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Original research article

On Existence and Uniqueness of Common Fixed Point in C^*-Algebra Valued Metric Spaces

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ABSTRACT

In the present article, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying rational type contractive condition in the framework of C^*-algebra valued metric spaces. The proved results extend and generalize some of the results in the literature.

Keywords: C^*-algebra valued metric space; Common fixed points; (CLR) property; (E.A.) property; Weakly compatible maps

1. Introduction and Preliminaries

The Banach contraction principle [1] is a very useful and effective tool of fixed point theory. The principle has numerous applications in pure as well as in applied sciences. For the last few decades, many researchers have investigated several types of mapping for the existence and uniqueness of a fixed point. (For reference see [2–13]). The principle is also useful for solving different kinds of equations such as integral, differential, fraction and partial differential equations (For reference see [14–19]).

Recently, Ma et al. [20] have extended Banach contraction principle to C^*-algebra valued metric spaces by replacing the set of real numbers with the set of all positive members of unital C^*-algebra. Afterward, many researcher have proved results in the framework of C^*-algebra valued metric spaces (For reference see [21–25]).

In the present manuscript, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying rational type contractive condition in the framework of C^*-algebra valued metric spaces. The proved results extend and generalize some of the results in the literature.

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For proving the results, we use some notation and definitions given in [23, 24].
A $*$-algebra $\mathbb{A}$ is a complex algebra with linear involution $*$ such that for all $\alpha, \beta \in \mathbb{A}$. $(\alpha, \beta)^* = \alpha^*\beta^*$ and $\alpha^{**} = \alpha$. The pair $(\mathbb{A}, *)$ is called a unital $*$-algebra if it contains the unity element $I_{\mathbb{A}}$. If a unital $*$-algebra satisfy $\|\alpha^*\| = \|\alpha\|$, for all $\alpha \in \mathbb{A}$, then $\mathbb{A}$ is called Banach $*$-algebra. A Banach C*-algebra satisfying $\|\alpha^*\alpha\| = \|\alpha\|^2$, for all $\alpha \in \mathbb{A}$ is called C*-algebra. If $\alpha = \alpha^*$ and $\sigma(\alpha) = \gamma \in \mathcal{R} : \beta I_{\mathbb{A}} - \alpha$ is non-invertible, α is called the positive element of $\mathbb{A}$. If $\alpha \in \mathbb{A}$ is positive, we write it as $\alpha > 0_{\mathbb{A}}$. The partial ordering on $\mathbb{A}$ can be defined as follows: $\alpha > \beta$ if and only if $\alpha - \beta > 0_{\mathbb{A}}$. Throughout the paper, by $\mathbb{A}$, we denote a unital C^*- algebra with the unity element $I_{\mathbb{A}}$.

Definition 1.1 ([20])**.** Suppose X is a non-empty set. The mapping $d : X \times X \to \mathbb{A}$ is called a C^*-algebra valued metric on X if it satisfies :

(i) $d(p, q) \geq 0_{\mathbb{A}}$ and $d(p, q) = 0_{\mathbb{A}}$ iff $p = q$;

(ii) $d(p, q) = d(q, p)$;

(iii) $d(p, r) \leq d(p, q) + d(q, r)$

for all $p, q, r \in X$. Then d is called a C^*-algebra metric on X and the triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 1.2 ([20])**.** A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to be

(i) convergent with respect to $\mathbb{A}$, if for given $\epsilon > 0$, there exists a positive integer k such that $\|d(x_n, x)\| < \epsilon$, for all $n > k$;

(ii) a Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \epsilon$, for all $n, m > k$.

The triplet $(X, \mathbb{A}, d)$ is called a complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 1.3 ([28])**.** Let f and g be two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to be weakly compatible if they commute at coincidence points.

Aamri and El Moutawakil [29] introduced the concept of $(E.A.)$ property in metric space.

Definition 1.4 ([29])**.** Let f and g be two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some} \quad t \in X.$$

The (E.A.) property allows to replace the completeness requirement of the space with a more natural condition of closeness of the range.
Sintunavarat and Kumam introduced [30] the concept of (CLR) property in which there is no requirement of closeness of space.

Definition 1.5 ([30])**.** Let f and g be two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to satisfy (CLR_f) property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = ft \quad \text{for some} \quad t \in X.$$

2. Main Results

2.1 Common fixed point theorem

Theorem 2.1. *Let $(X, \mathbb{A}, d)$ be C^*-algebra valued metric space and A, B, f and g are four self mapping on X satisfying the following conditions:*

(i) $A(X) \subseteq g(X)$ *and* $B(X) \subseteq f(X)$;

(ii) *for every* $x, y \in X$, $\alpha \in \mathbb{A}$ *with* $\|\alpha\| \leq 1$,

$$d(Ax, By) \preceq \alpha^*\Big(\big(d(fx, Ax)d(fx, By) + d(gy, By)d(gy, Ax)\big)/\big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha.$$

