P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : https://tci-thaijo.org/index.php/SciTechAsia
Science & Technology Asia
Vol.26 No.3 July - September 2021
Page: [14-21]
Original research article

Some Fixed Point Theorems for $R_{\tilde{n}}$-Contraction and $R_{\tilde{n}}$-Kannan Mappings in Metric Spaces

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Received 16 July 2020; Received in revised form 16 January 2021
Accepted 4 March 2021; Available online 6 September 2021

ABSTRACT

The purpose of this paper is to extend and improve some results concerning of R'-max-Kannan and R''-Kannan mappings to $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. Second, we establish new mapping, that is a $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. More than that, we prove the results of fixed point for such mappings in metric spaces.

Keywords: Fixed point; Metric spaces; b-metric spaces; $R_{\tilde{n}}$-contraction; $R_{\tilde{n}}$-function

1. Introduction

Let (X, d) be a metric space and T be a mapping from X into itself. A mapping T is a contraction if there exists a number $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y) \qquad (1.1)$$

for all $x, y \in X$. The well-known Banach contraction principle is the following: If $T : X \to X$ is a contraction mapping of a complete metric space X into itself, then

1. there is $x*$ in X which is a unique fixed-point,
2. $T^n x \to x*$ for all $x \in X$,
3. $d(T^n x, x) \leq \frac{r^n}{1-r} d(x, Tx), \forall\ x \in X$.

The theorem of Banach and its extensions usually are proved by the fact that the geometrical series $\sum_{n=0}^{\infty} r^n$ is convergent. Some different proof of the Banach theorem is given by Kannan [1], where he investigated properties of subsets of X, defined as $S_r = \{x \in X : d(x, Tx) \leq r\}, 0 < r < +\infty$. Fur-

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doi: 10.14456/scitechasia.2021.42

ther, Kannan [2] showed the following: If X is a complete metric space and mapping $T : X \to X$ satisfies the following condition

$$d(Tx, Ty) \leq r(d(x, Tx) + d(y, Ty)) \quad (1.2)$$

for all $x, y \in X$, where $0 < r < \frac{1}{2}$. Then T has exactly a fixed point in X. The condition (1.1) and (1.2) are independent, as it was shown by two examples in [2].

In 1972, Bianchini [3] introduced generalized Kannan mapping which generalized the concept of Kannan [2] as follows: Let T be a self-mapping on a metric space X. A mapping T is called a generalized Kannan mapping or Bianchini mapping if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\} \quad (1.3)$$

for all $x, y \in X$.

In 2015, Khojasteh et al. [4] introduced the notion of Z-contraction defined by simulation function. Then, they proved a new fixed point theorem concerning Z-contraction which generalizes Banach's contraction principle. Recently, Roldan-López-de-Hierro and Shahzad [5] introduced the concept of R-contraction defined by R-function in order to generalize the previous results.

In 2017, Mongkolkeha et al. [6] introduced a simulation function in the framework of b-metric spaces showed below:

Definition 1.1 ([6]). Let K be a given real number such that $K \geq 1$. A K-simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$,

(ζ_2) $\zeta(Kt, s) \leq s - Kt$, for all $t, s > 0$,

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $[0, \infty)$ such that $\lim_{n\to\infty} Kt_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \zeta(Kt_n, s_n) < 0.$$

The class of all K-simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is denoted by Z^* .

Example 1.2 ([6]). Let $\lambda, K \in \mathbb{R}$ such that $\lambda < 1$ and $K \geq 1$. Define the mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(Kt, s) = \begin{cases} s - Kt & \text{if } s < t, \\ \frac{\lambda s - Kt}{Ks+1} & \text{if } s \geq t. \end{cases}$$

Then $\zeta \in Z^*$ but $\zeta \notin Z$, where $Z$is simulation functions and Z^* is K-simulation functions.

