force that $\sigma_s \in P(2)$. This implies that the $\mathcal{I}^T-$class of σ_{x_i} is $P(2)$.

Let $\sigma_s \in Hyp(2)$ such that $\sigma_s \mathcal{D}^T \sigma_{x_i}$. Then $\sigma_s \mathcal{L}^T \sigma_t$ and $\sigma_t \mathcal{R}^T \sigma_{x_i}$

for some $\sigma_t \in Hyp(2)$. Since σ_{x_i} is $\mathcal{R}^T$−related to all elements in $P(2)$, this implies that $\sigma_t \in P(2)$. So, we obtain that $\sigma_s \in P(2)$. Hence the $\mathcal{D}^T$−class of σ_{x_i} is $P(2)$. Since $\mathcal{R}^T \subseteq \mathcal{D}^T$, then the $\mathcal{R}^T$−class of σ_{x_i} is $P(2)$. □

Lemma 3.6. *Let $\sigma_s, \sigma_t \in Hyp(2) \setminus P(2)$. Then σ_t is $\mathcal{R}^T$−related to σ_s if and only if $s = t$ or $s = \bar{t}$.*

Proof. Assume that $\sigma_t \mathcal{R}^T \sigma_s$. Then there exist $\sigma_g, \sigma_j, \sigma_k, \sigma_w \in Hyp(2)$ such that $\sigma_t = \sigma_s \circ_h \sigma_g \circ_h \sigma_j$ and $\sigma_s = \sigma_t \circ_h \sigma_k \circ_h \sigma_w$. Since $\sigma_s, \sigma_t \notin P(2)$, then we have $\sigma_g, \sigma_j, \sigma_k, \sigma_w \notin P(2)$. So, $g = f(u,v), j = f(m.n), k = f(r,l)$ and $w = f(p,q)$ where $u,v,m,n,r,l,p,q \in W_{(2)}(X_2)$. Suppose that $t = f(a,b), s = f(c,d)$ where $a,b,c,d \in W_{(2)}(X_2)$, then, by the assumption above, we have to consider the following two equations

$$\begin{aligned} f(a,b) &= S^2(f(c,d), \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[m], \\ &\quad \hat{\sigma}_{f(u,v)}[n])], \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(v, \hat{\sigma}_{f(u,v)}[m], \\ &\quad \hat{\sigma}_{f(u,v)}[n])]) \qquad (3.5) \end{aligned}$$

$$\begin{aligned} f(c,d) &= S^2(f(a,b), \\ &\quad \hat{\sigma}_{f(a,b)}[S^2(r, \hat{\sigma}_{f(r,l)}[p], \\ &\quad \hat{\sigma}_{f(r.l)}[q])], \\ &\quad \hat{\sigma}_{f(a,b)}[S^2(l, \hat{\sigma}_{f(r,l)}[p], \\ &\quad \hat{\sigma}_{f(r.l)}[q])]). \qquad (3.6) \end{aligned}$$

Thus, by (3.5) and (3.6), we obtain that $vb(f(a,b)) = vb(f(c,d))$. We will consider the following cases.

Case 1 : $f(c,d) \in W_2(\{x_1,x_2\})$. Suppose that $m \notin X_2$ or $n \notin X_2$. Then $\hat{\sigma}_{f(u,v)}[m] \notin X_2$ or $\hat{\sigma}_{f(u,v)}[n] \notin X_2$. This forces that $u \notin X_2$ or $v \notin X_2$. From (3.5) and $x_1, x_2 \in f(c,d)$, we obtain that $vb(f(a,b)) > vb(f(c,d))$ which is a contradiction. So $m,n \in X_2$. Suppose that $m = n = x_1$. Then $\hat{\sigma}_{f(u,v)}[m] = \hat{\sigma}_{f(u,v)}[n] = x_1$. So, we have $f(u,v) \in W(\{x_1\})$, thus $f(a,b) \in W(\{x_1\})$. Suppose that $p \notin X_2$ or $q \notin X_2$. So, $\hat{\sigma}_{f(r,l)}[p] \notin X_2$ or $\hat{\sigma}_{f(r,l)}[q] \notin X_2$. Then $\hat{\sigma}_{f(a,b)}[S^2(r, \hat{\sigma}_{f(r,l)}[p], \hat{\sigma}_{f(r.l)}[q])] \notin X_2$. Since $x_1 \in var(f(a,b))$, then, from (3.6), we obtain that $vb(f(c,d)) > vb(f(a,b))$ and it is a contradiction. Hence $p,q \in X_2$, so we have $\hat{\sigma}_{f(a,b)}[S^2(r, \hat{\sigma}_{f(r,l)}[p], \hat{\sigma}_{f(r.l)}[q])] \in X_2$. Since $f(a,b) \in W(\{x_1\})$, from (3.6) again we get $x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$ which contradicts to $f(c,d) \in W(\{x_1,x_2\})$.

