P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia*
Science & Technology Asia
Vol.27 No.1 January - March 2022
Page: [20-30]
Original research article

On Answer to Kirk-Shahzad's Question for Strong b-TVS Cone Metric Spaces

Doan Trong Hieu[1], Bui The Hung[2],
Muhammad Sirajo Abdullahi[3,4], Poom Kumam[3,*]

[1] *Faculty of Basic Sciences, Quang Ninh University of Industry, Quang Ninh 01000, Vietnam*
[2] *Department of Mathematics, Thai Nguyen University of Education, Thai Nguyen 24000, Vietnam*
[3] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand*
[4] *Department of Mathematics, Faculty of Science, Usmanu Danfodiyo University, Sokoto 234, Nigeria*

Received 18 June 2020; Received in revised form 22 May 2021
Accepted 17 September 2021; Available online 28 March 2022

ABSTRACT

In this paper, we introduced the concept of strong b-TVS cone metric space. Furthermore, we proved that any strong b-TVS cone metric space has a completion. Our results gave an analogue answer to Kirk-Shahzad's question in the case of strong b-TVS cone metric spaces.

Keywords: Fixed point; Completion; Neighborhood property; Strong b-TVS cone metric space

1. Introduction

Recently, the generalization of metric space attracted several authors (see. [1–8] and references therein). For example, in 2007, Huang and Zhang [9] introduced the concept of cone metric space as a generalization of metric space and proved some results of contraction mapping in this space.

In 2010, Wei-Shih Du [10] showed using the scalarization method that the Banach Contraction Principle and some other fixed point results both in the settings of classical metric space and in TVS-cone metric space are equivalent. In 2014, W. Kirk and N. Shahzad [11] surveyed the concept of strong b-metric space and its related problems:

Definition 1.1 ([11]). Let X be a nonempty set and $K \geq 1$. A mapping $d : X \times X \to [0, +\infty)$ is called a *strong b-metric* on X if

(m_1) $d(x, y) = 0$ if and only if $x = y$;

(m_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

***Corresponding author:** poom.kumam@mail.kmutt.ac.th
doi: 10.14456/scitechasia.2022.3

(m_3) $d(x, y) \leqslant d(x, z) + Kd(z, y)$ for all $x, y, z \in X$.

Then (X, d, K) is called a *strong b-metric space*.

The authors further asked the following question:

Question 1.2 ([11]). Is every strong b-metric space dense in a complete strong b-metric space?

The Kirk-Shahzad's question was answered by An and Dung [12] in 2016 for strong $b-$metric spaces, where the authors proved the following:

Theorem 1.3 ([12]). *Let (X, d, K) be a strong b-metric spaces. Then*

(i) (X, d, K) *has a completion* (X^*, d^*, K).

(ii) The completion of (X, d, K) is unique in the sense that if (X_1^*, d_1^*, K_1) and (X_2^*, d_2^*, K_2) are two completions of (X, d, K) then there is a bijective isometry $\phi : X_1^* \to X_2^*$ which restricts to the identity on X.

Theorem 1.3 is a completion theorem type for a strong b-metric space. It's shown that every strong b-metric space has a completion and that the completion is unique.

In the same vein, we introduce the concept of strong b-TVS cone metric space which is a generalization of the notion of strong b-metric space. Furthermore, we provide an analogue answer to W. Kirk and N. Shahzad's question in the case of strong b-TVS cone metric space. Namely, we shall prove a completion theorem for strong b-TVS cone metric spaces. In fact, we introduced the concept of strong b-TVS cone metric space and presented some properties of the cone in topological vector space and properties of strong b-TVS cone metric spaces in Section 2. In Section 3, we proved a completion theorem for strong b-TVS cone metric spaces, this generalized and improved Theorem 1.3.

2. Strong b-TVS Cone Metric Spaces

Let E always be a real Hausdorff locally convex topological vector spaces with its zero vector θ and C be a subset of E. We say that C is a cone in E if

(i) C is closed, nonempty and $C \neq \{\theta\}$;

(ii) $ax + by \in C$, for all $x, y \in C$ and non-negative real numbers a, b;

(iii) $C \cap (-C) = \{\theta\}$.

For a given cone C in E, we can define a partial order $\preceq$ with respect to C by $x \preceq y$ if and only if $y - x \in C$, while $x \ll y$ will stand for $y - x \in intC$, where $intC$ denotes the interior of C. We write $x \prec y$ if and only if $x \preceq y$ and $x \neq y$.

In this paper, we always suppose E is a real Hausdorff locally convex topological vector space, C is a cone in E with $intC \neq \emptyset$ and $\preceq$ is partial ordering with respect to C.