In 2018, Wiriyapongsanon and Phudolsitthiphat [7] defined a generalization of R-contraction in b-metric spaces, called R'-contractions, via $R\prime$-functions and proved the existence and uniqueness of fixed point for such classes of mappings in complete b-metric spaces.

Definition 1.3 ([7]). Let K be a given real number such that $K \geq 1$. A function $\tilde{n} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called R'-function if it satisfies the following two conditions:

- ($\tilde{n}'_1$) If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\tilde{n}(Ka_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.
- ($\tilde{n}'_2$) If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n = \limsup_{n\to\infty} b_n = L \geq 0$ and verifying that $L < Ka_n$ and $\tilde{n}(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$. The class of all R'-functions $\tilde{n} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is denoted by R^*. We also consider the following property.
- ($\tilde{n}'_3$) If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $b_n \to 0$ and

$\tilde{n}(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Lemma 1.4 ([7]). *Every K-simulation function is a R-function that also verifies* $(\tilde{n}'_3)$.

Definition 1.5 ([7]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called R-contraction if there exists an R-function $\tilde{n} : A \times A \to \mathbb{R}$ such that $ran(d) \subseteq A$ and $\tilde{n}(d(Tx, Ty), d(x, y)) > 0$ for all $x, y \in X$ such that $x \neq y$.

Notice that if we take $\tilde{n}(t, s) = \lambda s - t$ for all $s, t \geq 0$ and $\lambda \in [0, 1)$ in Definition 1.5, then R-contraction become the Banach contraction.

Theorem 1.6 ([7]). *Let (X, d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R'-contraction with respect $\tilde{n} \in R^*$. If $\tilde{n}(Kt, s) \leq s - Kt$ for all $s, t \in (0, \infty)$ then T has a unique fixed point.*

In 2019, Cholatis et al. [8] improved R'-contractions and via R'-functions mappings to R'-Max-Kanan and R''-Kanan mappings by using the concept of Kanan mappings. Second, who establish new mapping, that is R'-Max-Kanan and R''-Kanan mappings and prove the results of fixed point for R'-Max-Kanan and R''-Kanan mappings in b-metric spaces. Moreover, who obtain fixed point theorems for R'-Max-Kanan and R''-Kanan mappings in b-metric spaces

Theorem 1.7 ([8]). *Let (X, d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R'-Max-Kanan mapping , i.e., $\tilde{n}(2Kd(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}) > 0$, with respect to $\tilde{n} \in R^*$. If $\tilde{n}(2Kt, s) \leq s - 2Kt$ for all $s, t \in (0, \infty)$ then T has a unique fixed point.*

Definition 1.8 ([8]). Let K be a given real number such that $K \geq 1$. A function $\tilde{n} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called *R''-function* if it satisfies the following two conditions:

- $(\tilde{n}'_1)$ If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\tilde{n}(2Ka_{n+1}, a_n + a_{n+1}) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.
- $(\tilde{n}'_2)$ If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n = \limsup_{n\to\infty} b_n = L \geq 0$ and verifying that $L < Ka_n$ and $\tilde{n}(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then L = 0. The class of all R''-functions $\tilde{n} : [0, \infty) \times [0, \infty) \to \mathbb{R}$. is denoted by R^{**}.We also consider the following property.
- $(\tilde{n}'_3)$ If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $b_n \to 0$ and $\tilde{n}(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Theorem 1.9 ([8]). *Let (X, d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R''-Kannan mapping, i.e., $\tilde{n}(2Kd(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}) > 0$, with respect to $\tilde{n} \in R^*$. If $\tilde{n}(2Kt, s) \leq s - 2Kt$ for all $s, t \in (0, \infty)$ then Thas a unique fixed point.*

The purpose of this paper is to extend and improve some results concerning of R'-max-Kannan and R''-Kannan mappings to $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. Second, we establish new mapping, that is a $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. More than that, we prove the results of fixed point for such mappings in metric spaces.