Similarly, we can prove the case $m = n = x_2$.

If $m = x_1$ and $n = x_2$, then $\hat{\sigma}_{f(u,v)}[m] = x_1$ and $\hat{\sigma}_{f(u,v)}[n] = x_2$. So, from (3.5), we have

$$\begin{aligned} f(a,b) &= S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \\ &\quad \hat{\sigma}_{f(c,d)}[v]) \qquad (3.7) \end{aligned}$$

Suppose that $u \notin X_2$ or $v \notin X_2$. Then $\hat{\sigma}_{f(c,d)}[u] \notin X_2$ or $\hat{\sigma}_{f(c,d)}[v] \notin X_2$. Since $x_1, x_2 \in var(f(c,d))$, so $vb(f(a,b)) > vb(f(c,d))$ which is a contradiction. So $u,v \in X_2$. If $u = v = x_1$ or $u = v = x_2$, then, by the same proof as the case $m = n = x_1$, we have $x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$. If $u = x_1, v = x_2$, then $\hat{\sigma}_{f(c,d)}[u] = x_1$ and $\hat{\sigma}_{f(c,d)}[v] = x_2$. So, from (3.7), we obtain that $f(a,b) = f(c,d)$. If $u = x_1, v = x_2$, then $\hat{\sigma}_{f(c,d)}[u] = x_2$ and $\hat{\sigma}_{f(c,d)}[v] = x_1$. So, from (3.7) again we obtain that $f(a,b) = \overline{f(c,d)}$.

If $m = x_2$ and $n = x_1$, then $\hat{\sigma}_{f(u,v)}[m] = x_2$ and $\hat{\sigma}_{f(u,v)}[n] = x_1$. So, from (3.5) we have

$$f(a,b) = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[\overline{u}], \hat{\sigma}_{f(c,d)}[\overline{v}]) \quad (3.8)$$

Then, by the same proof as the case $m = x_1$ and $n = x_2$ and equation (3.8), we have $f(a,b) = f(c,d)$ or $f(a,b) = \overline{f(c,d)}$.

Case 2 : $f(c,d) \in W(\{x_1\})$. Suppose that $m \notin X_2$ or $n \notin X_2$. Then $\hat{\sigma}_{f(u,v)}[m] \notin X_2$ or $\hat{\sigma}_{f(u,v)}[n] \notin X_2$. So $\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[m], \hat{\sigma}_{f(u,v)}[n])] \notin X_2$. Since $x_1 \in f(c,d)$ and $\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[m], \hat{\sigma}_{f(u,v)}[n])] \notin X_2$, then, from (3.5), we have $vb(f(a,b)) > vb(f(c,d))$ and it is a contradiction. So $m, n \in X_2$ and thus $\hat{\sigma}_{f(u,v)}[m] = m$ and $\hat{\sigma}_{f(u,v)}[n] = n$.

If $m = n = x_1$, then $\hat{\sigma}_{f(u,v)}[m] = \hat{\sigma}_{f(u,v)}[n] = x_1$. Thus $S^2(u, \hat{\sigma}_{f(u,v)}[m], \hat{\sigma}_{f(u,v)}[n]) = u \in W(\{x_1\})$. So $\hat{\sigma}_{f(c,d)}[u] \in W(\{x_1\})$.

Suppose that $u \neq x_1$, then $\hat{\sigma}_{f(c,d)}[u] \in W(\{x_1\}) \setminus \{x_1\}$. From (3.5) and $x_1 \in var(f(c,d))$, we have $vb(f(a,b)) > vb(f(c,d))$ which is a contradiction. So $u = x_1$, and thus $\hat{\sigma}_{f(c,d)}[u] = x_1$. Again from (3.5) and $x_1 \in var(f(c,d))$, we have $f(a,b) = f(c,d)$. By the same proof as this case, if $m = n = x_2$, then we have $f(a,b) = \overline{f(c,d)}$.