Definition 2.1. Let C be a cone in E. We say that C is has neighborhood property if for any neighborhood U of θ in E, there is neighborhood V of θ in E such that

$$(V + C) \cap (V - C) \subset U.$$

Remark 2.2. *If C has a closed convex bounded base then C has neighborhood property (see. Proposition 1.8 in* [13]).

Example 2.3. Let $E = \mathbb{R}$ and $C = \mathbb{R}_+$. Then, it is easy to see that C has the neighborhood property.

Proposition 2.4. *Assume that C has the neighborhood property. Then for any* *neighborhood U of θ in E, there is $e \in E, \theta \ll e$ such that*

$$C \cap (e - C) \subset U.$$

Proof. Assume that C has the neighborhood property. Let U be an arbitrary neighborhood of θ in E, there is neighborhood V of θ in E such that

$$(V + C) \cap (V - C) \subset U.$$

Since $intC \neq \emptyset$, we can choose $a \in E$, $\theta \ll a$. From $\lim_{n\to\infty} \frac{1}{n}a = \theta$, there is n_0 such that

$$\frac{1}{n}a \in V \text{ for all } n \geq n_0.$$

Set $e = \frac{1}{n_0}a$. Hence $e \in E, \theta \ll e$ and

$$C \cap (e - C) \subset (V + C) \cap (V - C) \subset U.$$

□

Lemma 2.5. *Suppose that C has the neighborhood property in a real Hausdorff locally convex topological vector space E. Let $\{u_n\}$, $\{v_n\}, \{w_n\}$ be sequences in E such that $w_n \preceq u_n \preceq v_n$ for all $n \geq 1$ and $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = \theta$. Then $\lim_{n\to\infty} u_n = \theta$.*

Proof. Let U be an arbitrary neighborhood of θ in E. Since C has the neighborhood property, there is neighborhood V of θ in E such that

$$(V + C) \cap (V - C) \subset U.$$

Since $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = \theta$, there exists n_0 such that $v_n, w_n \in V$ for all $n \geq n_0$.

On the other hand, since $w_n \preceq u_n \preceq v_n$ for all $n \geq 1$, we have $u_n \in (w_n + C) \cap (v_n - C)$ for all $n \geq 1$. Thus $u_n \in (V + C) \cap (V - C)$ for all $n \geq n_0$. Therefore, $u_n \in U$ for all $n \geq n_0$. Hence $\lim_{n\to\infty} u_n = \theta$. □

Example 2.6. Let $E = C^1_{[0,1]}$ with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

and consider the cone

$$C = \{f \in E : f(t) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Then, C has no neighborhood property. Indeed, consider $f_n(t) = \frac{t^n}{n}$ and $g_n(t) = \frac{1}{n}$ for all $t \in [0, 1]$. Then $\theta \preceq f_n \preceq g_n$ for all n and $\lim_{n\to\infty} g_n = \theta$.

On the other hand, we have

$$\|f_n\| = \max_{t\in[0,1]} \frac{t^n}{n} + \max_{t\in[0,1]} t^{n-1} = 1 + \frac{1}{n} > 1,$$

for all $n \geq 1$. Hence f_n does not converge to θ. By Lemma 2.5, C has no neighborhood property.

Definition 2.7. (See [9]) Let C be a cone in normed space E. We say that C is normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq M\|y\|.$$

Proposition 2.8. *Let C be a normal cone in normed space E. Then C has the neighborhood property.*

Proof. Assume that C has no neighborhood property. Then there exists $\epsilon > 0$ such that for any $n \geq 1$, we have

$$[B(\theta, \frac{1}{n}) + C] \cap [B(\theta, \frac{1}{n}) - C] \not\subset B(\theta, \epsilon),$$

where $B(\theta, \delta) = \{x \in E : \|x\| < \delta\}$. Thus, for any $n \geq 1$, there exists $u_n \in C, v_n \in B(\theta, \frac{1}{n})$ such that $u_n \preceq v_n$ and $u_n \notin B(\theta, \epsilon)$. Since C is normal cone and $\lim_{n\to\infty} v_n = \theta$, then $\lim_{n\to\infty} u_n = \theta$. Hence $\theta \notin B(\theta, \epsilon)$. This is a contradiction. □

Definition 2.9. Let X be a nonempty set and $K \geq 1$. The mapping $d : X \times X \to E$ is called a *strong b-cone metric* on X if

(d_1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3) $d(x,y) \preceq d(x,z) + Kd(z,y)$ for all $x, y, z \in X$.

Then (X, E, C, K, d) is called a *strong b-TVS cone metric space*.

Definition 2.10. Let (X, E, C, K, d) be a strong b-TVS cone metric space. Let $\{x_n\}$ be a sequence in X. We say that

(i) x is the limit of $\{x_n\}$ if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x) \ll e$ for all $n \geq n_0$. We denote this by $x_n \to x$ or $\lim_{n\to\infty} x_n$.

