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New Exponential Passivity Analysis of Neutral System with Time-Varying Delays and Nonlinear Perturbations

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ABSTRACT

In this work, we study the delay-dependent exponential passivity and stability analysis of neutral system with time-varying delays and nonlinear perturbations. We derive new delay-dependent exponential passivity and stability conditions for the system in terms of linear matrix inequalities (LMIs) by constructing new class of augmented Lyapunov-Krasovskii functional, utilization of zero equation, descriptor model transformation, Leibniz-Newton formula and various inequalities. We also demonstrate the effectiveness and improvement of the results by giving some numerical examples.

Keywords: Exponential passivity analysis; Linear matrix inequality; Model transformation; Neutral system

1. Introduction

During the last decade, there has been interesting in the time-delay of neutral system due to their extensive applications, for example, in industrial systems, in communication, in engineering and so on. Stability of time-delay systems of neutral type has been divided into two classes depending on the size of the time-delay, delay-independent and delay-dependent. In [1–4], convex optimization algorithms has been treated. Moreover, in [5–7], free-weighting method within convex optimization approach has been presented. Some results on robust stability for uncertain neutral system have been derived in [8–11].

The passivity is an essential tool in stability analysis. The stability of the feedback can be established by using the passivity theorem. Then, there has been an increasing of reseachers, who are interesting in applying passivity to time-delay system

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[12].

The results mentioned in the literatures are studied with the asymtotic stability ([2, 10, 11, 13, 14]) or only concerned with the exponential passivity of neural networks ([12,15,16]). However, the exponential passivity of neutral problem is also essential since it can estimate the convergence rate.

Motivated by the statement above, in this paper, the exponential passivity and stability analysis of neutral system with time-varying delays and nonlinear perturbations are investigated using new class of augmented Lyapunov-Krasovskii functional, utilization of zero equation, Leibniz-Newton formula and various inequalities. Two numerical examples are given to demonstate the method.

2. Problem Formula and Preliminaries

We introduce some definitions and lemmas which we used throughout the paper and we also introduce the neutral system with time-varying delays and nonlinear perturbations of the form

$$\begin{aligned} \dot{y}(t) &= Ay(t) + By(t-r(t)) \\ &\quad + C\dot{y}(t-k(t)) + g_1(t, y(t)) \\ &\quad + g_2(t, y(t-r(t))) \\ &\quad + g_3(t, \dot{y}(t-k(t))) + D_1\omega(t), \\ &\quad t \geq 0, \qquad (2.1) \\ \tilde{z}(t) &= \tilde{C}_1 y(t) + \tilde{C}_2 y(t-r(t)) \\ &\quad + D_2\omega(t), \;\; t \geq 0, \qquad (2.2) \\ y(t) &= \phi(t), \\ &\quad \forall t \in [-\max\{r_2, k_2\}, 0], \qquad (2.3) \end{aligned}$$

where $y(t) \in \mathbb{R}^n$ is the state vector, $\omega(t)$ is the disturbance input, $\tilde{z}(t)$ is the output of the system and $\phi(t)$ is a vector valued initial function. $A, B, C, D_1, \tilde{C}_1, \tilde{C}_2$ and $D_2 \in \mathbb{R}^{n\times n}$ are real constant matrices with appropriate dimensions. The delays are time-varying delays satisfying

$$0 \leq r(t) \leq r_2, \qquad (2.4)$$

$$0 \leq k(t) \leq k_2, \quad \dot{k}(t) \leq k_d, \qquad (2.5)$$

where r_2, k_2 and k_d are real constants. The uncertainties $g_1(t, y(t)), g_2(t, y(t - r(t))), g_3(t, \dot{y}(t - k(t)))$ are nonlinear disturbances satisfying

$$g_1^T(t, y(t))g_1(t, y(t)) \leq l_1^2 y^T(t)y(t), \quad (2.6)$$

$$\begin{aligned} & g_2^T(t, y(t-r(t)))g_2(t, y(t-r(t))) \\ & \leq l_2^2 y^T(t-r(t))y(t-r(t)), \qquad (2.7) \\ & g_3^T(t, \dot{y}(t-k(t)))g_3(t, \dot{y}(t-k(t))) \\ & \leq l_3^2 \dot{y}^T(t-k(t))\dot{y}(t-k(t)), \qquad (2.8) \end{aligned}$$

where l_1, l_2, l_3 are given nonnegative constants.

Definition 2.1 ([1]). If there exist positive real constants α, M such that, for all $\phi(t)$, the solution $y(t, \phi)$ of the system satisfies

$$\|y(t,\phi)\| \leq M\|\phi\|e^{-\alpha t}, \quad t \geq 0,$$

then the system $Eq.$ (2.1) – $Eq.$ (2.3) is called exponentially stable.

Definition 2.2 ([12]). If there exists an exponential Lyapunov function $V(t)$ defined on $\mathbb{R}$, and a constant $\alpha > 0$ such that for all $\omega(t)$, all initail condition $y(0)$ satisfy:

$$\dot{V}(t) + 2\alpha V(t) \leq 2\tilde{z}(t)\omega(t), \quad t \geq 0, \qquad (2.9)$$

where $\dot{V}(t)$ denotes the total derivative of $V(t)$ along the state, then $Eq.$ (2.1) – $Eq.$ (2.3) is said to be exponentially passive from input $\omega(t)$ to output $\tilde{z}(t)$, trajectories $y(t)$ of $Eq.$ (2.1) – $Eq.$ (2.3).