If $m = x_1, n = x_2$, then $\hat{\sigma}_{f(u,v)}[m] = x_1$ and $\hat{\sigma}_{f(u,v)}[n] = x_2$. So $\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[m], \hat{\sigma}_{f(u,v)}[n])] = \hat{\sigma}_{f(c,d)}[u]$. If $u \notin X_2$, then $\hat{\sigma}_{f(c,d)}[u] \notin X_2$. So from (3.5) and $x_1 \in f(c,d)$, $vb(f(a,b)) > vb(f(c,d))$ which is a contradiction. Thus $u \in X_2$. Since $u = x_1$ or $u = x_2$, then, from (3.5), we have $f(a,b) = f(c,d)$ or $f(a,b) = \overline{f(c,d)}$, respectively.

If $m = x_2, n = x_1$, then $\hat{\sigma}_{f(u,v)}[m] = x_2$ and $\hat{\sigma}_{f(u,v)}[n] = x_1$. So $\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[m], \hat{\sigma}_{f(u,v)}[n])] = \hat{\sigma}_{f(c,d)}[\overline{u}]$. By the same proof as the case $m = x_1, n = x_2$, we have $f(a,b) = f(c,d)$ or $f(a,b) = \overline{f(c,d)}$.

Case 3 : $f(c,d) \in W(\{x_2\})$. The proof is similarly to the case $f(c,d) \in W(\{x_1\})$. Thus $f(a,b) = f(c,d)$ or $f(a,b) = \overline{f(c,d)}$.

Conversely, assume that $s = t$ or $s = \bar{t}$. Then we consider $\sigma_{\bar{t}} = \sigma_t \circ_h \sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)}$ and $\sigma_t = \sigma_{\bar{t}} \circ_h \sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_1,x_2)}$. Hence, we have $\sigma_s \mathcal{R}^T \sigma_t$. □

Proposition 3.7. *(i) If $c, d \in W(\{x_1\})$ (or dually both in $W(\{x_2\})$), then for any $u, v, p, q \in W_{(2)}(X_2)$ the term corresponding to $\sigma_{f(u,v)} \circ_h \sigma_{f(p,q)} \circ_h \sigma_{f(c,d)}$ is in $W(\{x_1\})$ (or dually $W(\{x_2\})$).*

(ii) If $f(c,d) \in W(\{x_1\}) \cup W(\{x_2\})$, then for any $p, q \in W_{(2)}(X_2)$ the term t corresponding to $\sigma_{f(c,d)} \circ_h \sigma_{f(u,v)} \circ_h \sigma_{f(p,q)}$ is in $W(\{x_1\}) \cup W(\{x_2\})$ if and only if $u \in W(\{x_1\}) \cup W(\{x_2\})$ or $v \in W(\{x_1\}) \cup W(\{x_2\})$.

Proof. (i) Assume that $c, d \in W(\{x_1\})$. We will consider the term

$$w = S^2(f(u,v), \hat{\sigma}_{f(u,v)}[S^2(p, \hat{\sigma}_{f(p,q)}[c], \hat{\sigma}_{f(p,q)}[d])], \hat{\sigma}_{f(u,v)}[S^2(q, \hat{\sigma}_{f(p,q)}[c], \hat{\sigma}_{f(p,q)}[d])])$$

where $u, v, p, q \in W_{(2)}(X_2)$.

Since $c, d \in W(\{x_1\})$, then $\hat{\sigma}_{f(p,q)}[c], \hat{\sigma}_{f(p,q)}[d] \in W(\{x_1\})$. This implies that $\hat{\sigma}_{f(u,v)}[S^2(p, \hat{\sigma}_{f(p,q)}[c], \hat{\sigma}_{f(p,q)}[d])]$ and $\hat{\sigma}_{f(u,v)}[S^2(q, \hat{\sigma}_{f(p,q)}[c], \hat{\sigma}_{f(p,q)}[d])]$ are in $W(\{x_1\})$. Therefore, $w \in W(\{x_1\})$. Similarly, we can prove the case $c, d \in W(\{x_2\})$.