(ii) $\{x_n\}$ is Cauchy sequence if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x_m) \ll e$ for all $n, m \geq n_0$.

Now, we prove some results in strong b-TVS cone metric spaces for orders generated by the cone in a real Hausdorff locally convex topological vector spaces.

Lemma 2.11. *Let (X, E, C, K, d) be a strong b-TVS cone metric space and $\{x_n\}$ be a sequence in X. Then we have:*

(i) If $\{x_n\}$ converges to $x \in X$ then $\{x_n\}$ is a Cauchy sequence.

(ii) If $\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$, then $x = y$.

Proof. (i) Suppose that $\lim_{n\to\infty} x_n = x \in X$. For any $e \in E, \theta \ll e$, there is n_0 such that

$$d(x_n, x) \ll \frac{e}{2} \text{ and } d(x_m, x) \ll \frac{e}{2K}$$

for all $n, m \geq n_0$. This implies

$$d(x_n, x_m) \preceq d(x_n, x) + Kd(x_m, x) \ll e,$$

for all $n, m \geq n_0$. Hence $\{x_n\}$ is a Cauchy sequence.
(ii) Let $e \in E, 0 \ll \theta$. Since $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} x_n = y$, for any $k \geq 1$, there exists n_0 such that

$$d(x_n, x) \ll \frac{e}{2k} \text{ and } d(x_n, y) \ll \frac{e}{2kK},$$

for all $n \geq n_0$. This implies

$$d(x, y) \preceq d(x_n, x) + Kd(x_n, y) \ll \frac{e}{k},$$

for all $n \geq n_0$. Hence

$$\frac{e}{k} - d(x, y) \in C \text{ for all } k \geq 1.$$

Then, by the closedness of C, $-d(x, y) \in C$. Therefore, $x = y$. □

Lemma 2.12. *Let (X, E, C, K, d) be a strong b-TVS cone metric space, C has the neighborhood property and $\{x_n\}$, $\{y_n\}$ be two sequences in X. Then*

(i) $\lim_{n\to\infty} x_n = x \in X$ *if and only if* $\lim_{n\to\infty} d(x_n, x) = \theta$.

(ii) $\{x_n\}$ *is a Cauchy sequence if and only if* $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

(iii) If $\lim_{n\to\infty} x_n = x \in X$ *and* $\lim_{n\to\infty} y_n = y \in X$, *then* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Proof. (i) Suppose that $\lim_{n\to\infty} x_n = x \in X$. Let U be an arbitrary neighborhood of θ in E. Since C has the neighborhood property, then by Proposition 2.4, there exists $e \in E, \theta \ll e$ such that

$$C \cap (e - C) \subset U.$$

By $\lim_{n\to\infty} x_n = x \in X$, there is n_0 such that

$$d(x_n, x) \ll e \text{ for all } n \geq n_0.$$

This implies

$$d(x_n, x) \in e - intC \subset e - C \text{ for all } n \geq n_0.$$

Hence

$$d(x_n, x) \in C \cap (e - C) \text{ for all } n \geq n_0.$$

Thus, $d(x_n, x) \in U$ for all $n \geq n_0$. This means $\lim_{n\to\infty} d(x_n, x) = \theta$.

Conversely, suppose that $\lim_{n\to\infty} d(x_n, x) = \theta$. For $e \in E, \theta \ll e$, then there is a neighborhood U of θ in E such that

$$e - U \subset intC.$$

Since $\lim_{n\to\infty} d(x_n, x) = \theta$, then there is n_0 such that

$$d(x_n, x) \in U \text{ for all } n \geq n_0.$$

This implies

$$e - d(x_n, x) \in intC \text{ for all } n \geq n_0.$$

This means that $d(x_n, x) \ll e$ for all $n \geq n_0$. Hence $\lim_{n\to\infty} x_n = x$.

(ii) Suppose $\{x_n\}$ is a Cauchy sequence. Let U be an arbitrary neighborhood of θ in E. Because C has neighborhood property and by Proposition 2.4, there is $e \in E, \theta \prec e$ such that

$$C \cap (e - C) \subset U.$$

Since $\{x_n\}$ is a Cauchy sequence, there is n_0 such that

$$d(x_n, x_m) \ll e \text{ for all } n, m \geq n_0.$$

This implies

$$d(x_n, x_m) \in e - intC \subset e - C \text{ for all } n, m \geq n_0.$$

Hence

$$d(x_n, x_m) \in C \cap (e - C) \text{ for all } n, m \geq n_0.$$

Thus, $d(x_n, x_m) \in U$ for all $n, m \geq n_0$. This means $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

Conversely, suppose that $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$. Let $e \in E$ and $\theta \ll e$. Then there is a neighborhood U of the θ in E such that

$$e - U \subset intC.$$

Since $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$, then there is n_0 such that

$$d(x_n, x_m) \in U \text{ for all } n, m \geq n_0.$$

This implies

$$e - d(x_n, x_m) \in intC \text{ for all } n, m \geq n_0.$$

This means that $d(x_n, x_m) \ll e$ for all $n, m \geq n_0$. Hence $\{x_n\}$ is a Cauchy sequence.