2. Main Results

In this section, we prove fixed point theorems for $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings in metric spaces.

Definition 2.1. A function $\tilde{n} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called $R_{\tilde{n}}$-function if it satisfies the following two conditions:

- $(\tilde{n}_1)$ If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\tilde{n}(2a_{n+1}, a_n + 2a_{n+1} + a_{n+2}) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.
- $(\tilde{n}_2)$ If $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two sequences such that $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = L \geq 0$ and verifying that $L < a_n$ and $\tilde{n}(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.
- $(\tilde{n}_3)$ If $s \geq l$, then $\tilde{n}(t,s) \geq \tilde{n}(t,l)$.

Theorem 2.2. *Let (X,d) be a complete metric and suppose that let $T : X \to X$ be $R_{\tilde{n}}$-contraction mapping with respect to $\tilde{n} \in R^*$, i.e.*

$$\tilde{n}(2d(Tx,Ty), d(x,Ty) + d(y,Tx)) > 0$$

for all $x \in X$. If $\tilde{n}(t,s) \leq s - t$ for all $s,t \in (0,\infty)$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is $x_{n+1} = Tx_n$ for all $n \geq 1$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ which implies that x_{n_0} is a fixed point. Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0,\infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By $R_{\tilde{n}}$-contraction mapping, $(\tilde{n}_1)$ and $(\tilde{n}_3)$, we get

$$\begin{aligned}
&\tilde{n}(2a_{n+1}, a_n + 2a_{n+1} + a_{n+2}) \\
&= \tilde{n}(2d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3})) \\
&\geq \tilde{n}(2d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+3})) \\
&= \tilde{n}(2d(Tx_n, Tx_{n+1}), d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)) > 0.
\end{aligned}$$

By using the condition $(\tilde{n}_1)$, we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \text{ and } d(x_{n_k}, x_{m_{k-1}}) \leq \varepsilon_0, \tag{2.1}$$

for all $m_k > n_k \geq k$. We consider, for any $m_k > n_k \geq k$,

$$\begin{aligned}
\varepsilon_0 &< d(x_{n_k}, x_{m_k}) \\
&\leq (d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})) \\
&\leq \varepsilon_0 + d(x_{m_{k-1}}, x_{m_k}).
\end{aligned}$$

Taking limit superior form k to infinity, we have

$$\varepsilon_0 \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \varepsilon_0. \tag{2.2}$$

So,

$$\limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon_0. \tag{2.3}$$

Since

$$\begin{aligned}
d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}) \\
&\leq \varepsilon_0 + d(x_{m_{k-1}}, x_{m_k}),
\end{aligned}$$

taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_k}, x_{m_{k-1}}) = \varepsilon_0. \tag{2.4}$$

Since $d(x_{n_{k-1}}, x_{m_k}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k})$, taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_k}) \le \varepsilon_0. \qquad (2.5)$$

By $R_{\tilde{n}}$-contraction mapping,

$$\begin{aligned} 0 &< \tilde{n}(2d(x_{n_k}, x_{m_k}), d(x_{n_{k-1}}, Tx_{m_{k-1}}) + \\ &\quad d(x_{m_{k-1}}, Tx_{n_{k-1}})) \\ &< \tilde{n}(2d(x_{n_k}, x_{m_k}), d(x_{n_{k-1}}, x_{m_k}) + \\ &\quad d(x_{m_{k-1}}, x_{n_k})) \\ &\le [d(x_{n_{k-1}}, x_{m_k}) + d(x_{m_{k-1}}, x_{n_k})] \\ &\quad - 2d(x_{n_k}, x_{m_k}). \end{aligned}$$

By (2.1)-(2.5), we get that

$$\limsup_{k\to\infty} 2d(x_{n_{k-1}}, x_{m_k}) = 2\varepsilon_0.$$

And, so

$$\begin{aligned} &\limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \\ &= \limsup_{k\to\infty}(d(x_{n_{k-1}}, x_{m_k}) + d(x_{n_k}, x_{m_{k-1}})) \\ &= \varepsilon_0. \end{aligned}$$