(ii) Assume that $f(c,d) \in W(\{x_1\}) \cup W(\{x_2\})$. We consider the term

$$t = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[p],$$

$\hat{\sigma}_{f(u,v)}[q])], \hat{\sigma}_{f(c,d)}[S^2(v, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q])])$

where $p, q \in W_{(2)}(X_2)$.

Suppose that $t \in W(\{x_1\}) \cup W(\{x_2\})$. Since $f(c,d) \in W(\{x_1\}) \cup W(\{x_2\})$ this forces that $\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q])] \in W(\{x_1\}) \cup W(\{x_2\})$ or $\hat{\sigma}_{f(c,d)}[S^2(v, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q])] \in W(\{x_1\}) \cup W(\{x_2\})$. Since $p, q \in W_{(2)}(X_2)$, then we obtain that $u \in W(\{x_1\}) \cup W(\{x_2\})$ or $v \in W(\{x_1\}) \cup W(\{x_2\})$.

Conversely, we suppose that $u \in W(\{x_1\}) \cup W(\{x_2\})$ or $v \in W(\{x_1\}) \cup W(\{x_2\})$. Assume that $f(c,d) \in W(\{x_1\})$.

If $p, q \in X$, then we have $\hat{\sigma}_{f(u,v)}[p] = p$ and $\hat{\sigma}_{f(u,v)}[q] = q$. Since $u \in W(\{x_1\}) \cup W(\{x_2\})$, we obtain that $S^2(u, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q]) \in W(\{x_1\}) \cup W(\{x_2\})$. Since $f(c,d) \in W(\{x_1\})$, we have
$\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q])] \in W(\{x_1\}) \cup W(\{x_2\})$ which implies that $t \in W(\{x_1\}) \cup W(\{x_2\})$.

If $p = f(p_1, p_2), q = f(q_1, q_2)$ where $p_i, q_i \in W_{(2)}(X_2); i = 1, 2$ and assume that $f(u,v) \in W(\{x_1\}) \cup W(\{x_2\})$, then we have $\hat{\sigma}_{f(u,v)}[p]$ and $\hat{\sigma}_{f(u,v)}[q]$ are in $W(\{x_1\}) \cup W(\{x_2\})$. So $S^2(u, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q]) \in W(\{x_1\}) \cup W(\{x_2\})$. Since $f(c,d) \in W(\{x_1\})$, we obtain that
$\hat{\sigma}_{f(c,d)}[S^2(u, \hat{\sigma}_{f(u,v)}[p], \hat{\sigma}_{f(u,v)}[q])] \in W(\{x_1\}) \cup W(\{x_2\})$.
Hence $t \in W(\{x_1\}) \cup W(\{x_2\})$.

In the same way, if $f(c,d) \in W(\{x_2\})$, then $t \in W(\{x_1\}) \cup W(\{x_2\})$.

□

Lemma 3.8. *Let $\sigma_{f(c,d)} \in Hyp(2) \setminus \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ and $u, v \in W_{(2)}(X_2) \setminus X_2$. If $x_1, x_2 \in var(f(c,d))$, then $vb((\sigma_{f(c,d)} \circ_h \sigma_v \sigma_u)(f)) > vb(u)$.*

Proof. Let $\sigma_{f(c,d)} \in Hyp(2) \setminus \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. Since $x_1, x_2 \in var(f(c,d))$ and $f(c,d) \notin \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$, thus $c \notin X_2$ or $d \notin X_2$ and $vb(f(c,d)) \geq 3$.

Case 1: Let $u = f(x_i, x_j)$ where $x_1, x_2 \in X_2$. Then $vb(u) = 2$. If $v = f(s,t)$ where $s, t \in X_2$, then $vb(w) = vb(\hat{\sigma}_{f(c,d)}[S^2(f(s,t), x_i, x_j)]) = vb(\hat{\sigma}_{f(c,d)}[f(x_m, x_n)]) = vb(S^2(f(c,d), x_m, x_n)) \geq 3 > 2 = vb(u)$ where $x_m, x_n \in X_2$. If $v = f(s,t)$ where $s \in X_2$ and $t \notin X_2$, then $vb(v) \geqslant 3$ and $\hat{\sigma}_{f(c,d)}[S^2(s, x_i, x_j)] = s \in X_2$. Assume that $vb(\hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)]) > vb(t)$. So

$$\begin{aligned} vb(w) &= vb(\hat{\sigma}_{f(c,d)}[S^2(f(s,t), x_i, x_j)]) \\ &= vb(\hat{\sigma}_{f(c,d)}[f(S^2(s, x_i, x_j), \\ &\quad S^2(t, x_i, x_j))]) \\ &= vb(S^2(f(c,d), s, \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)])). \end{aligned}$$