(iii) We have

$$d(x_n, y_n) \preceq Kd(x_n, x) + d(x, y) + Kd(y_n, y),$$

and

$$d(x, y) \preceq Kd(x_n, x) + d(x_n, y_n) + Kd(y_n, y)$$

for all $n \geq 1$. This implies

$$\begin{aligned} -K[d(x_n, x) + d(y_n, y)] & \\ &\preceq d(x, y) - d(x_n, y_n) \\ &\preceq K[d(x_n, x) + d(y_n, y)], \end{aligned}$$

for all $n \geq 1$.

Since $\lim_{n\to\infty}[d(x_n, x) + d(y_n, y)] = \theta$ and by Lemma 2.5, we have

$$\lim_{n\to\infty} [d(x, y) - d(x_n, y_n)] = \theta.$$

Hence $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. □

Remark 2.13. *Note that in* [9]*, Lemma 2.12 was proved with a condition that C is a normal cone. Whereas in our own case, we suppose that the cone C has neighborhood property instead. Further, Lemma 2.12 may fail to hold if C has no neighborhood property in E. The following example illustrates this.*

Example 2.14. Let $E = C^1_{[0,1]}$ with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

and consider the cone $C = \{f \in E : f(t) \geq 0 \text{ for all } t \in [0,1]\}$. Then we have C not has neighborhood property (see, Example 2.6).

Let $X = \{0, \frac{1}{n} : n \geq 1\}$ and $d : X \times X \to E$ by

$$d(x,y) = \begin{cases} \theta, \text{ if } x = y, \\ |f_n - f_m|, \text{ if } x \neq y \in \{\frac{1}{n}, \frac{1}{m}\}, \\ f_n, \text{ if } x \neq y \in \{\frac{1}{n}, 0\}, \end{cases}$$

where $f_n(t) = \frac{t^n}{n}$ for all $t \in [0,1]$, $n \geq 1$ and θ is zero vector in E .

We can check that d is a strong b-TVS cone metric on X with $K = 1$ and $(X, E, C, K = 1, d)$ is a strong b- TVS cone metric space. Moreover, we have

$$d(\frac{1}{n}, 0) = f_n \leq \frac{I_E}{n} \text{ for all } n \geq 1,$$

where $I_E \in E$ by $I_E(t) = t$ for all $t \in [0,1]$.

Since $\lim_{n\to\infty} \frac{I_E}{n} = \theta$, by Proposition 1.4 in [14], for any $e \in E, \theta \ll e$, there exists n_0 such that

$$\frac{I_E}{n} \ll e \text{ for all } n \geq n_0.$$

Thus $d(\frac{1}{n}, 0) \ll e$ for all $n \geq n_0$. Hence $\lim_{n\to\infty} \frac{1}{n} = 0$.

On the other hand, we have

$$\begin{aligned} \|d(\frac{1}{n}, 0) - \theta\| &= \|f_n - \theta\| \\ &= \max_{t\in[0,1]} \frac{t^n}{n} + \max_{t\in[0,1]} t^{n-1} \\ &= 1 + \frac{1}{n} > 1, \end{aligned}$$

for all $n \geq 1$. Hence $d(\frac{1}{n}, 0)$ does not converge to θ in E. Thus, Lemma 2.12 does not holds.

3. Completion of Strong b-TVS Cone Metric Spaces

Let (X, E, C, K, d) be a strong b-TVS cone metric space. For $x_0 \in X$ and $e \in E, \theta \ll e$, the subset

$$B(x_0, e) := \{x \in X : d(x_0, x) \ll e\}$$

of X will be called an open ball centered at x_0 with radius e. We say that $A \subset X$ is an open set if for any $x \in A$, there is $e_x \in E, \theta \ll e_x$ such that $B(x, e_x) \subset A$. A set $B \subset X$ is closed if it's complement is open. The closure of A, denoted by $\bar{A}$ is defined to be the intersection of closed sets containing A. A is said to be dense in X if $\bar{A} = X$.