By using condition $(\tilde{n}_2)$, $\varepsilon_0 = 0$. That is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. By definition of convergence sequence, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, z) < \varepsilon \text{ for all } n > N. \qquad (2.6)$$

Next, we will show that z is fixed point. Let $\Omega := \{n \in \mathbb{N} : d(x_n, z) = 0\}$. Assume that Ω is not finite, then we can find $n_0 > N$ such that $d(x_{n_0}, z) = 0$ i.e. $x_{n_0} = z$. Since $x_{n_0} \neq x_{n_0+1}$ and $x_{n_0+1} = Tx_{n_0} = Tz, z \neq Tz$. Let $\varepsilon = \frac{d(z,Tz)}{2} > 0$. By (2.6), we get

$$\begin{aligned} \varepsilon &> d(x_{n_0+1}, z) \\ &= d(Tx_{n_0}, z) = d(Tz, z) = 2\varepsilon, \end{aligned}$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n, z) > 0$ for all $n > n_0$. Since T is a $R_{\tilde{n}}$-contraction mapping,

$$\begin{aligned} 0 &< \tilde{n}(2d(Tx_n, Tz), d(x_n, Tz) + d(z, Tx_n)) \\ &\le d(x_n, Tz) + d(z, Tx_n) - 2d(Tx_n, Tz). \end{aligned}$$

Hence,

$$\begin{aligned} 2d(x_{n+1}, Tz) &= 2d(Tx_n, Tz) \\ &\le d(x_n, Tz) + d(z, x_{n+1}) \\ &\le d(x_n, x_{n+1}) \\ &\quad + d(x_{n+1}, Tz) + d(z, x_{n+1}). \end{aligned}$$

And, so

$$d(x_{n+1}, Tz) \le d(x_n, x_{n+1}) + d(z, x_{n+1}).$$

Taking limit n to infinity, $\lim_{n\to\infty} d(x_{n+1}, Tz) = 0$. That is $x_{n+1} \to Tz$. By the uniqueness of the limit in a b-metric space and $x_{n+1} \to z$, we get that $Tz = z$. Finally, let us show that z is unique fixed point of T. Assume $x = Tx$ and $y = Ty$ such that $x \neq y$. Let $a_n = d(x, y) > 0$ for all $n \in \mathbb{N}$. By assumption, we have

$$\begin{aligned} &\tilde{n}(2a_{n+1}, a_n + 2a_{n+1} + a_{n+2}) \\ &= \tilde{n}(2d(x, y), d(x, Ty) + d(y, Tx)) > 0. \end{aligned}$$

By using $(\tilde{n}_1)$, we get $a_n \to 0$, which imply that $d(x, y) = 0$, which is a contradiction. So $x = y$. □

Theorem 2.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be $R_{\tilde{n}}$-Kannan mapping with respect to $\tilde{n} \in R^*$, i.e.,*

$$\tilde{n}(2d(Tx, Ty), \max\{d(x, Ty), d(y, Tx)\}) > 0$$

for all $x \in X$. If $\tilde{n}(t, s) \le s - t$ for all $s, t \in (0, \infty)$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$ for all $n \ge 1$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then

$Tx_{n_0} = x_{n_0}$ which implies that x_{n_0} is a fixed point. Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0, \infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By $R_{\tilde{n}}$-Kannan contractive condition, $(\tilde{n}_1)$ and $(\tilde{n}_3)$, we get

$$\begin{aligned}
&\tilde{n}(2a_{n+1}, a_n + 2a_{n+1} + a_{n+2}) \\
&\geq \tilde{n}(2a_{n+1}, \max\{a_n + a_{n+1}, a_{n+1} + a_{n+2}\}) \\
&= \tilde{n}(2d(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}) \\
&\quad + d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2}) \\
&\quad + d(x_{n+2}, x_{n+3})\}) \\
&\geq \tilde{n}(2d(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+2}), \\
&d(x_{n+1}, x_{n+3})\}) \\
&= \tilde{n}(2d(Tx_n, Tx_{n+1}), \max\{d(x_n, Tx_{n+1}), \\
&d(x_{n+1}, Tx_n)\}) \\
&> 0.
\end{aligned}$$