Lemma 2.3 ([2]). *For all constant matrix $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$, positive real constant r_2 and vector-valued function $\dot{y} : [-r_2, 0] \to \mathbb{R}^n$ such that the integrations concerned are well defined,*

$$-r_2 \int_{t-r_2}^{t} \dot{y}^T(s)M\dot{y}(s)ds \leq$$

$$-\Big(\int_{t-r_2}^{t}\dot{y}(s)ds\Big)^T M\Big(\int_{t-r_2}^{t}\dot{y}(s)ds\Big).$$

Replacing the term $\int_{t-r_2}^{t}\dot{y}(s)ds$ *with* $y(t)-y(t-r_2)$*, we can obtain the following inequality*

$$-r_2\int_{t-r_2}^{t}\dot{y}^T(s)M\dot{y}(s)ds \leq \begin{bmatrix} y(t)\\ y(t-r_2)\end{bmatrix}^T \begin{bmatrix} -M & M\\ * & -M\end{bmatrix}\begin{bmatrix} y(t)\\ y(t-r_2)\end{bmatrix}.$$

Lemma 2.4 ([10])**.** *For all constant matrix* $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$*,* $r(t)$ *is time-varying delays with* $0 \leq r(t) \leq r_2$*,* $r_2 \in \mathbb{R}$ *and a vector-valued function* $y : [-r_2, 0] \to \mathbb{R}^n$ *such that the integrations concerned are well defined,*

$$\begin{aligned}-r_2\int_{t-r_2}^{t}y^T(s)My(s)ds &\leq\\ -\int_{t-r(t)}^{t}y^T(s)dsM\int_{t-r(t)}^{t}y(s)ds&\\ -\int_{t-r_2}^{t-r(t)}y^T(s)dsM\int_{t-r_2}^{t-r(t)}y(s)ds.&\end{aligned}$$

Lemma 2.5 ([10])**.** *For all constant matrices* $M_1, M_2, M_3 \in \mathbb{R}^{n\times n}$*,* $M_1 \geq 0, M_3 > 0$*,* $\begin{bmatrix} M_1 & M_2\\ * & M_3\end{bmatrix} \geq 0$*,* $r(t)$ *is time-varying delays with* $0 \leq r(t) \leq r_2$*,* $r_2 \in \mathbb{R}$*, vector-valued functions* y *and* $\dot{y} : [-r_2, 0] \to \mathbb{R}^n$ *such that the following integration is well defined,*

$$-r_2\int_{t-r_2}^{t}\begin{bmatrix} y(s)\\ \dot{y}(s)\end{bmatrix}^T\begin{bmatrix} M_1 & M_2\\ * & M_3\end{bmatrix}\begin{bmatrix} y(s)\\ \dot{y}(s)\end{bmatrix}ds$$

$$\leq \begin{bmatrix} y(t)\\ y(t-r(t))\\ y(t-r_2)\\ \int_{t-r(t)}^{t}y(s)ds\\ \int_{t-r_2}^{t-r(t)}y(s)ds\end{bmatrix}^T$$

$$\times\begin{bmatrix} -M_3 & M_3 & 0 & -M_2^T & 0\\ * & -M_3-M_3^T & M_3 & M_2^T & -M_2^T\\ * & * & -M_3 & 0 & M_2^T\\ * & * & * & -M_1 & 0\\ * & * & * & * & -M_1\end{bmatrix}$$

$$\times\begin{bmatrix} y(t)\\ y(t-r(t))\\ y(t-r_2)\\ \int_{t-r(t)}^{t}y(s)ds\\ \int_{t-r_2}^{t-r(t)}y(s)ds\end{bmatrix}.$$

Lemma 2.6 ([1])**.** *The following inequality holds, for a positive matrix* M*:*

$$\begin{aligned}&-\frac{(\alpha-\beta)^2}{2}\int_{\beta}^{\alpha}\int_{s}^{\alpha}y^T(u)My(u)duds\\ &\leq -\Big(\int_{\beta}^{\alpha}\int_{s}^{\alpha}y(u)duds\Big)^T M\\ &\quad\Big(\int_{\beta}^{\alpha}\int_{s}^{\alpha}y(u)duds\Big).\end{aligned}$$

Lemma 2.7 ([10])**.** *Let* $y(t) \in \mathbb{R}^n$ *be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any constant matrices* $N, M_i \in \mathbb{R}^{n\times n}, i = 1, 2, \ldots, 5$ *and* $r(t)$ *is time-varying delays with* $0 \leq r(t) \leq r_2$*,* $r_2 \in \mathbb{R}^+$*,*

$$-\int_{t-r_2}^{t}\dot{y}^T(s)Q\dot{y}(s)ds \leq \Phi^T\omega\Phi + r_2\Phi^T\Omega\Phi,$$

where

$$\Phi = \begin{bmatrix} y(t)\\ y(t-r(t))\\ y(t-r_2)\end{bmatrix},$$

$$\Omega = \begin{bmatrix} M_3 & M_4 & 0\\ * & M_3+M_5 & M_4\\ * & * & M_5\end{bmatrix},$$

$$\omega = \begin{bmatrix} M_1+M_1^T & -M_1^T+M_2 & 0\\ * & \varrho & -M_1^T+M_2\\ * & * & -M_2-M_2^T\end{bmatrix},$$

$$\varrho = M_1 + M_1^T - M_2 - M_2^T$$

and $\begin{bmatrix} N & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5\end{bmatrix} \geq 0.$