Since $x_1, x_2 \in var(f(c,d))$ and $vb(\hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)]) > vb(t)$, so we have $vb(w) = vb(S^2(f(c,d), s, \hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)])) > vb(f(s,t)) \geqslant 3 > 2 = vb(u)$. Similarly, we can prove the case $v = f(s,t)$ where $s \notin X_2$ and $t \in X_2$. If $v = f(s,t)$ where $s, t \notin X_2$, then $vb(f(s,t)) \geqslant 4$. Assume that $vb(\hat{\sigma}_{f(c,d)}[S^2(s, x_i, x_j)]) > vb(s)$ and $vb(\hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)]) > vb(t)$. Since $x_1, x_2 \in var(f(c,d))$, then $vb(w) = vb(\hat{\sigma}_{f(c,d)}[S^2(f(s,t), x_i, x_j)]) = vb(S^2(f(c,d), \hat{\sigma}_{f(c,d)}[S^2(s, x_i, x_j)], \hat{\sigma}_{f(c,d)}[S^2(t, x_i, x_j)])) > vb(f(s,t)) \geqslant 4 > 2 = vb(u)$.

Case 2: Let $u = f(p,q)$ where $p \in X_2$ and $q \notin X_2$. Then $\hat{\sigma}[p] = p \in X_2$. If $v = f(s,t)$ where $s, t \in X_2$, then we assume that $vb(\hat{\sigma}_{f(s,t)}[q]) > vb(q)$. So $vb(\hat{\sigma}_{f(c,d)}[S^2(s, p, \hat{\sigma}_{f(s,t)}[q])]) > vb(q)$ or $vb(\hat{\sigma}_{f(c,d)}[S^2(t, p, \hat{\sigma}_{f(s,t)}[q])]) >$

$vb(q)$. Since $x_1, x_2 \in var(f(c,d))$, then

$$\begin{aligned} vb(w) &= vb(\hat{\sigma}_{f(c,d)}[S^2(f(s,t), p, \\ &\quad \hat{\sigma}_{f(s,t)}[q])]) \\ &= vb(\hat{\sigma}_{f(c,d)}[f(S^2(s,p,\hat{\sigma}_{f(s,t)}[q]), \\ &\quad S^2(t,p,\hat{\sigma}_{f(s,t)}[q]))]) \\ &= vb(S^2(f(c,d), \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(t,p,\hat{\sigma}_{f(s,t)}[q], \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(t,p,\hat{\sigma}_{f(s,t)}[q])])) \\ &> vb(f(p,q)) = vb(u). \end{aligned}$$

Similarly, we can prove the case $v = f(s,t)$ where $s \in X_2, t \notin X_2$ and the case $v = f(s,t)$ where $s,t \notin X_2$.

Case 3: Let $u = f(p,q)$ where $p,q \notin X_2$. If $v = f(s,t)$ where $s,t \in X_2$, then we assume that $vb(\hat{\sigma}_{f(s,t)}[p]) > vb(p)$ and $vb(\hat{\sigma}_{f(s,t)}[q]) > vb(q)$. So $vb(\hat{\sigma}_{f(c,d)}[S^2(s,\hat{\sigma}_{f(s,t)}[p],\hat{\sigma}_{f(s,t)}[q])]) > vb(p)$ or $vb(\hat{\sigma}_{f(c,d)}[S^2(t, \hat{\sigma}_{f(s,t)}[p],\hat{\sigma}_{f(s,t)}[q])]) > vb(p)$ and $vb(\hat{\sigma}_{f(c,d)}[S^2(s,\hat{\sigma}_{f(s,t)}[p],\hat{\sigma}_{f(s,t)}[q])]) > vb(q)$ or $vb(\hat{\sigma}_{f(c,d)}[S^2(t,\hat{\sigma}_{f(s,t)}[p], \hat{\sigma}_{f(s,t)}[q])]) > vb(q)$. Since $x_1, x_2 \in var(f(c,d))$, we obtain that