If one of $f(X)$, $g(X)$, $A(X)$ and $B(X)$ is a complete subspace of X, the pairs (A, f) and (B, g) have a coincidence point. Moreover, if the pairs (A, f) and (B, g) are weakly compatible then the mapping A, B, f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. From (i), we can construct a sequence $\{y_n\}$ in X as follows: $y_{2n+1} = Ax_{2n} = gx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = fx_{2n+2}$.
Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ i.e. $d(y_{2n}, y_{2n+1}) = 0$ for some n. Then $Ax_{2n} = gx_{2n+1} = Bx_{2n-1} = fx_{2n}$. Thus A and f have coincidence point. Hence the result. Now, suppose that $d_{2n} > 0$ for all $n \in \mathbb{N}$. Then, put $x = x_{2n}$ and $y = x_{2n+1}$ in condition (ii), we have

$$d(Ax_{2n}, Bx_{2n+1}) \preceq \alpha^*\Big(\big(d(fx_{2n}, Ax_{2n})d(fx_{2n}, Bx_{2n+1}) + d(gx_{2n+1}, Bx_{2n+1})d(gx_{2n+1}, Ax_{2n})\big)/\big(1+d(fx_{2n}, Bx_{2n+1})+d(gx_{2n+1}, Ax_{2n})\big)\Big)\alpha,$$

or

$$d(y_{2n+1}, y_{2n+2}) \preceq \alpha^*\Big(\big(d(y_{2n}, y_{2n+1})d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+2})d(y_{2n+1}, y_{2n+1})\big)/\big(1 + d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})\big)\Big)\alpha,$$

or

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\preceq \alpha^* \frac{d(y_{2n}, y_{2n+1})d(y_{2n}, y_{2n+2})}{1 + d(y_{2n}, y_{2n+2})}\alpha \\ &\preceq \alpha^* d(y_{2n}, y_{2n+1})\alpha. \end{aligned}$$

Thus, we have

$$d_{2n+1} \preceq \alpha^* d_{2n}\alpha.$$

On the same argument, we can conclude that $d_{2n} \preceq \alpha^*(d_{2n-1})\alpha$, $d_{2n-1} \preceq \alpha^*(d_{2n-2})\alpha$ and so on. In general, we have

$$d_n \preceq \alpha^*(d_{n-1})\alpha \quad \text{for all} \quad n \in \mathbb{N},$$

i.e

$$\begin{aligned} d(y_n, y_{n+1}) &\preceq (\alpha^*)d(y_{n-1}, y_n)\alpha \\ &\preceq (\alpha^*)^2 d(y_{n-2}, y_{n-1})\alpha^2 \\ &\dots \\ &\preceq (\alpha^*)^n d(y_0, y_1)\alpha^n. \end{aligned}$$

For any $p \in \mathbb{N}$ and by using triangle inequality, we get

$$\begin{aligned} d(y_{n+p}, y_n) &\preceq d(y_{n+p}, y_{n+p-1}) \\ &\quad + d(y_{n+p-1}, y_{n+p-2}) \\ &\quad + \dots + d(y_{n+1}, y_n) \\ &\preceq \sum_{m=n}^{n+p-1} (\alpha^*)^m d(y_0, y_1)\alpha^m \\ &\preceq \sum_{m=n}^{n+p-1} (\beta\alpha^m)^*\beta\alpha^m \end{aligned}$$

$$
\begin{aligned}
&\preceq \sum_{m=n}^{n+p-1} |\beta\alpha^m)|^2 \\
&\preceq \sum_{m=n}^{n+p-1} \||\beta\alpha^m)|^2\| I_{\mathbb{A}} \\
&\preceq \|\beta\|^2 I_{\mathbb{A}} \sum_{m=n}^{n+p-1} (\alpha^m)^2 \\
&\quad \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

where $|\beta|^2 = d(y_0, y_1)$ for some $\beta \in \mathbb{A}_+$ and $I_{\mathbb{A}}$ is the unity element in $\mathbb{A}$. Hence, $\{y_n\}$ is a Cauchy sequence since fX is complete subspace of X. Therefore $\{y_n\}$ is contained in fX and has a limit in fX, say u. Let $v \in f^{-1}u$, then $fv = u$. Next, we shall show that $Av = u$. Assume that, $Av \neq u$. Substituting $x = v$ and $y = x_{n-1}$ in contractive condition (ii), we get

$$
\begin{aligned}
d(Av, Bx_{n-1}) \preceq \\
&\alpha^*\Big(\big(d(fv, Av)d(fv, Bx_{n-1}) \\
&+ \ d(gx_{n-1}, Bx_{n-1})d(gx_{n-1}, Av)\big)/ \\
&\big(1 + d(fv, Bx_{n-1}) + d(gx_{n-1}, Av)\big)\Big)\alpha
\end{aligned}
$$

or

$$
\begin{aligned}
d(Av, y_n) \preceq \\
&\alpha^*\Big(\big(d(fv, Av)d(fv, y_n) \\
&+ \ d(y_{n-1}, y_n)d(y_{n-1}, Av)\big)/ \\
&\big(1 + d(fv, y_n) + d(y_{n-1}, Av)\big)\Big)\alpha.
\end{aligned}
$$

Taking limit as $n \to \infty$ on both sides, we get

$$
\begin{aligned}
d(Av, u) \preceq \\
&\alpha^*\Big(\big(d(u, Av)d(u, u) \\
&+ \ d(u, u)d(u, Av)\big)/ \\
&\big(1 + d(u, u) + d(u, Av)\big)\Big)\alpha
\end{aligned}
$$