Lemma 2.8 ([14])**.** *For all constant matrix* $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$*, nonnegative real constants* r_2 *and a vector-valued function* $\dot{y} : [-r_2, 0] \to \mathbb{R}^n$ *such that the following integrals are well defined,*

$$-r_2\int_{t-r_2}^{t}\dot{y}^T(s)M\dot{y}(s)ds \leq \omega^T(t)\ominus\omega(t),$$

where
$\omega(t) = \left[y^T(t), y^T(t-r_2), \frac{1}{r_2}\int_{t-r_2}^{t}y^T(s)ds\right]^T$

and $\ominus = \begin{bmatrix} -4M & -2M & 6M\\ * & -4M & 6M\\ * & * & -12M\end{bmatrix}.$

Lemma 2.9 ([13,17]). *For any constant matrices $M, N \in \mathbb{R}^{n\times n}$, $M \geq 0$, $\begin{bmatrix} M & N \\ * & M \end{bmatrix} \geq 0$, $r(t)$ is time-varying delay with Eq. (2.5), a vector-valued function $\dot{y} : [-r_2, 0] \to \mathbb{R}^n$ such that the concerned integrations are well defined,*

$$-r_2 \int_{t-r_2}^{t} \dot{y}^T(s) M \dot{y}(s) ds \leq \tilde{\omega}^T(t) \tilde{\ominus} \tilde{\omega}(t),$$

where
$\tilde{\omega}(t) = \left[y^T(t), y^T(t-r(t)), y^T(t-r_2)\right]^T$
and $\tilde{\ominus} = \begin{bmatrix} -M & M-N & N \\ * & -2M+N+N^T & M-N \\ * & * & -M \end{bmatrix}$.

3. Main Results

In this section, we will indicate the exponential passivity criteria dependent on time-varying delays. We define a new parameter

$$\sum = [\Xi_{i,j}]_{16\times 16}, \qquad (3.1)$$

where $\Xi_{i,j} = \Xi_{j,i}^T$, $\quad i, j = 1, 2, 3, ..., 16,$

$$\begin{aligned}
S_{16} &= P_1 G, \\
\Xi_{1,1} &= -S_5 - S_5^T + r_2^2 S_{10} + P_2 + r_2^2 P_7 \\
&\quad -M_2 - M_2^T, \\
\Xi_{1,2} &= P_1^T - S_4^T + S_5 A + r_2^2 S_9^T, \\
\Xi_{1,3} &= -M_1^T + M_2, \\
\Xi_{1,4} &= S_5 B - S_6^T, \\
\Xi_{1,6} &= S_5 C, \\
\Xi_{1,11} &= S_5, \\
\Xi_{1,12} &= S_5, \\
\Xi_{1,13} &= S_5, \\
\Xi_{2,2} &= S_1 A + S_2 + A^T S_1 + S_2^T + 2\alpha S_1 \\
&\quad + S_{16} + S_{16}^T + S_4 A + A^T S_4^T + r_2^2 S_8 \\
&\quad + r_2^2 P_4 - e^{-2\alpha r_2} S_{10} + S_{11} + S_{11}^T \\
&\quad + r_2 S_{13} + r_2 C_1 + r_2 C_1^T + r_2 C_4 \\
&\quad + \epsilon_1 l_1^2 I + 2\alpha P_1 + P_3 + r_2^2 P_5 \\
&\quad - e^{-4\alpha r_2} r_2^2 P_9 + r_2^2 P_{10} + r_2 C_4^T \\
&\quad - 4e^{-2\alpha r_2} P_{11} - e^{-2\alpha r_2} P_{12}, \\
\Xi_{2,3} &= A^T J_1^T, \\
\Xi_{2,4} &= S_1 B S_{16} - S_4 B + A^T S_6^T - r_2 C_4 \\
&\quad - S_{11}^T + S_{12} + r_2 S_{14} + r_2 C_5^T \\
&\quad + e^{-2\alpha r_2} P_{12} - e^{-2\alpha r_2} P_{13} \\
&\quad + e^{-2\alpha r_2} S_{10}, \\
\Xi_{2,5} &= -S_2 + r_2 C_2^T - r_2 C_1 - 2e^{-2\alpha r_2} P_{11} \\
&\quad + e^{-2\alpha r_2} P_{13}, \\
\Xi_{2,6} &= S_1 C + S_4 C + A^T J_2^T, \\
\Xi_{2,7} &= A^T S_2 + S_3 + 2\alpha S_2 - e^{-2\alpha r_2} S_9^T, \\
\Xi_{2,8} &= A^T S_2 + S_3 + 2\alpha S_2, \\
\Xi_{2,9} &= -S_{16} + S_7^T + r_2 C_6^T - r_2 C_4, \\
\Xi_{2,10} &= r_2 C_3^T - r_2 C_1, \\
\Xi_{2,11} &= S_1 + S_4, \\
\Xi_{2,12} &= S_1 + S_4, \\
\Xi_{2,13} &= S_1 + S_4, \\
\Xi_{2,14} &= e^{-4\alpha r_2} r_2^2 P_9 + 6e^{-2\alpha r_2} P_{11}, \\
\Xi_{2,16} &= S_1 D_1 - \tilde{C}_1^T, \\
\Xi_{3,1} &= -M_1 + M_2^T, \\
\Xi_{3,3} &= M_1 + M_1^T - J_1 - J_1^T + \frac{\frac{r_2^4}{2}}{2} P_8 \\
&\quad + \frac{\frac{r_2^4}{2}}{2} P_9 - e^{-2\alpha r_2} L_2 + r_2^2 P_{11} \\
&\quad + r_2^2 P_{12}, \\
\Xi_{3,4} &= J_1 B, \\
\Xi_{3,6} &= J_1 C - J_2^T, \\
\Xi_{3,11} &= J_1, \\
\Xi_{3,12} &= J_1, \\
\Xi_{3,13} &= J_1, \\
\Xi_{4,4} &= S_6 B + B^T S_6^T - e^{-2\alpha r_2} S_{10} + S_{11} \\
&\quad - e^{-2\alpha r_2} S_{10}^T + S_{11}^T - S_{12} - S_{12}^T \\
&\quad + r_2 S_{13} + r_2 S_{15} - r_2 C_5 - r_2 C_5^T \\
&\quad - 2e^{-2\alpha r_2} P_{12} + e^{-2\alpha r_2} P_{13} \\
&\quad + e^{-2\alpha r_2} P_{13}^T + \epsilon_2 l_2^2 I, \\
\Xi_{4,5} &= e^{-2\alpha r_2} S_{10} - S_{11}^T + S_{12} + r_2 S_{14} \\
&\quad + e^{-2\alpha r_2} P_{12} - e^{-2\alpha r_2} P_{13}, \\
\Xi_{4,6} &= S_6 C + B^T J_2^T, \\
\Xi_{4,7} &= B^T S_2 + e^{-2\alpha r_2} S_9^T,
\end{aligned}$$