Remark 3.1. *If (X, E, C, K, d) is a strong b-TVS cone metric space, then the following statements are true:*

(i) The empty set $\emptyset$ and the space X are open and closed;

(ii) The open ball $B(x_0, e)$ is open and the closed ball $\bar{B}(x_0, e)$ is closed, where

$$\bar{B}(x_0, e) := \{x \in X : d(x_0, x) \leq e\},$$

with $e \in E, \theta \ll e$ and $x_0 \in X$;

(iii) The union of open sets is open, the intersection of an arbitrary number of closed sets is closed;

(iv) The intersection of an finite number of open sets is open, the union of an finite number of closed sets is closed.

Lemma 3.2. *A sequence $\{x_n\}$ is convergent to x in (X, E, C, K, d) if and only if for any open set W containing x there exists n_0 such that*

$$x_n \in W \text{ for all } n \geq n_0.$$

Proof. Suppose that $\lim_{n\to\infty} x_n = x$. Let W be an open set containing x. Then there is $e \in E, \theta \ll e$ such that $B(x,e) \subset W$. Since $\lim_{n\to\infty} x_n = x$, there exists n_0 such that $d(x,x_n) \ll e$ for all $n \geq n_0$. Hence $x_n \in W$ for all $n \geq n_0$.

Conversely, let any $e \in E, \theta \ll e$. Then $B(x,e)$ is open set containing x. By hypothesis, there is n_0 such that $x_n \in B(x,e)$ for all $n \geq n_0$. Hence $d(x,x_n) \ll e$ for all $n \geq n_0$. Thus, $\lim_{n\to\infty} x_n = x$. □

Lemma 3.3. *Let A be a subset of strong b-TVS cone metric space (X,E,C,K,d). Then we have $x \in \bar{A}$ if and only if there exists an sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = x$.*

Proof. Suppose that $x \in \bar{A}$. Then for each $n \geq 1$, $e \in E, \theta \ll e$, we have

$$B(x, \frac{e}{n}) \cap A \neq \emptyset.$$

We choose $x_n \in B(x, \frac{e}{n}) \cap A$ for all $n \geq 1$. This implies that $\{x_n\} \subset A$ and $\lim_{n\to\infty} x_n = x$.

Conversely, assume that there exists a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = x$. We prove that $x \in \bar{A}$. If not, we have $x \in X\backslash\bar{A}$. If $X\backslash\bar{A}$ is an open set, then there exists n_0 such that $x_n \in X\backslash\bar{A}$ for all $n \geq n_0$, which a contradiction with $\{x_n\} \subset A$. □

Lemma 3.4. *Let A be a subset of strong b-TVS cone metric space (X,E,C,K,d). Then A is dense in X if and only if for each $x \in X$ and $e \in E, \theta \ll e$, there exists $a \in A$ such that $d(x,a) \ll e$.*

Proof. Suppose A is dense in X. Let $x \in X$ and $e \in E, \theta \ll e$. By Lemma 3.3, there exists a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = x$. This implies that there is n_0 such that $d(x,x_n) \ll e$ for all $n \geq n_0$. Set $a = x_{n_0}$. Then we have $d(x,a) \ll e$.

Conversely, let $x \in X$ and $\theta \ll e$. For each $n \geq 1$, there exists $a_n \in A$ such that $d(a_n,x) \ll \frac{e}{n}$. This implies $\{a_n\} \subset A$ such that $\lim_{n\to\infty} a_n = x$. Hence $x \in \bar{A}$ and A is dense in X. □

Definition 3.5. Let (X,E,C,K,d) be a strong b-TVS cone metric space.

(i) If every Cauchy sequence is convergent in X, then (X,E,C,K,d) is called a complete strong b-TVS cone metric space.

(ii) A map $f : X \to Y$ from a strong b-TVS cone metric space (X,E,C,K,d) into a strong b-TVS cone metric space (Y,E,C,K',d') is called an isometry if

$$d'(f(x), f(y)) = d(x,y)$$

for all $x, y \in X$.

(iii) A complete strong b-TVS cone metric space (X^*,E,C,K^*,d^*) is said to be a completion of the strong b-TVS cone metric space (X,E,C,K,d) if there exists an isometry $f : X \to X^*$ such that $\overline{f(X)} = X^*$.

We can now state and prove the theorem that every strong b-TVS cone metric space can be completed.

Theorem 3.6. *Let (X,E,C,K,d) be a strong b-TVS cone metric space and the cone C has neighborhood property in a complete real Hausdorff locally convex topological vector space E. Then*

(i) (X,E,C,K,d) has a completion (X^*,E,C,K,d^*);

(ii) The completion of (X,E,C,K,d) is unique in the sense that if (X_1^*,E,C,K_1,d_1^*) and

$(X_2^*, E, C, K_2, d_2^*)$ are two completions of (X, E, C, K, d) then there is a bijective isometry $\phi : X_1^* \to X_2^*$ which restricts to the identity on X.