Then, $\|d(Av, u)\| \leq 0$; hence $Av = u$. Thus, we have $fv = u = Av$, where v is the coincidence point of the pair(A, f). Since $AX \subseteq gX$, $Av = u$, this implies that $u \in gX$. Let $w \in g^{-1}u$, then $gw = u$. Using same argument as above, we can easily verify that $Bw = gw = u$, i.e., w is the coincidence point of the pair (B, g). The same result can be verified assuming gX complete instead of fX. Now, if $B(X)$ is complete, then by (i) $u \in B(X) \subseteq f(X)$. Similarly, if $A(X)$ is complete $u \in A(X) \subseteq g(X)$. Since the pair (A, f) and (B, g) are weakly compatible, therefore

$$u = Av = fv = gw = Bw,$$

then

$$
\begin{aligned}
gu &= gBw = Bgw = Bu, \\
fu &= fAv = Afv = Au.
\end{aligned}
$$

We claim that $Bu = u$. If possible, let $Bu \neq u$.

$$
\begin{aligned}
d(u, Bu) = d(Av, Bu) \preceq \\
&\alpha^*\Big(\big(d(fv, Av)d(fv, Bu) \\
&+ \ d(gu, Bu)d(gu, Av)\big)/ \\
&\big(1 + d(fv, Bu) + d(gu, Av)\big)\Big)\alpha \\
&\preceq \ 0.
\end{aligned}
$$

Then, $\|d(u, Bu)\| \leq 0$; hence $Bu = u$. On the same lines, we can show that $Au = u$. Thus, we get $Au = fu = gu = Bu = u$. Hence, u is a common fixed point of A, B, f and g.

Next, to prove uniqueness let z be another common fixed point different from u. i.e. $z \neq u$ of A, B, f and g.

$$
\begin{aligned}
d(z,u) = d(Az,Bu) \preceq \\
&\alpha^*\Big(\big(d(fz,Az)d(fz,Bu)\\
+ &\ d(gu,Bu)d(gu,Az)\big)/\\
&\big(1+d(fz,Bu)+d(gu,Az)\big)\Big)\alpha\\
\preceq &\ \alpha^*\Big(\big(d(z,z)d(z,u)\\
+ &\ (u,u)d(u,z)\big)/\\
&\big(1+d(z,u)+d(u,z)\big)\Big)\alpha\\
\preceq &\ 0.
\end{aligned}
$$

Then $\|d(z,u)\| \leq 0$; hence $z = u$. This implies uniqueness. □

Corollary 2.2. *Let $(X, \mathbb{A}, d)$ be C^*-algebra valued metric space and A and B are two self mapping on X satisfying*

$$
\begin{aligned}
&\|d(Ax,By)\|\\
&\quad \leq \ \|\alpha\|\big(\|d(x,Ax)\| + \|d(y,By)\|\big)
\end{aligned}
$$

for any $x, y \in X$, where $\alpha \in \mathbb{A}$ with $\|\alpha\| \leq 1$. Then A and B have a unique common fixed point in X.

Proof. Substituting $f = g = I_X$ in Theorem 2.1, one can easily verify the result. □

2.2 Common fixed point theorem using (E.A.) property

Theorem 2.3. *Let $(X, \mathbb{A}, d)$ be C^*-algebra valued metric space and A, B, f and g are four self mapping on X satisfying the following conditions:*

(i) *$A(X) \subseteq g(X)$ and $B(X) \subseteq f(X)$;*

(ii) *for every $x, y \in X$, $\alpha \in \mathbb{A}$ with $\|\alpha\| \leq 1$,*

$$
\begin{aligned}
d(Ax,By) \preceq \\
&\alpha^*\Big(\big(d(fx,Ax)d(fx,By)\\
+ &\ d(gy,By)d(gy,Ax)\big)/\\
&\big(1+d(fx,By)+d(gy,Ax)\big)\Big)\alpha;
\end{aligned}
$$

(iii) *The pair (A,f) and (B,g) are weakly compatible;*

(iv) *one of the pair (A,f) or (B,g) satisfy E.A. property.*

If the range of one of the mapping $f(X)$ or $g(X)$ is a closed subspace of X, then the mapping A, B, f and g have a unique common fixed point in X.

Proof. Firstly, we assume that the pair (B, g) satisfies E.A. property. Then, by Definition 1.4, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Further, $B(X) \subseteq f(X)$, there exists a sequence $\{y_n\}$ in X such that $B(x_n) = f(y_n)$. Hence, $\lim_{n\to\infty} f(y_n) = t$. We claim that $\lim_{n\to\infty} A(y_n) = t$. Let if possible $\lim_{n\to\infty} A(x_n) = t_1 \neq t$. Then, putting $x = y_n$ and $y = x_n$ in condition (ii), we have

$$
\begin{aligned}
d(Ay_n,Bx_n) \preceq \\
&\alpha^*\Big(\big(d(fy_n,Ay_n)d(fy_n,Bx_n)\\
+ &\ d(gx_n,Bx_n)d(gx_n,Ay_n)\big)/\\
&\big(1+d(fy_n,Bx_n)+d(gx_n,Ay_n)\big)\Big)\alpha.
\end{aligned}
$$