$$
\begin{aligned}
\Xi_{4,8} &= B^T S_2 - e^{-2\alpha r_2} S_9^T,\\
\Xi_{4,9} &= -S_7^T - r_2 C_6^T - r_2 C_5,\\
\Xi_{4,11} &= S_6,\\
\Xi_{4,12} &= S_6,\\
\Xi_{4,13} &= S_6,\\
\Xi_{5,5} &= -e^{-2\alpha r_2} S_{10} - S_{12} - S_{12}^T + r_2 S_{15}\\
&\quad -r_2 C_2 - r_2 C_2^T - e^{-2\alpha r_2} P_3\\
&\quad -4e^{-2\alpha r_2} P_{11} - e^{-2\alpha r_2} P_{12},\\
\Xi_{5,7} &= -S_3,\\
\Xi_{5,8} &= -S_3 + e^{-2\alpha r_2} S_9^T,\\
\Xi_{5,10} &= -r_2 C_3^T - r_2 C_2,\\
\Xi_{5,14} &= 6e^{-2\alpha r_2} P_{11},\\
\Xi_{6,6} &= -e^{-2\alpha k_2} W + k_d W + \epsilon_3 l_3^2 I + J_2 C\\
&\quad +C^T J_2^T,\\
\Xi_{6,7} &= C^T K_{12},\\
\Xi_{6,8} &= C^T K_{12},\\
\Xi_{6,11} &= J_2,\\
\Xi_{6,12} &= J_2,\\
\Xi_{6,13} &= J_2,\\
\Xi_{7,7} &= 2\alpha S_3 - e^{-2\alpha r_2} S_8 - e^{-2\alpha r_2} P_4\\
&\quad -e^{-2\alpha r_2} P_5,\\
\Xi_{7,8} &= 2\alpha S_3 - e^{-2\alpha r_2} P_4,\\
\Xi_{7,11} &= S_2^T,\\
\Xi_{7,12} &= S_2^T,\\
\Xi_{7,13} &= S_2^T,\\
\Xi_{7,16} &= S_2^T D_1,\\
\Xi_{8,7} &= 2\alpha S_3 - e^{-2\alpha r_2} P_4,\\
\Xi_{8,8} &= 2\alpha S_3 - e^{-2\alpha r_2} S_8 - e^{-2\alpha r_2} P_4\\
&\quad -e^{-2\alpha r_2} P_5,\\
\Xi_{8,11} &= S_2^T,\\
\Xi_{8,12} &= S_2^T,\\
\Xi_{8,13} &= S_2^T,\\
\Xi_{8,16} &= S_2^T D_1,\\
\Xi_{9,9} &= -S_7 S_7^T - r_2 C_6 - r_2 C_6^T\\
&\quad -e^{-2\alpha r_2} P_6,\\
\Xi_{10,10} &= r_2 C_3 - r_2 C_3^T - e^{-2\alpha r_2} P_6,\\
\Xi_{11,11} &= -\epsilon_1 I,\\
\Xi_{12,12} &= -\epsilon_2 I,\\
\Xi_{13,13} &= -\epsilon_3 I,\\
\Xi_{14,14} &= -e^{-4\alpha r_2} r_2^2 P_9 - r_2^2 e^{-2\alpha r_2} P_{10}\\
&\quad -12 e^{-2\alpha r_2} P_{11},\\
\Xi_{15,15} &= -e^{-4\alpha r_2} P_8,\\
\Xi_{16,16} &= -D_2 - D_2^T
\end{aligned}
$$

and others are equal to zero.