Proof. (i) Let S denote the set of all Cauchy sequences in (X, E, C, K, d). Define a relation $\sim$ on S as follows:

For $\{x_n\}, \{y_n\} \in S$, we say $\{x_n\} \sim \{y_n\}$ if $\lim_{n\to\infty} d(x_n, y_n) = \theta$.

It is not difficult to see that $\sim$ is an equivalence relation on S. Denote $X^* = \{x^* = [\{x_n\}] : \{x_n\} \in S\}$ where $x^* = [\{x_n\}]$ is the equivalence class of $\{x_n\}$ under the relation $\sim$. Define the function $d^* : X^* \times X^* \to E$ by

$$d^*(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n).$$

We first show that $\lim_{n\to\infty} d(x_n, y_n)$ exists. Indeed, we see that, for all n, m

$$\begin{aligned} d(x_n, y_n) &\preceq Kd(x_n, x_m) + d(x_m, y_n) \\ &\preceq Kd(x_n, x_m) \\ &\quad + d(x_m, y_m) + Kd(y_m, y_n) \end{aligned}$$

This implies that

$$\begin{aligned} &d(x_n, y_n) - d(x_m, y_m) \\ &\qquad \preceq Kd(x_n, x_m) + Kd(y_m, y_n) \end{aligned}$$

for all n, m. Similarly, we have

$$\begin{aligned} &d(x_m, y_m) - d(x_n, y_n) \\ &\qquad \preceq Kd(x_n, x_m) + Kd(y_m, y_n) \end{aligned}$$

for all n, m. Hence

$$\begin{aligned} &-K[d(x_n,x_m) + d(y_m, y_n)] \\ &\qquad \preceq d(x_n, y_n) - d(x_m, y_m) \\ &\qquad \preceq K[d(x_n, x_m) + d(y_m, y_n)] \end{aligned}$$

for all n, m. Since

$$\begin{aligned} &\lim_{n,m\to\infty} -K[d(x_n, x_m) + d(y_m, y_n)] \\ &\quad = \lim_{n,m\to\infty} K[d(x_n, x_m) + d(y_m, y_n)] \\ &\quad = \theta, \end{aligned}$$

and by Lemma 2.5, we have

$$\lim_{n,m\to\infty} [d(x_n, y_n) - d(x_m, y_m)] = \theta.$$

Therefore, $\{d(x_n, y_n)\}$ is a Cauchy sequence in E. Since E is complete, thus $\lim_{n\to\infty} d(x_n, y_n)$ exists.

Next, we show that d^* is well defined. Indeed, if $\{x_n\} \sim \{z_n\}$ and $\{y_n\} \sim \{w_n\}$ then

$$\lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(y_n, w_n) = \theta.$$

From the relation

$$\begin{aligned} d(x_n, y_n) &\preceq Kd(x_n, z_n) \\ &\quad + d(z_n, w_n) + Kd(w_n, y_n) \end{aligned}$$

and the relation

$$\begin{aligned} d(z_n, w_n) &\preceq Kd(z_n, x_n) \\ &\quad + d(x_n, y_n) + Kd(y_n, w_n), \end{aligned}$$

it follows that

$$\begin{aligned} &-K[d(x_n,z_n) + d(w_n, y_n)] \\ &\qquad \preceq d(x_n, y_n) - d(z_n, w_n) \\ &\qquad \preceq K[d(x_n, z_n) + d(w_n, y_n)]. \end{aligned}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, w_n).$$

Therefore, the function d^* is well defined.

We next show that d^* is a strong b-TVS cone metric on X^*. Since $\theta \preceq d(x_n, y_n)$ for all n and by the closedness of C, it follows that

$$\theta \preceq \lim_{n\to\infty} d(x_n, y_n) = d^*(x^*, y^*).$$

If $d^*(x^*, y^*) = \theta$ then $\lim_{n\to\infty} d(x_n, y_n) = \theta$. This is equivalent to $\{x_n\} \sim \{y_n\}$ and thus $x^* = y^*$. Since $d(x_n, y_n) = d(y_n, x_n)$

for all n, we have $d^*(x^*, y^*) = d^*(y^*, x^*)$. Finally,

$$\begin{aligned} d^*(x^*, y^*) &= \lim_{n\to\infty} d(x_n, y_n) \\ &\preceq \lim_{n\to\infty} d(x_n, z_n) \\ &\quad + K \lim_{n\to\infty} d(z_n, y_n) \\ &= d^*(x^*, z^*) + Kd^*(z^*, y^*), \end{aligned}$$

where $\{x_n\} \in x^*, \{y_n\} \in y^*$ and $\{z_n\} \in z^*$. Thus, d^* is a strong b-TVS cone metric on X^*.