Taking norm and $\lim_{n\to\infty}$ on both sides, we get

$$
\begin{aligned}
\|d(t_1,t)\| \ \leq \ & \|\alpha\|^2\Big\|\Big(\big(d(t,t_1)d(t,t)\\
+ &\ d(t,t)d(t,t_1)]\big)/
\end{aligned}
$$

$$\left(1+d(t,t)+d(t,t_1)\right)\Big)\Big\|$$
$$\text{with } \|\alpha\| \leq 1$$
$$\leq \quad 0.$$

Then $\|d(t_1,t)\| = 0$; hence $t_1 = t$ i.e., $\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} B(x_n) = t$. Now, we suppose that $f(X)$ is closed subspace of X and $fu = t$ for some $u \in X$. Subsequently, we get

$$\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(y_n) = t = fu.$$

We claim that $Au = fu$. Put $x = u$ and $y = x_n$ in condition (ii), we have

$$d(Au, Bx_n) \leq \alpha^*\Big(\big(d(fu,Au)d(fu,Bx_n) + d(gx_n,Bx_n)d(gx_n,Au)\big)/\left(1+d(fu,Bx_n)+d(gx_n,Au)\right)\Big)\alpha.$$

Taking norm and $\lim_{n\to\infty}$ on both sides, we get

$$\lim_{n\to\infty} \|d(Au,t)\| \leq \lim_{n\to\infty} \|\alpha\|^2\Big(\Big\|\big(d(t,Au)d(t,t) + d(t,t)d(t,Au)\big)/\left(1+d(t,t)+d(t,Au)\right)\Big\|\Big) \quad \text{with } \|\alpha\| \leq 1,$$

or

$$\|d(Au,t)\| \leq 0.$$

Then $\|d(Au,t)\| = 0$; hence $Au = t = fu$ i.e u is the coincidence point of the pair (A, f).

Now the weak compatibility of the pair (A, f) implies that $Afu = fAu$ or $At = ft$.

Since $A(X) \subseteq g(X)$, there exits $v \in X$ such that $Au = gv = fu = t$. Now, we prove that v is coincidence point of pair (B, g), i.e $Bv = gv = t$. Put $x = u$ and $y = v$ in condition (ii), we get

$$d(Au, Bv) \leq \alpha^*\Big(\big(d(fu,Au)d(fu,Bv) + d(gv,Bv)d(gv,Au)\big)/\left(1+d(fu,Bv)+d(gv,Au)\right)\Big)\alpha.$$

Taking norm on both sides, we get

$$\|d(t,Bv)\| \leq \|\alpha\|^2\Big(\Big\|\big(d(t,t)d(t,Bv) + d(t,Bv)d(t,t)\big)/\left(1+d(t,Bv)+d(t,t)\right)\Big\|\Big) \quad \text{with } \|\alpha\| \leq 1,$$
$$\leq \quad 0.$$

Then $\|d(Bv,t)\| = 0$; hence $Bv = t$. Thus $Bv = gv = t$ and v is coincidence point of B and g.

Further, the weak compatibility of pair (B, g) implies that $Bgv = gBv$, or $Bt = gt$. Therefore, t is a common coincidence point of A, B, f and g.

Now, we prove that t is a common fixed point of A, B, f and g. Put $x = u$ and $y = t$ in condition (ii), we get

$$d(Au, Bt) \leq \alpha^*\Big(\big(d(fu,Au)d(fu,Bt) + d(gt,Bt)d(gt,Au)\big)/\left(1+d(fu,Bt)+d(gt,Au)\right)\Big)\alpha.$$

Taking norm on both sides, we get

$$\begin{aligned}\|d(Bt,t)\| &\\ \leq\ & \|\alpha\|^2\Big(\Big\|\big(d(t,t)d(t,Bt)\\ +\ & d(t,Bt)d(t,t)\big)/\\ & \big(1+d(t,Bt)+d(t,t)\big)\Big\|\Big)\\ & \quad\text{with}\quad \|\alpha\|\leq 1,\\ \leq\ & 0.\end{aligned}$$

Then $\|d(Bt,t)\| = 0$; hence $Bt = t$. Thus, $At = Bt = ft = gt = t$.

Similar arguments arise if we assume that $g(X)$ is closed subspace of X. Similarly, the (E.A.) property of the pair (A, f) will give a similar result.

For uniqueness, let w be another common fixed point of A,B,f and g. Then, put $x = w$ and $y = t$ in condition (ii), we get

$$\begin{aligned}d(w,t) &= d(Aw,Bt)\\ \leq\ & \alpha^*\Big(\big(d(fw,Aw)d(fw,Bt)\\ +\ & d(gt,Bt)d(gt,Aw)\big)/\\ & \big(1+d(fw,Bt)+d(gt,Aw)\big)\Big)\alpha,\\ \leq\ & \alpha^*\Big(\big(d(w,w)d(w,t)+d(t,t)d(t,w)\big)/\\ & \big(1+d(w,t)+d(t,w)\big)\Big)\alpha\\ \leq\ & 0.\end{aligned}$$

Then $\|d(w,t)\| \leq 0$; hence $d(w,t) = 0$ i.e $w = t$. Hence t is a unique common fixed point of A, B, f and g. □

2.3 Common fixed point theorem using (CLR) property

Theorem 2.4. *Let $(X, \mathbb{A}, d)$ be C^*-algebra valued metric space and A, B, f and g are four self mapping on X satisfying the following conditions:*

(i) *$A(X) \subseteq g(X)$ and $B(X) \subseteq f(X)$;*

(ii) *for every $x, y \in X$*

$$\begin{aligned}d(Ax,By) \leq &\\ & \alpha^*\Big(\big(d(fx,Ax)d(fx,By)\\ +\ & d(gy,By)d(gy,Ax)\big)/\\ & \big(1+d(fx,By)+d(gy,Ax)\big)\Big)\alpha;\end{aligned}$$

(iii) *The pair (A, f) and (B, g) are weakly compatible;*

(iv) *one of the pair satisfy (CLR) property.*

then the mapping A, B, f and g have a unique common fixed point in X.