Theorem 3.1. *For* $||C|| + l_3 < 1$, *the system* $Eq.$ (2.1) $-$ $Eq.$ (2.3) *is exponentially passive with a decay rate* $\alpha > 0$ *if there exist positive definite symmetric matrices* $P_m, m = 1, 2, ..., 13$, *any appropriate dimensional matrices* $S_l, l = 1, 2, ..., 16$, *and positive real constants* ϵ_n, $n = 1, 2, 3$, *such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0, \quad (3.2)$$

$$\begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} > 0, \quad (3.3)$$

$$\begin{bmatrix} e^{-2\alpha r_2} P_7 & S_{11} & S_{11} \\ * & S_{13} & S_{13} \\ * & * & S_{15} \end{bmatrix} \geq 0, \quad (3.4)$$

$$\begin{bmatrix} P_{12} & P_{13} \\ * & P_{12} \end{bmatrix} > 0, \quad (3.5)$$

$$\sum < 0. \quad (3.6)$$

Proof. Initially, we use descriptor system to rewrite the system $Eq.$ (2.1)

$$
\begin{aligned}
\dot{y}(t) &= z(t), &(3.7)\\
0 &= -z(t) + Ay(t) + By(t - r(t))\\
&\quad +C\dot{y}(t - k(t)) + g_1(t, y(t))\\
&\quad +g_2(t, y(t - r(t)))\\
&\quad +g_3(t, \dot{y}(t - k(t)))\\
&\quad +D_1\omega(t). &(3.8)
\end{aligned}
$$

Apply the following utilization of zero equation, we have

$$0 = Gy(t) - Gy(t - r(t))$$

$$-G\int_{t-r(t)}^{t} z(s)ds, \qquad (3.9)$$

where $G \in \mathbb{R}^{n\times n}$ will be be chosen to assure the exponential passivity of system $Eq.$ (2.1) – $Eq.$ (2.3). By $Eq.$ (3.9), the system $Eq.$ (3.7) and $Eq.$ (3.8) can be represented in the form of the descriptor delayed system

$$\begin{aligned}\dot{y}(t) &= z(t) + Gy(t) - Gy(t-r(t)) \\ &\quad -G\int_{t-r(t)}^{t} z(s)ds, \qquad (3.10)\\ 0 &= -z(t) + Ay(t) + By(t-r(t)) \\ &\quad +C\dot{y}(t-k(t)) + g_1(t,y(t)) \\ &\quad +g_2(t,y(t-r(t))) \\ &\quad +g_3(t,\dot{y}(t-k(t))) \\ &\quad +D_1\omega(t). \qquad (3.11)\end{aligned}$$

Construct a Lyapunov-Krasovskii functional candidate for the system $Eq.$ (2.1) $-$ $Eq.$ (2.3), $Eq.$ (3.10) $-$ $Eq.$ (3.11) of the form

$$V(t) = \sum_{i=1}^{9} V_i(t), \qquad (3.12)$$

where

$$\begin{aligned}V_1(t) &= \begin{bmatrix} y(t) \\ \int_{t-r_2}^{t} y(s)ds \end{bmatrix}^T \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \\ &\quad \times \begin{bmatrix} y(t) \\ \int_{t-r_2}^{t} y(s)ds \end{bmatrix}, \\ V_2(t) &= y^T(t)P_1y(t) \\ &= \begin{bmatrix} y(t) \\ z(t) \\ y(t-r(t)) \\ \int_{t-r(t)}^{t} z(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S_4 & S_5 & S_6 & S_7 \end{bmatrix} \\ &\quad \times \begin{bmatrix} y(t) \\ z(t) \\ y(t-r(t)) \\ \int_{t-r(t)}^{t} z(s)ds \end{bmatrix},\end{aligned}$$

$$\begin{aligned}V_3(t) &= r_2\int_{-r_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)} \begin{bmatrix} y(\theta) \\ \dot{y}(\theta) \end{bmatrix}^T \\ &\quad \times \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(\theta) \\ \dot{y}(\theta) \end{bmatrix} d\theta ds, \\ V_4(t) &= \int_{t-k(t)}^{t} e^{2\alpha(s-t)}\dot{y}^T(s)P_2\dot{y}(s)ds \\ &\quad + \int_{t-r_2}^{t} e^{2\alpha(s-t)}y^T(s)P_3y(s), \\ V_5(t) &= r_2\int_{-r_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_4 \\ &\quad \times y(\theta)d\theta ds + r_2^2\int_{-r_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)} \\ &\quad y^T(\theta)P_5y(\theta)d\theta ds \\ &\quad +r_2^2\int_{-r_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}z^T(\theta)P_6 \\ &\quad \times z(\theta)d\theta ds, \\ V_6(t) &= \int_{-r_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}z^T(\theta)P_7z(\theta)d\theta ds, \\ V_7(t) &= \Big[\frac{(r_2)^2}{2}\int_{-r_2}^{0}\int_{\theta}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)} \\ &\quad z^T(\theta)P_8z(\theta)d\theta ds d\lambda\Big] \\ &\quad +\Big[\frac{(r_2)^2}{2}\int_{-r_2}^{0}\int_{\theta}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)} \\ &\quad z^T(\theta)P_9z(\theta)d\theta ds d\lambda\Big], \\ V_8(t) &= r_2\int_{-r2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_{10} \\ &\quad \times y(\theta)d\theta ds, \\ V_9(t) &= r_2\int_{-r2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{y}^T(\theta)P_{11} \\ &\quad \times \dot{y}(\theta)d\theta ds + r_2\int_{-r2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)} \\ &\quad \dot{y}^T(\theta)P_{12}\dot{y}(\theta)d\theta ds.\end{aligned}$$