$$\begin{aligned} vb(w) &= vb(\hat{\sigma}_{f(c,d)}[S^2(f(s,t), \\ &\quad \hat{\sigma}_{f(s,t)}[p],\hat{\sigma}_{f(s,t)}[q])]) \\ &= vb(\hat{\sigma}_{f(c,d)}[f(S^2(s,\hat{\sigma}_{f(s,t)}[p], \\ &\quad \hat{\sigma}_{f(s,t)}[q]), S^2(t,\hat{\sigma}_{f(s,t)}[p], \\ &\quad \hat{\sigma}_{f(s,t)}[q]))]) \\ &= vb(S^2(f(c,d),\hat{\sigma}_{f(c,d)}[S^2(t, \\ &\quad \hat{\sigma}_{f(s,t)}[p],\hat{\sigma}_{f(s,t)}[q], \\ &\quad \hat{\sigma}_{f(c,d)}[S^2(t,\hat{\sigma}_{f(s,t)}[p], \\ &\quad \hat{\sigma}_{f(s,t)}[q])])) \\ &> vb(f(p,q)) = vb(u). \end{aligned}$$

Similarly, we can prove the case $v = f(s,t)$ where $s \in X_2, t \notin X_2$ and the case $v = f(s,t)$ where $s,t \notin X_2$.

□

Lemma 3.9. *Let $t = f(u,v)$ be a term which use both x_1 and x_2. The only elements which are $\mathcal{L}^T$−related to σ_t are σ_t itself and $\sigma_{t'}$.*

Proof. Let $t = f(u,v)$ where $u,v \in W_{(2)}(X_2)$. Assume that $\sigma_s \in Hyp(2)$ where $\sigma_s \mathcal{L}^T \sigma_t$. Then, by Lemma 3.5, we have $s \notin X_2$. So $s = f(c,d)$ for some $c,d \in W_{(2)}(X_2)$. Since $s,t \notin X_2$ and $\sigma_s \mathcal{L}^T \sigma_t$, then there exist $e,g,h,k,l,r,m,n \in W_{(2)}(X_2)$ such that $\sigma_{f(c,d)} = \sigma_{f(e,g)} \circ_h \sigma_{f(h,k)} \circ_h \sigma_{f(u,v)}$ and $\sigma_{f(u,v)} = \sigma_{f(l,r)} \circ_h \sigma_{f(m,n)} \circ_h \sigma_{f(c,d)}$. Suppose that $f(e,g), f(h,k), f(l,r)$, and $f(m,n)$ are not in $\{f(x_1,x_2), f(x_2,x_1)\}$ and all of that are in $W_{(2)}(X_2)$. Then, by Lemma 3.8, we obtain that $vb(f(c,d)) > vb(f(u,v))$ and $vb(f(u,v)) > vb(f(c,d))$, which is a contradiction. Suppose that $f(e,g), f(h,k) \in W_{(2)} \setminus W_{(2)}(\{x_1,x_2\})$. Then, by Proposition 3.7 (ii), we obtain that $x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$. Since $x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$, then we get $x_1 \notin var(f(u,v))$ or $x_2 \notin var(f(u,v))$, which contradicts to $x_1,x_2 \in var(f(u,v))$. Similarly, if $f(l,r), f(m,n) \in W_{(2)}(X_2) \setminus W_{(2)}(\{x_1,x_2\})$, then $x_1 \notin var(f(u,v))$ or $x_2 \notin var(f(u,v))$, which contradicts to $x_1,x_2 \in var(f(u,v))$. So $f(e,g), f(h,k), f(l,r)$, $f(m,n) \in \{f(x_1,x_2), f(x_2,x_1)\}$. This forces that $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. □

Corollary 3.10. *For any $\sigma_t \in Hyp(2) \setminus P(2)$ where $x_1,x_2 \in var(t)$, we have the $\mathcal{D}^T$−class of σ_t is the set $\{\sigma_t, \sigma_{t'}, \sigma_{\bar{t}}, \sigma_{\overline{t'}}\}$.*

Proof. The proof is directly from Lemma 3.6 and Lemma 3.9. □

Definition 3.11 ([8])**.** A band is a ternary semigroup where any element is an idempotent. A left zero band (resp. right) S is a band satisfying the equation $x \cdot x \cdot y = x$ (resp. $y \cdot x \cdot x = x$) for all $x, y \in S$.