By using the condition $(\tilde{n}_1)$, we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \text{ and } d(x_{n_k}, x_{m_{k-1}}) \leq \varepsilon_0 \tag{2.7}$$

for all $m_k > n_k \geq k$. We consider, for any $m_k > n_k \geq k$,

$$\begin{aligned}
\varepsilon_0 &< d(x_{n_k}, x_{m_k}) \\
&\leq (d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})) \\
&\leq \varepsilon_0 + d(x_{m_{k-1}}, x_{m_k}).
\end{aligned}$$

Taking limit superior form k to infinity, we have

$$\varepsilon_0 \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \varepsilon_0.$$

So,

$$\limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon_0. \tag{2.8}$$

Since

$$\begin{aligned}
\varepsilon_0 &< d(x_{n_k}, x_{m_k}) \\
&\leq d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}) \\
&\leq \varepsilon_0 + d(x_{m_{k-1}}, x_{m_k}),
\end{aligned}$$

taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_k}, x_{m_{k-1}}) = \varepsilon_0. \tag{2.9}$$

Since $d(x_{n_{k-1}}, x_{m_k}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k})$, taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_k}) \leq \varepsilon_0. \tag{2.10}$$

By $R_{\tilde{n}}$-Kannan contractive condition,

$$\begin{aligned}
0 &< \tilde{n}(2d(x_{n_k}, x_{m_k}), \\
&\max\{d(x_{n_{k-1}}, Tx_{m_{k-1}}), d(x_{m_{k-1}}, Tx_{n_{k-1}})\}) \\
&< \tilde{n}(2d(x_{n_k}, x_{m_k}), \\
&\max\{d(x_{n_{k-1}}, x_{m_k}), d(x_{m_{k-1}}, x_{n_k})\}) \\
&\leq [\max\{d(x_{n_{k-1}}, x_{m_k}), d(x_{m_{k-1}}, x_{n_k})\}] \\
&- 2d(x_{n_k}, x_{m_k}).
\end{aligned}$$

So, we have, for any $k \in \mathbb{N}$,

$$\begin{aligned}
2\varepsilon_0 &< 2d(x_{n_k}, x_{m_k}) \\
&\leq \max\{d(x_{n_{k-1}}, x_{m_k}), d(x_{m_{k-1}}, x_{n_k})\} \\
&\leq \max\{d(x_{n_{k-1}}, x_{m_k}), \varepsilon_0\} \\
&\leq d(x_{n_{k-1}}, x_{m_k}) + \varepsilon_0.
\end{aligned}$$

By (2.9)-(2.10), we get that

$$\limsup_{k\to\infty} 2d(x_{n_{k-1}}, x_{m_k}) = 2\varepsilon_0.$$

And, so

$$\begin{aligned}
&\limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \\
&= \limsup_{k\to\infty} (d(x_{n_{k-1}}, x_{m_k}) + d(x_{n_k}, x_{m_{k-1}})) \\
&= \varepsilon_0.
\end{aligned}$$

By using condition ($\tilde{n}_2$) $\varepsilon_0 = 0$. That is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. By definition of convergence sequence,

$$\text{for any } \varepsilon > 0 \text{ there exists } N, \tag{2.11}$$

such that $d(x_n, z) < \varepsilon$ for all $n > N$.