The time derivative of $V(t)$ along the trajectory of $Eq.$ (2.1), $Eq.$ (3.10) $-$ $Eq.$ (3.11) is given by

$$\dot{V}(t) = \sum_{i=1}^{9} \dot{V}_i(t). \qquad (3.13)$$

The time derivatives of $V_1(t), V_2(t)$ are computed as

$$\begin{aligned}
\dot{V}_1(t) &= \begin{bmatrix} y(t) \\ \int_{t-r(t)}^{t} y(s)ds + \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) - y(t-r_2) \end{bmatrix} \\
&\quad + \begin{bmatrix} \xi(t) \\ y(t) - y(t-r_2) \end{bmatrix}^T \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \\
&\quad \times \begin{bmatrix} y(t) \\ \int_{t-r(t)}^{t} y(s)ds + \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix} \\
&\quad +2\alpha V_1(t) - 2\alpha V_1(t),
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2(t) &= 2 \begin{bmatrix} y(t) \\ z(t) \\ y(t-r(t)) \\ \int_{t-r(t)}^{t} z(s)ds \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} P_1^T & 0 & 0 & S_4^T \\ 0 & 0 & 0 & S_5^T \\ 0 & 0 & 0 & S_6^T \\ 0 & 0 & 0 & S_7^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} I^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \\ \dot{y}(t-r(t)) \\ \int_{t-r(t)}^{t} \dot{z}(s)ds \end{bmatrix} \\
&= 2y^T R\big[\phi(t)\big] + 2y^T(t)S_4\big[\beta(t)\big] \\
&\quad +2z^T(t)S_5\big[\beta(t)\big] + 2\alpha V_2(t) \\
&\quad +2y^T(t-r(t))S_6\big[\beta(t)\big] \\
&\quad +2\int_{t-r(t)}^{t} z^T(s)dsS_7\big[\psi(t)\big] \\
&\quad -2\alpha V_2(t),
\end{aligned}$$

where

$$\begin{aligned}
\phi(t) &= z(t) + Gy(t) - Gy(t-r(t)) \\
&\quad -G\int_{t-r(t)}^{t} z(s)ds, \\
\psi(t) &= y(t) - y(t-r(t)) - \int_{t-r(t)}^{t} z(s)ds, \\
\xi(t) &= Ay(t) + By(t-r(t)) \\
&\quad +C\dot{y}(t-k(t)) + g_1(t, y(t)) \\
&\quad +g_2(t, y(t-r(t))) \\
&\quad +g_3(t, \dot{y}(t-k(t))) + D_1\omega(t)
\end{aligned}$$

and

$$\begin{aligned}
\beta(t) &= -z(t) + Ay(t) + By(t-r(t)) \\
&\quad +C\dot{y}(t-k(t)) + g_1(t, y(t)) \\
&\quad +g_2(t, y(t-r(t))) \\
&\quad +g_3(t, y(t-k(t))) + D_1\omega(t).
\end{aligned}$$

Obviously, we get $e^{-2\alpha r_2} \leq e^{2\alpha(s-t)} \leq 1$, for any a scalar $s \in [t-r_2, t]$, and apply with Lemma 2.5, we get $\dot{V}_3(t)$ as follows

$$\begin{aligned}
\dot{V}_3 &= r_2 \int_{-r_2}^{0} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} ds \\
&\quad -r_2 \int_{-r_2}^{0} e^{2\alpha s} \begin{bmatrix} y(t+s) \\ \dot{y}(t+s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\
&\quad \times \begin{bmatrix} y(t+s) \\ \dot{y}(t+s) \end{bmatrix} ds - 2\alpha V_3(t) \\
&= r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\
&\quad -r_2 \int_{t-r_2}^{t} e^{2\alpha(s-t)} \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\
&\quad \times \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds - 2\alpha V_3(t) \\
&\leq r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\
&\quad -r_2 e^{-2\alpha r_2} \int_{t-r_2}^{t} \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\
&\quad \times \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds - 2\alpha V_3(t) \\
&\leq r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\
&\quad +e^{-2\alpha r_2}\Upsilon^T \varrho \Upsilon - 2V_3(t) \\
&= r_2^2 \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \\
&\quad +e^{-2\alpha r_2}\Upsilon^T \varrho \Upsilon - 2V_3(t),
\end{aligned}$$

where

$$\Upsilon = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \\ \int_{t-r(t)}^{t} y(s)ds \\ \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix}$$

and

$$\varrho = \begin{bmatrix} -S_{10} & S_{10} & 0 & -S_9^T & 0 \\ * & -S_{10} - S_{10}^T & S_{10} & S_9^T & -S_9^T \\ * & * & -S_{10} & 0 & S_9^T \\ * & * & * & -S_8 & 0 \\ * & * & * & * & -S_8 \end{bmatrix}.$$