Proposition 3.12. *E_{x_1} is a left zero band.*

Proof. Let $\sigma_{f(x_1,s)}$ and $\sigma_{f(x_1,t)}$ be in E_{x_1}. Since $s \in W_{(2)}(\{x_1\})$, so $(\sigma_{f(x_1,s)} \circ_h \sigma_{f(x_1,s)} \circ_h \sigma_{f(x_1,t)})(f) = \hat{\sigma}_{f(x_1,s)}[S^2(f(x_1,s), x_1, \hat{\sigma}_{f(x_1,s)}[t])] = \hat{\sigma}_{f(x_1,s)}[f(x_1, S^2(s, x_1, \hat{\sigma}_{f(x_1,s)}[t]))] = S^2(f(x_1,s), x_1, \hat{\sigma}_{f(x_1,s)}[S^2(s, x_1, \hat{\sigma}_{f(x_1,s)}[t])]) = f(x_1,s)$. Since E_{x_1} is a band and $\sigma_{f(x_1,s)} \circ_h \sigma_{f(x_1,s)} \circ_h \sigma_{f(x_1,t)} = \sigma_{f(x_1,s)}$ for any $\sigma_{f(x_1,s)}, \sigma_{f(x_1,t)} \in E_{x_1}$, hence E_{x_1} is a left zero band. □

Lemma 3.13. *The $\mathcal{L}^T$−class of the element $\sigma_{f(x_1,x_1)}$ is the set $E_{x_1} \cup \overline{E_{x_2}}$.*

Proof. Let e, f be any two idempotent elements in a ternary semigroup T. So $e\mathcal{L}^T f$ if and only if $eef = e$ and $ffe = f$ [See [2]]. If E_{x_1} is a left zero band, then $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}^T$−related to every elements in E_{x_1}. By Proposition 3.3, we have $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}^T$−related to $\overline{E'_{x_1}} = \overline{E_{x_2}}$. Thus the $\mathcal{L}^T$−class of $\sigma_{f(x_1,x_1)}$ contains at least $E_{x_1} \cup \overline{E_{x_2}}$. Assume that $\sigma_t \in Hyp(2)$ such that $\sigma_t\mathcal{L}^T \sigma_{f(x_1,x_1)}$. Then Lemma 3.5 tell us that $t \notin X_2$, so $t = f(u,v)$ for some $u, v \in W_{(2)}(X_2)$. Since $\sigma_t\mathcal{L}^T \sigma_{f(x_1,x_1)}$, then there exist $\sigma_p, \sigma_q, \sigma_r, \sigma_s \in Hyp(2)$ such that $\sigma_t = \sigma_p \circ_h \sigma_q \circ_h \sigma_{f(x_1,x_1)}$ and $\sigma_{f(x_1,x_1)} = \sigma_r \circ_h \sigma_s \circ_h \sigma_t$. Since $\sigma_t, \sigma_{f(x_1,x_1)} \notin X_2$, this forces that $p, q, r, s \notin X_2$. So we get $p = f(p_1,p_2)$, $q = f(q_1,q_2)$ and $s = f(s_1,s_2)$ where $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 \in W_{(2)}(X_2)$. Then we have $\sigma_{f(u,v)} = \sigma_{f(p_1,p_2)} \circ_h \sigma_{f(q_1,q_2)} \circ_h \sigma_{f(x_1,x_1)}$ and $\sigma_{f(x_1,x_1)} = \sigma_{f(r_1,r_2)} \circ_h \sigma_{f(s_1,s_2)} \circ_h \sigma_{f(u,v)}$. From $\sigma_{f(u,v)} = \sigma_{f(p_1,p_2)} \circ_h \sigma_{f(q_1,q_2)} \circ_h \sigma_{f(x_1,x_1)}$, then, by Proposition 3.7(i), we have $x_2 \notin var(f(u,v))$. Since $\sigma_{f(x_1,x_1)} = \sigma_{f(r_1,r_2)} \circ_h \sigma_{f(s_1,s_2)} \circ_h \sigma_{f(u,v)}$, we have