Proof. Firstly, we suppose that the pair (B, g) satisfies (CLR_B) property. By Definition 1.5 there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} g(x_n) = Bx = t,$$

for some $x \in X$.

Since $B(X) \subseteq f(X)$, we have $Bx = fu$, for some $u \in X$. We claim that $Au = fu = t$(say). Put $x = u$ and $y = x_n$ in condition (ii), we get

$$\begin{aligned}d(Au,Bx_n) \leq &\\ & \alpha^*\Big(\big(d(fu,Au)d(fu,Bx_n)\\ +\ & d(gx_n,Bx_n)d(gx_n,Au)\big)/\\ & \big(1+d(fu,Bx_n)+d(gx_n,Au)\big)\Big)\alpha.\end{aligned}$$

Taking norm and $\lim_{n\to\infty}$ on both sides, we get

$$\begin{aligned}\|d(Au,Bx)\| &\\ \leq\ & \|\alpha\|^2\Big(\Big\|\big(d(Bx,Au)d(Bx,Bx)\end{aligned}$$

$$
\begin{aligned}
+ \quad & d(Bx,Bx)d(Bx,Au)\big)/ \\
& \big(1+d(Bx,Bx)+d(Bx,Au)\big)\Big\|\Big) \\
& \quad \text{with} \quad \|\alpha\| \le 1, \\
\le \quad & 0.
\end{aligned}
$$

Then $\|d(Au,Bx)\| = 0$; hence $Au = fu = Bx = t$.

Since $A(X) \subseteq g(X)$, there exists $v \in X$ such that $gv = Au = fu = t$.

Now, we prove that $gv = Bv = t$ i.e., v is the coincidence point of (B,g). Put $x = u$ and $y = v$ in condition (ii), we get

$$
\begin{aligned}
& d(Au,Bv) \le \\
& \quad \alpha^*\Big(\big(d(fu,Au)d(fu,Bv) \\
+ \quad & d(gv,Bv)d(gv,Au)\big)/ \\
& \big(1+d(fu,Bv)+d(gv,Au)\big)\Big)\alpha.
\end{aligned}
$$

Taking norm on both side, we get

$$
\begin{aligned}
& \|d(Au,Bv)\| \le \\
& \quad \|\alpha\|^2\Big(\Big\|\big(d(fu,Au)d(fu,Bv) \\
+ \quad & d(gv,Bv)d(gv,Au)\big)/ \\
& \big(1+d(fu,Bv)+d(gv,Au)\big)\Big\|\Big) \\
& \quad \text{with} \quad \|\alpha\| \le 1, \\
\le \quad & 0.
\end{aligned}
$$

Then $\|d(t,Bv)\| = 0$; hence $Bv = t$ i.e $Bv = gv = t$ and v is the coincidence point of B and g.

Further, the weak compatibility of pair (B,g) implies that $Bgv = gBv$ or $Bt = gt$. Therefore, t is a common coincidence point of A, B, f and g. Now, we prove that t is common fixed point of A, B, f and g. Put $x = u$ and $y = t$ in condition (ii), we get

$$
\begin{aligned}
& d(Au,Bt) \le \\
& \quad \alpha^*\Big(\big(d(fu,Au)d(fu,Bt) \\
+ \quad & d(gt,Bt)d(gt,Au)\big)/ \\
& \big(1+d(fu,Bt)+d(gt,Au)\big)\Big)\alpha.
\end{aligned}
$$

Taking norm on both side, we get

$$
\begin{aligned}
& \|d(t,Bt)\| \le \\
& \quad \|\alpha\|^2\Big(\Big\|\big(d(t,t)d(t,Bt) \\
+ \quad & d(t,Bt)d(t,t)\big)/ \\
& \big(1+d(t,Bt)+d(t,t)\big)\Big\|\Big) \\
& \quad with \quad \|\alpha\| \le 1, \\
\le \quad & 0.
\end{aligned}
$$

Then $\|d(t,Bt)\| = 0$; hence $Bt = t$. Thus $At = Bt = ft = gt = t$. i.e t is common fixed point of A, B, f and g.