For each $x \in X$, we put $f(x) = [\{x, x, ...\}] \in X^*$. Since $d^*(f(x), f(y)) = \lim_{n\to\infty} d(x, y) = d(x, y)$ for all $x, y \in X$ then we have f is an isometry from (X, E, C, K, d) into (X^*, E, C, K, d^*).

Let e be a fixed point of $intC$. Next, we will prove that $f(X)$ is dense in X^*. Indeed, if $x^* = [\{x_n\}] \in X^*$ then $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$. By Lemma 2.12, for each $i \geq 1$, there exists n_0^i such that $d(x_n, x_m) \ll \frac{e}{i}$ for all $n, m \geq n_0^i$. This implies that

$$\theta \preceq d^*(f(x_{n_0^i}), x^*) = \lim_{n\to\infty} d(x_{n_0^i}, x_n) \preceq \frac{e}{i}.$$

Hence $\lim_{i\to\infty} d^*(f(x_{n_0^i}), x^*) = \theta$. Thus $\lim_{i\to\infty} f(x_{n_0^i}) = x^*$, that is, $f(X)$ is dense in X^*.

We now prove (X^*, E, C, K, d^*) is complete. Let $\{x_n\}$ be a Cauchy sequence in X^*, where $x_n^* = [\{x_i^n\}_i]$ with $\{x_i^n\}_i \in S$. We have

$$\lim_{n,m\to\infty} d^*(x_n^*, x_m^*) = \theta.$$

Since $f(X)$ is dense in X^*, for each n, there exists $y_n \in X$ such that

$$d^*(f(y_n), x_n^*) \ll \frac{e}{Kn}.$$

This implies

$$\begin{aligned} \theta &\preceq d(y_n, y_m) \\ &= d^*(f(y_n), f(y_m)) \\ &\preceq Kd^*(f(y_n), x_n^*) + d^*(x_n^*, f(y_m)) \\ &\preceq Kd^*(f(y_n), x_n^*) + d^*(x_n^*, x_m^*) \\ &\quad + Kd^*(x_m^*, f(y_m)) \\ &\ll \frac{e}{n} + \frac{e}{m} + d^*(x_n^*, x_m^*). \end{aligned}$$

Since $\lim_{n,m\to\infty}[\frac{e}{n} + \frac{e}{m} + d^*(x_n^*, x_m^*)] = \theta$, then we have $\lim_{n,m\to\infty} d(y_n, y_m) = \theta$ and $\{y_n\}$ is a Cauchy sequence in (X, E, C, K, d).

Put $y^* = [\{y_n\}] \in X^*$. Then, we have

$$\begin{aligned} \theta \preceq d^*(x_n^*, y^*) &\preceq Kd^*(x_n^*, f(y_n)) \\ &\quad + d^*(f(y_n), y^*) \\ &\ll \frac{e}{n} + \lim_{m\to\infty} d(y_n, y_m). \end{aligned}$$

Now, letting $n \to \infty$, we have $\lim_{n\to\infty} d^*(x_n^*, y^*) = \theta$, which further means that $\lim_{n\to\infty} x_n^* = y^*$. Therefore, (X^*, E, C, K, d^*) is complete.

(ii) We now prove the uniqueness of the completion. Indeed, let $(X_1^*, E, C, K_1, d_1^*)$ and $(X_2^*, E, C, K_2, d_2^*)$ be two completions of (X, E, C, K, d). Let $f_1 : X \to X_1^*$ and $f_2 : X \to X_2^*$ be isometries. For each $x_1^* \in X_1^*$, there exists $\{x_n\} \subset X$ such that $\lim_{n\to\infty} f_1(x_n) = x_1^*$. This implies that $\{f_1(x_n)\}$ is a Cauchy sequence in X_1^*. Since f_1 is an isometry, $\{x_n\}$ is a Cauchy sequence in X. Similarly, since f_2 is an isometry, then $\{f_2(x_n)\}$ is a Cauchy sequence in X_2^*. Thus, there exists $x_2^* \in X_2^*$ such that $\lim_{n\to\infty} f_2(x_n) = x_2^*$.

Now, we define $\phi : X_1^* \to X_2^*$ by $\phi(x_1^*) = x_2^*$. We first show that ϕ is bijective. Indeed, let $y_2^* \in X_2^*$ be arbitrary. Then there exists $\{y_n\} \subset X$ such that $\lim_{n\to\infty} f_2(y_n) = y_2^*$. Since f_2 is an isometry, $\{y_n\}$ is a Cauchy sequence in X. By the isometry of f_1, $\{f_1(y_n)\}$ is a Cauchy sequence in X_1^*. This implies there exists $y_1^* \in X_1^*$ such that $\lim_{n\to\infty} f_1(y_n) = y_1^*$. Hence $\phi(y_1) = y_2$. This proves that ϕ is bijective. On the other hand, for every $x^*, y^* \in X_1^*$ with $\lim_{n\to\infty} f_1(x_n) = x^*$ and $\lim_{n\to\infty} f_1(y_n) = y^*$, it follows from Lemma 2.12 that

$$\begin{aligned} d_1^*(x^*, y^*) &= \lim_{n\to\infty} d_1^*(f_1(x_n), f_1(y_n)) \\ &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} d_2^*(f_1(x_n), f_1(y_n)) \\ &= d_2^*(\phi(x^*), \phi(y^*)). \end{aligned}$$