Next, we will show that z is fixed point. Let $\Omega := \{n \in \mathbb{N} : d(x_n, z) = 0\}$. Assume that Ω is not finite, then we can find $n_0 > N$ such that $d(x_{n_0}, z) = 0$ i.e. $x_{n_0} = z$. Since $x_{n_0} \neq x_{n_0+1}$ and $x_{n_0+1} = Tx_{n_0} = Tz, z \neq Tz$. Let $\varepsilon = \frac{d(z,Tz)}{2} > 0$. By (2.11), we have

$$\varepsilon > d(x_{n_0+1}, z) = d(Tx_{n_0}, z) = d(Tz, z) = 2\varepsilon,$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n, z) > 0$ for all $n > n_0$. Since T is a $R_{\tilde{n}}$-kannan mapping,

$$\begin{aligned} 0 &< \tilde{n}(2d(Tx_n, Tz), \\ &\quad \max\{d(x_n, Tz), d(z, Tx_n)\}) \\ &\leq \max\{d(x_n, Tz), d(z, Tx_n)\} \\ &\quad - 2d(Tx_n, Tz) \\ &\leq d(x_n, Tz) + d(z, Tx_n) - 2d(Tx_n, Tz). \end{aligned}$$

Hence,

$$\begin{aligned} 2d(Tx_n, Tz) &\leq d(x_n, Tz) + d(z, x_{n+1}) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tz) + d(z, x_{n+1}). \end{aligned}$$

And, so

$$d(Tx_n, Tz) \leq d(x_n, x_{n+1}) + d(z, x_{n+1}).$$

Taking limit n to infinity, $\{x_{n+1} = Tx_n\} \to Tz$. By the uniqueness of the limit, $Tz = z$. Finally, we show that z is unique fixed point of T. Assume $x = Tx$ and $y = Ty$ such that $x \neq y$. Let $a_n = d(x, y) > 0$ for all $n \in \mathbb{N}$. We consider

$$\begin{aligned} 0 &< \varrho(2d(Tx, Ty), \\ &\quad \max\{d(x, Ty), d(y, Tx)\}) \\ &< \max\{d(x, Ty), d(y, Tx)\} - 2kd(x, y) \\ &< d(x, y) - 2d(x, y) \\ &= -d(x, y), \end{aligned}$$

which is a contradiction. So $x = y$.

□

3. Conclusion

The purpose of this paper is to extend and improve some results concerning of R'-max-Kannan and R''-Kannan mappings to $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. Second, we establish new mapping, that is a $R_{\tilde{n}}$-contraction and $R_{\tilde{n}}$-Kannan mappings. More than that, we prove the results of fixed point for such mappings in metric spaces as follows:

1.) Let (X, d) be a complete metric and let $T : X \to X$ be $R_{\tilde{n}}$-contraction mapping with respect to $\tilde{n} \in R^*$, i.e.

$$\tilde{n}(2d(Tx, Ty), d(x, Ty) + d(y, Tx)) > 0$$

for all $x \in X$. If $\tilde{n}(t, s) \leq s - t$ for all $s, t \in (0, \infty)$ then T has a unique fixed point.

2.) Let (X, d) be a complete metric space and let $T : X \to X$ be $R_{\tilde{n}}$-Kannan mapping with respect to $\tilde{n} \in R^*$, i.e.,

$$\tilde{n}(2d(Tx, Ty), \max\{d(x, Ty), d(y, Tx)\}) > 0$$

for all $x \in X$. If $\tilde{n}(t, s) \leq s - t$ for all $s, t \in (0, \infty)$, then T has a unique fixed point.

4. Discussion

Future research directions may also be possible.

Open problems 1:

If T satisfies

$\tilde{n}(5d(Tx, Ty), \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}) > 0$, then T has a unique fixed point.

Open problems 2:

If T satisfies $\tilde{n}(5d(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}) > 0$, then T has a unique fixed point.

Acknowledgements

The authors would like to thank the Science and Applied Science center, Kamphaeng Phet Rajabhat University, which provides funding for research.

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