For uniqueness, let w be another common fixed point of A,B,f and g. Then, put $x = w$ and $y = t$ in condition (ii), we get

$$
\begin{aligned}
d(w,t) &= d(Aw,Bt) \\
\le \quad & \alpha^*\Big(\big(d(fw,Aw)d(fw,Bt) \\
+ \quad & d(gt,Bt)d(gt,Aw)\big)/ \\
& \big(1+d(fw,Bt)+d(gt,Aw)\big)\Big)\alpha, \\
\le \quad & \alpha^*\Big(\big(d(w,w)d(w,t) \\
+ \quad & d(t,t)d(t,w)\big)/ \\
& \big(1+d(w,t)+d(t,w)\big)\Big)\alpha \\
\le \quad & 0.
\end{aligned}
$$

Then $\|d(w,t)\| \le 0$; hence $d(w,t) = 0$ i.e $w = t$. Hence t is a unique common fixed point of A, B, f and g. □

Example 2.5. Let $X = [0,1]$ and $\mathbb{A} = \mathbb{C}$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y) = |x - y|.$$

Then, $(X, \mathbb{A}, d)$ is C^*-algebra-valued metric space.

Define four self maps A, B, f and g on X by

$$Ax = x, \quad Bx = \frac{x}{2}, \\ gx = 2x, \quad fx = 4x \ \forall\, x \in X.$$

Clearly,

$$AX = [0,1] \subset [0,2] = gX, \\ BX = \left[0, \frac{1}{2}\right] \subset [0,4] = fX.$$

Also, fX is a complete subspace of X and the pair (A, f) and (B, g) are weakly compatible.

$$d(Ax, By) = \left|x - \frac{y}{2}\right|, \\ d(fx, Ax) = |4x - x|, \\ d(fx, By) = \left|4x - \frac{y}{2}\right|, \\ d(gy, Ax) = |2y - x|, \\ \text{and} \quad d(gy, By) = \left|2y - \frac{y}{2}\right|.$$

Observe that

$$d(Ax, By) \preceq \\ \alpha^*\Big(\big(d(fx, Ax)d(fx, By) \\ + \ d(gy, By)d(gy, Ax)\big)/ \\ \big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha \\ \forall\, x, y \in X \text{ with } \quad \|\alpha\| \leq 1.$$

Here, 0 is the unique common fixed point of A, B, f and g.

Example 2.6. Let $X = [0,2]$ and $\mathbb{A} = \mathbb{C}$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y) = \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Then, $(X, \mathbb{A}, d)$ is C^*-algebra-valued metric space.

Define four self maps A, B, f and g on X by

$$A(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2 & \text{if } x \in (1,2] \end{cases},$$

$$B(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,2] \end{cases},$$

$$g(x) = \begin{cases} 4x & \text{if } x \in [0,1] \\ 5 & \text{if } x \in (1,2] \end{cases}$$

and

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 3 & \text{if } x \in (1,2] \end{cases}.$$

Following cases arises
Case (i): Let $x, y \in [0,1]$, clearly $AX \subset gX$ and $BX \subset fX$.
Now,

$$d(Ax, By) = x + \frac{y}{2}, \quad d(fx, Ax) = 3x, \\ d(fx, By) = 2x + \frac{y}{2}, \quad d(gy, By) = \frac{9y}{2} \\ \text{and} \quad d(gy, Ax) = x + 4y.$$

Therefore,

$$\alpha^*\Big(\big(d(fx, Ax)d(fx, By) \\ + \ d(gy, By)d(gy, Ax)\big)/ \\ \big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha \\ = \ \alpha^*\Big(\big(3x(2x + \frac{y}{2}) + \frac{9y}{2}(x + 4y)\big)/$$

$$\Big(1+3x+\frac{9y}{2})\Big)\alpha$$

$$= \quad ||\alpha||^2\Big(\frac{12x^2+12xy+36y^2}{6x+9y+2}\Big)$$

$$\geq \quad ||\alpha||^2\Big(\frac{12x^2+12xy+36y^2}{6x+9y+3}\Big)$$

$$= \quad ||\alpha||^2\Big(\frac{4x^2+4xy+12y^2}{2x+3y+1}\Big)$$

$$= \quad ||\alpha||^2\Big(\big((2x+3y+1)^2+3y^2$$

$$- \quad 8xy-6y-4x-1\big)/$$

$$\big(2x+3y+1\big)\Big)$$

$$= \quad ||\alpha||^2\Big((2x+3y+1)$$

$$+ \quad \frac{3y^2-8xy-6y-4x-1}{2x+3y+1}\Big)$$

$$\geq \quad (x+\frac{y}{2}) = d(Ax,By).$$

Thus,

$$d(Ax,By) \preceq$$

$$\alpha^*\Big(\big(d(fx,Ax)d(fx,By)$$

$$+ \quad d(gy,By)d(gy,Ax)\big)/$$

$$\big(1+d(fx,By)+d(gy,Ax)\big)\Big)\alpha$$

$$\forall\, x,y\in[0,1] \text{ with } \quad \|\alpha\|\leq 1.$$

Case (ii): Let $x,y\in(1,2]$, clearly $AX\subset gX$ and $BX\subset fX$.
Now,

$$d(Ax,By)=3,\ d(fx,Ax)=5,$$
$$d(fx,By)=4, d(gy,By)=6$$
$$\text{and} \quad d(gy,Ax)=7.$$