This implies that ϕ is a bijective isometry which restricts to the identity on X. □

Remark 3.7. *(i) In the proof of Theorem 3.6, we used Lemma 2.5 and 2.12. So, the neighborhood property of the cone C was used.*

(ii) When $E = \mathbb{R}$ and $C = \mathbb{R}_+$. Then Theorem 3.6 will reduce to Theorem 2.2 in [12].

The next example shows that our theorem can be useful, while other results in [12] cannot be applied.

Example 3.8. Let $X = \mathbb{Q}, E = \mathbb{R}^2, C = \{(t, 0) \in E : t \geq 0\}, K \geq 1$ and $d : X \times X \to E$ by

$$d(x, y) = (|x - y|, 0) \text{ for all } x, y \in X.$$

Since (X, E, C, K, d) is not strong b-metric space in the sense of An and Dung in [12], we cannot apply Theorem 2.2 of [12]. However, we can easily check that (X, E, C, K, d) is a strong b-TVS cone metric space.

Since C has a closed convex bounded base $B = \{(1, 0)\}$, then C has the neighborhood property. Therefore, the assumptions in Theorem 3.6 are satisfied. Moreover, taking $X^* = \mathbb{R}$ and $E = \mathbb{R}^2$, we have (X^*, E, C, K, d^*) as a completion of (X, E, C, K, d), where $d^*(x^*, y^*) = (|x^* - y^*|, 0)$ for all $x^*, y^* \in X^*$.

Acknowledgments

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. The third author was supported by the "Petchra Pra Jom Klao Ph.D. Research Scholarship at King Mongkut's University of Technology Thonburi".

References

[1] Abdullahi M. S, Kumam P. Partial $b_v(s)$-metric spaces and fixed point theorems. Journal of Fixed Point Theory and Applications. 2018;20(3):1-13.

[2] Abdullahi M. S, Sumalai P, Gopal D, Kumam P. Jungck-type Fixed Point Theorem in 0-complete Partial $b_v(s)$-Metric Spaces. Thai Journal of Mathematics. 2019;18(1):105-12.

[3] Rehman H.U, Kumam P, Dhompongsa S. Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness. Carpathian Journal of Mathematics. 2019;35(2):193-208.

[4] Kumam W, Sukprasert P, Kumam P, Shoaib A, Shahzad A, Mahmood Q. Some fuzzy fixed point results for fuzzy mappings in complete b-metric spaces. Cogent Mathematics & Statistics. 2018;5(1):1458933.

[5] Li X, Hussain A, Adeel M, Savas. Fixed Point Theorems for $\mathcal{Z}_\vartheta$-Contraction and Applications to Nonlinear Integral Equations. Ekrem. IEEE Access. 2019;7:120023-9.

[6] Phiangsungnoen S, Kumam P. On stability of fixed point inclusion for multivalued type contraction mappings in dislocated b-metric spaces with application.

Mathematical Methods in the Applied Sciences. 2018;41(17):7407-20.

[7] Shoaib A, Kumam P, Shahzad A, Phiangsungnoen S, Mahmood Q. Fixed point results for fuzzy mappings in a b-metric space. Fixed Point Theory and Applications. 2018;2018(1):1-12.

[8] Sintunavarat W, Kumam P. Common fixed points of f-weak contractions in cone metric spaces. Bulletin of the Iranian Mathematical Society. 2013;38(2):293-303.

[9] Huang L.G, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications. 2007;332(2):1468-76.

[10] Du W.S. A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis: Theory, Methods & Applications. 2010;72(5):2259-61.

[11] Kirk W, Shahzad N. Fixed point theory in distance spaces. vol.1. Springer; 2014.

[12] Van An T, Van Dung N. Answers to Kirk-Shahzad's questions on strong b-metric spaces. Taiwanese Journal of Mathematics. 2016;20(5):1175-84.

[13] Luc DT. Theory of Vector Optimization title: Lecture Notes in Economics and Mathematical Systems. Springer-VerlagBerlin; 1988.

[14] Alimohammady M, Balooee J, Radojević S, Rakočević V, Roohi M. Conditions of regularity in cone metric spaces. Applied Mathematics and Computation. 2011;217(13):6359-63.