$$\begin{aligned}
\sigma_{f(x_1,x_1)}(f) &= \hat{\sigma}_{f(r_1,r_2)}[S^2(f(s_1,s_2), \\
&\quad \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])] \\
&= S^2(f(r_1,r_2), \hat{\sigma}_{f(r_1,r_2)}[S^2(s_1, \\
&\quad \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])], \\
&\quad \hat{\sigma}_{f(r_1,r_2)}[S^2(s_2, \hat{\sigma}_{f(s_1,s_2)}[u], \\
&\quad \hat{\sigma}_{f(s_1,s_2)}[v])]).
\end{aligned}$$

Suppose that $u, v \neq x_1$, so $\hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[u] \neq x_1$. Then $\hat{\sigma}_{f(r_1,r_2)}[S^2(s_1, \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])]$, $\hat{\sigma}_{f(r_1,r_2)}[S^2(s_2, \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])] \neq x_1$, this implies that $S^2(f(r_1,r_2), \hat{\sigma}_{f(r_1,r_2)}[S^2(s_1, \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])], \hat{\sigma}_{f(r_1,r_2)}[S^2(s_2, \hat{\sigma}_{f(s_1,s_2)}[u], \hat{\sigma}_{f(s_1,s_2)}[v])]) \neq f(x_1,x_1)$. This is a contradiction, so $u = x_1$ or $v = x_1$. Since $x_2 \notin var(f(u,v))$ and $u = x_1$ or $v = x_1$, then $\sigma_t = \sigma_{f(u,v)} \in E_{x_1} \cup \overline{E_{x_2}}$. □

Corollary 3.14. *The $\mathcal{D}^T$−class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E_{x_1} \cup E_{x_2} \cup \overline{E_{x_1}} \cup \overline{E_{x_2}}$.*

Proof. Let $\sigma_t \in Hyp(2)$ where $\sigma_t\mathcal{D}^T \sigma_{f(x_1,x_1)}$. Then there exists $\sigma_s \in Hyp(2)$ such that $\sigma_t\mathcal{R}^T \sigma_s$ and $\sigma_s\mathcal{L}^T \sigma_{f(x_1,x_1)}$. From $\sigma_t\mathcal{R}^T \sigma_s$, by Lemma 3.5, we get $\sigma_t = \sigma_s$ or $\sigma_t = \sigma_{\overline{s}}$. Since $\sigma_s\mathcal{L}^T \sigma_{f(x_1,x_1)}$, then, by Lemma 3.13, we obtain that $\sigma_s \in E_{x_1} \cup \overline{E_{x_2}}$. If $\sigma_s \in E_{x_1}$, then $\sigma_t \in E_{x_1} \cup \overline{E_{x_1}} \subseteq E_{x_1} \cup E_{x_2} \cup \overline{E_{x_1}} \cup \overline{E_{x_2}}$. If $\sigma_s \in \overline{E_{x_2}}$, then $\sigma_t \in E_{x_2} \cup \overline{E_{x_2}} \subseteq E_{x_1} \cup E_{x_2} \cup \overline{E_{x_1}} \cup \overline{E_{x_2}}$.

For the opposite inclusion, assume that $\sigma_t \in E_{x_1} \cup E_{x_2} \cup \overline{E_{x_1}} \cup \overline{E_{x_2}}$. If $\sigma_t \in E_{x_1} \cup \overline{E_{x_2}}$, then we get $\sigma_t\mathcal{L}^T \sigma_{f(x_1,x_1)}$ by Lemma 3.13. Since $\mathcal{L}^T \subseteq \mathcal{D}^T$, thus $\sigma_t\mathcal{D}^T \sigma_{f(x_1,x_1)}$. If $\sigma_t \in E_{x_2} \cup \overline{E_{x_1}}$, then $\sigma_{\overline{t}} \in E_{x_1} \cup \overline{E_{x_2}}$. By Lemma 3.13 again, we have $\sigma_{\overline{t}}\mathcal{L}^T \sigma_{f(x_1,x_1)}$. From Lemma 3.6, we have $\sigma_t\mathcal{R}^T \sigma_{\overline{t}}$. So $\sigma_t\mathcal{D}^T \sigma_{f(x_1,x_1)}$. □

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