Therefore,

$$\alpha^*\Big(\big(d(fx,Ax)d(fx,By)$$

$$+ \quad d(gy,By)d(gy,Ax)\big)/$$

$$\big(1+d(fx,By)+d(gy,Ax)\big)\Big)\alpha$$

$$= \quad \alpha^*\Big(\frac{62}{12}\Big)\alpha$$

$$= \quad ||\alpha||^2\Big(\frac{62}{12}\Big)$$

$$\geq \quad 3 = d(Ax,By).$$

Thus,

$$d(Ax,By) \preceq$$

$$\alpha^*\Big(\big(d(fx,Ax)d(fx,By)$$

$$+ \quad d(gy,By)d(gy,Ax)\big)/$$

$$\big(1+d(fx,By)+d(gy,Ax)\big)\Big)\alpha$$

$$\forall\, x,y\in(1,2] \text{ with } \quad \|\alpha\|\leq 1.$$

Also, fX is a complete subspace of X, the pair (A,f) and (B,g) are weakly compatible. Hence, by Theorem 2.1 the mappings A,B,f and g have a unique common fixed point. Indeed, 0 is a common unique fixed point.

Example 2.7. Let $X=[0,2]$ and $\mathbb{A}=\mathbb{C}$. Define $d: X\times X\to\mathbb{A}$ by $d(x,y)=|x-y|$. Then, we can easily show that $(X,\mathbb{A},d)$ is C^*-algebra-valued metric space.

Define four self mappings A,B,f and g on X by

$$A(x)=\begin{cases}0 & \text{if } x\in[0,1],\\ 1 & \text{if } x\in(1,2],\end{cases}$$

$$B(x)=\begin{cases}0 & \text{if } x\in[0,1],\\ \frac{1}{2} & \text{if } x\in(1,2],\end{cases}$$

$$g(x)=\begin{cases}3x & \text{if } x\in[0,1],\\ 4 & \text{if } x\in(1,2],\end{cases}$$

and

$$f(x) = \begin{cases} x & \text{if } x \in [0,1], \\ 2 & \text{if } x \in (1,2]. \end{cases}$$

Firstly, we show that the pair (A, f) is satisfying $(E.A.)$ property. Taking $\{x_n\}$ be a sequence in X such that $\{x_n\} = \left(\frac{1}{\sqrt{n^2+2n}}\right)$. Then,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(\frac{1}{\sqrt{n^2+2n}}\right) = \lim_{n\to\infty}(0) = 0$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f\left(\frac{1}{\sqrt{n^2+2n}}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+2n}}\right) = 0.$$

So, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = 0 \in X$. Hence, the pair (A, f) satisfy $(E.A.)$ property. Similarly, we can show that the pair (B, g) satisfy $(E.A.)$ property.
Following cases arise
Case(i) : Let $x, y \in [0,1]$, clearly $AX \subset gX$ and $BX \subset fX$.
Now,

$$d(Ax,By) = 0,\ d(fx,Ax) = x,\ d(fx,By) = x,\ d(gy,By) = 3y \text{ and } d(gy,Ax) = 3y.$$

Therefore,

$$\begin{aligned} &\alpha^*\Big(\big(d(fx,Ax)d(fx,By) \\ &+ d(gy,By)d(gy,Ax)\big)/ \\ &\big(1 + d(fx,By) + d(gy,Ax)\big)\Big)\alpha \\ &= \alpha^*\left(\frac{(x*x)+(3y*3y)}{x+3y+1}\right)\alpha \\ &= ||\alpha||^2\left(\frac{x^2+9y^2}{x+3y+1}\right) \\ &\geq 0 = d(Ax,By). \end{aligned}$$

Thus,

$$\begin{aligned} d(Ax,By) \leq \ &\alpha^*\Big(\big(d(fx,Ax)d(fx,By) \\ &+ d(gy,By)d(gy,Ax)\big)/ \\ &\big(1 + d(fx,By) + d(gy,Ax)\big)\Big)\alpha \\ &\forall\, x, y \in [0,1] \text{ with } \|\alpha\| \leq 1. \end{aligned}$$

Case (ii): Let $x, y \in (1,2]$, clearly $AX \subset gX$ and $BX \subset fX$.
Now,

$$d(Ax,By) = \frac{1}{2},\ d(fx,Ax) = 1,\ d(fx,By) = \frac{3}{2},\ d(gy,By) = \frac{7}{2} \text{ and } d(gy,Ax) = 3.$$

Therefore,

$$\begin{aligned} &\alpha^*\Big(\big(d(fx,Ax)d(fx,By) \\ &+ d(gy,By)d(gy,Ax)\big)/ \\ &\big(1 + d(fx,By) + d(gy,Ax)\big)\Big)\alpha \\ &= \alpha^*\left(\frac{24}{11}\right)\alpha \\ &= ||\alpha||^2\left(\frac{24}{11}\right) \\ &\geq \frac{1}{2} = d(Ax,By). \end{aligned}$$

Thus,

$$\begin{aligned} d(Ax,By) \leq \ &\alpha^*\Big(\big(d(fx,Ax)d(fx,By) \\ &+ d(gy,By)d(gy,Ax)\big)/ \\ &\big(1 + d(fx,By) + d(gy,Ax)\big)\Big)\alpha \end{aligned}$$

$$\forall\, x, y \in (1, 2] \text{ with } \quad \|\alpha\| \leq 1.$$

Also, fX is a closed subspace of X, the pair (A, f) and (B, g) are weakly compatible. Hence, by Theorem 2.3 the mappings A, B, f and g have a unique common fixed point. Indeed, 0 is a common unique fixed point.