Taking the differential of $\dot{V}_4(t)$, we have

$$\begin{aligned}
\dot{V}_4(t) &= \dot{y}^T(t)P_2\dot{y}(t) + y^T P_3 y(t) \\
&\quad -(1-\dot{k}(t))e^{-2\alpha k(t)}\dot{y}^T(t-k(t)) \\
&\quad \times P_2\dot{y}(t-k(t)) - 2\alpha V_4(t) \\
&\quad -e^{-2\alpha r_2}y^T(t-r_2)P_3 y(t-r_2) \\
&\leq \dot{y}^T(t)P_2\dot{y}(t) + y^T(t)P_3 y(t) \\
&\quad -e^{-2\alpha k_2}\dot{y}(t-k(t))P_2\dot{y}(t-k(t)) \\
&\quad +k_d\dot{y}^T(t-k(t))P_2\dot{y}(t-k(t)) \\
&\quad -e^{-2\alpha r_2}y^T(t-r_2)P_3 y(t-r_2) \\
&\quad -2\alpha V_4(t) \\
&= z^T(t)P_2 z(t) - e^{-2\alpha k_2}\dot{y}(t-k(t)) \\
&\quad \times P_2\dot{y}(t-k(t)) + k_d\dot{y}^T(t-k(t)) \\
&\quad \times P_2\dot{y}(t-k(t)) + y^T(t)P_3 y(t) \\
&\quad -e^{-2\alpha r_2}y^T(t-r_2)P_3 y(t-r_2) \\
&\quad -2\alpha V_4(t).
\end{aligned}$$

From Lemma 2.3 and Lemma 2.4, we obtain $\dot{V}_5(t)$ as follows

$$\begin{aligned}
\dot{V}_5(t) &= r_2\int_{-r_2}^{0} y^T(t)P_4 y(t)ds \\
&\quad -r_2\int_{-r_2}^{0} e^{2\alpha s}y^T(t+s)P_4 y^T(t+s)ds \\
&\quad +r_2\int_{-r_2}^{0} y^T(t)P_5 y(t)ds \\
&\quad -r_2\int_{-r_2}^{0} e^{2\alpha s}y^T(t+s)P_5 y^T(t+s)ds \\
&\quad +r_2\int_{-r_2}^{0} z^T(t)P_6 z(t)ds - 2\alpha V_5(t) \\
&\quad -r_2\int_{-r_2}^{0} e^{2\alpha s}z^T(t+s)P_6 z^T(t+s)ds \\
&= r_2^2 y^T(t)P_4 y(t) \\
&\quad -r_2\int_{t-r_2}^{t} e^{2\alpha(s-t)}y^T(s)P_4 y(s)ds \\
&\quad +r_2^2 y^T(t)P_5 y(t) - 2\alpha V_5(t) \\
&\quad -r_2\int_{t-r_2}^{t} e^{2\alpha(s-t)}y^T(s)P_5 y(s)ds \\
&\quad +r_2^2 z^T(t)P_6 z(t) \\
&\quad -r_2\int_{t-r_2}^{t} e^{2\alpha(s-t)}z^T(s)P_6 z(s)ds \\
&\leq r_2^2 y^T(t)P_4 y(t) + r_2^2 y^T(t)P_5 y(t) \\
&\quad -r_2 e^{2\alpha r_2}\int_{t-r_2}^{t} y^T(s)P_4 y(s)ds \\
&\quad -r_2 e^{-2\alpha r_2}\int_{t-r_2}^{t} y^T(s)P_5 y(s)ds \\
&\quad +r_2^2 z^T(t)P_6 z(t) - 2\alpha V_5(t) \\
&\quad -r_2 e^{-2\alpha r_2}\int_{t-r_2}^{t} z^T(s)P_6 z(s)ds \\
&\leq r_2^2 y^T(t)P_4 y(t) + r_2^2 y^T(t)P_5 y(t) \\
&\quad -e^{-2\alpha r_2}\int_{t-r(t)}^{t} y^T(s)dsP_5 \\
&\quad \times\int_{t-r(t)}^{t} y^T(s)ds \\
&\quad -e^{2\alpha r_2}\Big(\Big[\int_{t-r(t)}^{t} y^T(s)ds \\
&\quad +\int_{t-r_2}^{t-r(t)} y^T(s)ds\Big]P_4 \\
&\quad \times\Big[\int_{t-r(t)}^{t} y^T(s)ds \\
&\quad +\int_{t-r_2}^{t-r(t)} y^T(s)ds\Big]\Big) \\
&\quad -e^{-2\alpha r_2}\int_{t-r_2}^{t-r(t)} y^T(s)dsP_5 \\
&\quad \times\int_{t-r_2}^{t-r(t)} y^T(s)ds + r_2^2 z^T(t)P_6 z(t) \\
&\quad -e^{-2\alpha r_2}\int_{t-r(t)}^{t} z^T(s)dsP_6 \\
&\quad \times\int_{t-r(t)}^{t} z^T(s)ds \\
&\quad -e^{-2\alpha r_2}\int_{t-r_2}^{t-r(t)} z^T(s)dsP_6 \\
&\quad \times\int_{t-r_2}^{t-r(t)} z^T(s)ds - 2\alpha V_5(t).
\end{aligned}$$