Example 2.8. Let $X = [0, 2]$ and $\mathbb{A} = \mathbb{C}$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x, y) = \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then, $(X, \mathbb{A}, d)$ is C^*-algebra-valued metric space.

Define four self maps A, B, f and g on X by

$$A(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ x & \text{if } x \in (1, 2] \end{cases},$$

$$B(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ \frac{3}{2} & \text{if } x \in (1, 2] \end{cases},$$

$$g(x) = \begin{cases} 3x & \text{if } x \in [0, 1] \\ 2x & \text{if } x \in (1, 2] \end{cases}$$

and

$$f(x) = \begin{cases} 4x & \text{if } x \in [0, 1] \\ 7 & \text{if } x \in (1, 2]. \end{cases}$$

Firstly, we show that the pair (A, f) is satisfying (CLR_A) property. Taking $\{x_n\}$ be a sequence in X such that $\{x_n\} = \left(\frac{1}{n^2+2n+3}\right)$. Then,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(\frac{1}{n^2+2n+3}\right)$$

$$= \lim_{n\to\infty} (0) = 0$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f\left(\frac{1}{n^2+2n+3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{n^2 + 2n + 3}\right) = 0.$$

So, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = A(0) = 0 \quad for\ 0 \in X$. Hence, the pair (A, f) satisfy (CLR_A) property. Similarly, we can show that the pair (B, g) satisfy (CLR_B) property.

Following cases arise

Case(i) : Let $x, y \in [0, 1]$, clearly $AX \subset gX$ and $BX \subset fX$.

Now,

$$d(Ax, By) = y, \quad d(fx, Ax) = 4x$$
$$d(fx, By) = 4x + y, d(gy, By) = 4y$$
$$\text{and} \quad d(gy, Ax) = 3y.$$

Therefore,

$$\alpha^*\Big(\big(d(fx, Ax)d(fx, By)$$
$$+ \; d(gy, By)d(gy, Ax)\big)/$$
$$\big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha$$
$$= \alpha^*\left(\frac{4x(4x + y) + 4y(3y)}{1 + 4x + y + 3y}\right)\alpha$$

$$= \|\alpha\|^2\left(\frac{16x^2 + 4xy + 12y^2}{4x + 4y + 1}\right)$$
$$\geq \|\alpha\|^2\left(\frac{16x^2 + 4xy + 12y^2}{4x + 4y + 4}\right)$$
$$= \|\alpha\|^2\left(\frac{4x^2 + xy + 3y^2}{x + y + 1}\right)$$
$$= \|\alpha\|^2\Big((x + y + 1)^2$$
$$+ \; \big(2y^2 + 3x^2 - y - x - 1\big)/$$

$$
\begin{aligned}
& \big(x+y+1\big)\Big) \\
= \;& ||\alpha||^2\Big(\big(x+y+1\big) \\
+ \;& \frac{3x^2+2y^2-y-x-1}{x+y+1}\Big) \\
\geq \;& y = d(Ax, By).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(Ax, By) \leq & \\
& \alpha^*\Big(\big(d(fx, Ax)d(fx, By) \\
+ \;& d(gy, By)d(gy, Ax)\big)/ \\
& \big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha \\
& \forall\, x, y \in [0, 1] \text{ with } \quad ||\alpha|| \leq 1.
\end{aligned}
$$

Case(ii) : Let $x, y \in (1, 2]$, clearly $AX \subset gX$ and $BX \subset fX$.
Now,

$$
\begin{aligned}
& d(Ax, By) = x + \frac{3}{2},\ d(fx, Ax) = 7 + x \\
& d(fx, By) = \frac{17}{2}, d(gy, By) = 2y + \frac{3}{2} \\
& \text{and} \quad d(gy, Ax) = 2y + x.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \alpha^*\Big(\big(d(fx, Ax)d(fx, By) \\
+ \;& d(gy, By)d(gy, Ax)\big)/ \\
& \big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha \\
= \;& \alpha^*\left(\frac{(\frac{17}{2}(7+x)) + (2y+\frac{3}{2})(2y+x)}{1+2y+x+\frac{17}{2}}\right)\alpha \\
= \;& ||\alpha||^2\left(\frac{17(7+x)+(4y+3)(2y+x)}{4y+2x+19}\right) \\
= \;& ||\alpha||^2\left(\frac{20x+6y+8y^2+4xy+119}{4y+2x+19}\right) \\
= \;& ||\alpha||^2\left(4 + \frac{12x-18y+8y^2+4xy+43}{4y+2x+19}\right) \\
\geq \;& x + \frac{3}{2} = d(Ax, By).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(Ax, By) \leq & \\
& \alpha^*\Big(\big(d(fx, Ax)d(fx, By) \\
+ \;& d(gy, By)d(gy, Ax)\big)/ \\
& \big(1 + d(fx, By) + d(gy, Ax)\big)\Big)\alpha \\
& \forall\, x, y \in (1, 2] \text{ with } \quad ||\alpha|| \leq 1.
\end{aligned}
$$

The pair (A, f) and (B, g) are weakly compatible. Hence, by the Theorem 2.4 the mappings A, B, f and g have a unique common fixed point. Indeed, 0 is a common unique fixed point.

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