Using Lemma 2.7, $\dot{V}_6(t)$ will be obtained as

$$\begin{aligned}
\dot{V}_6(t) &= \int_{-r_2}^{0} z^T(t)P_7 z(t)ds - 2\alpha V_6(t) \\
&\quad -\int_{-r_2}^{0} e^{2\alpha s}z^T(t+s)P_7 z(t+s)ds \\
&= r_2 z^T(t)P_7 z(t) - 2\alpha V_6(t) \\
&\quad -\int_{t-r_2}^{t} e^{2\alpha(s-t)}z^T(s)P_7 z(s)ds
\end{aligned}$$

$$\begin{aligned}
&\leq r_2 z^T(t)P_7 z(t) - 2\alpha V_6(t) \\
&\quad -e^{2\alpha r_2}\int_{t-r_2}^{t} z^T(s)P_7 z(s)ds \\
&\leq r_2 z^T(t)P_7 z(t) + \eta^T\Delta\eta \\
&\quad +r_2\eta^T\Theta\eta - 2\alpha V_6(t),
\end{aligned}$$

where

$$\eta = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \end{bmatrix},$$

$$\Delta = \begin{bmatrix} S_{11}+S_{11}^T & -S_{11}^T+S_{12} & 0 \\ * & \phi & -S_{11}^T+S_{12} \\ * & * & -S_{12}-S_{12}^T \end{bmatrix},$$

$$\phi = S_{11}+S_{11}^T - S_{12} - S_{12}^T$$

and

$$\Theta = \begin{bmatrix} S_{13} & S_{14} & 0 \\ * & S_{13}+S_{15} & S_{14} \\ * & * & S_{15} \end{bmatrix}.$$

From Lemma 2.6, $\dot{V}_7(t)$ can be estimated as

$$\begin{aligned}
\dot{V}_7(t) &\leq \frac{(r_2)^4}{2}\dot{y}^T(t)P_8\dot{y}(t) \\
&\quad +\frac{(r_2)^4}{2}\dot{y}^T(t)P_9\dot{y}(t) \\
&\quad -\frac{r_2^2}{2}e^{-4\alpha r_2}\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)P_8\dot{y}dsd\theta \\
&\quad -\frac{r_2^2}{2}e^{-4\alpha r_2}\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)P_9\dot{y}dsd\theta \\
&\quad -2\alpha V_7(t) \\
&\leq \frac{(r_2)^4}{2}\dot{y}^T(t)P_8\dot{y}(t) \\
&\quad +\frac{(r_2)^4}{2}\dot{y}^T(t)P_9\dot{y}(t) \\
&\quad -e^{-4\alpha r_2}\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta\Big)P_8 \\
&\quad \times\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}(s)dsd\theta\Big) \\
&\quad -e^{-4\alpha r_2}\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta\Big)P_9 \\
&\quad \times\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}(s)dsd\theta\Big) \\
&= \frac{(r_2)^4}{2}\dot{y}^T(t)P_8\dot{y}(t) \\
&\quad +\frac{(r_2)^4}{2}\dot{y}^T(t)P_9\dot{y}(t) \\
&\quad -e^{-4\alpha r_2}\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta\Big)P_8 \\
&\quad \times\Big(\int_{-r_2}^{0}\int_{t+\theta}^{t}\dot{y}(s)dsd\theta\Big) \\
&\quad -e^{-4\alpha r_2}\Big[r_2 y^T(t) - \int_{t-r_2}^{t} y^T(\theta)d\theta\Big] \\
&\quad \times P_9\Big[r_2 y(t) - \int_{t-r_2}^{t} y(\theta)d\theta\Big].
\end{aligned}$$

Using Lemma 2.3, we compute $\dot{V}_8(t)$ as follows

$$\begin{aligned}
\dot{V}_8(t) &= r_2^2 y^T(t)P_{10}y(t) - 2\alpha V_8(t) \\
&\quad -e^{-2\alpha r_2}\left(\frac{1}{r_2}\int_{t-r_2}^{t} y^T(s)ds\right)r_2^2 P_{10} \\
&\quad \times\left(\frac{1}{r_2}\int_{t-r_2}^{t} y(s)ds\right).
\end{aligned}$$

By Lemma 2.8 and Lemma 2.9, we get $\dot{V}_9(t)$ as

$$\begin{aligned}
\dot{V}_9(t) &\leq r_2^2\dot{y}(t)P_{11}\dot{y}(t) + e^{-2\alpha r_2}\Theta^T\Delta\Theta \\
&\quad +r_2^2\dot{y}(t)P_{12}\dot{y}(t) \\
&\quad +e^{-2\alpha r_2}\Psi^T\Omega\Psi - 2\alpha V_9(t),
\end{aligned}$$

where

$$\Theta = \begin{bmatrix} y(t) \\ y(t-r_2) \\ \frac{1}{r_2}\int_{t-r_2}^{t} y(s)ds \end{bmatrix},$$

$$\Delta = \begin{bmatrix} -4P_{11} & -2P_{11} & 6P_{11} \\ -2P_{11} & -4P_{11} & 6P_{11} \\ 6P_{11} & 6P_{11} & -12P_{11} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \end{bmatrix},$$

$$\Omega = \begin{bmatrix} -P_{12} & P_{12}-P_{13} & P_{13} \\ P_{12}-P_{13}^T & \phi & P_{12}-P_{13} \\ P_{13}^T & P_{12}-P_{13}^T & -P_{12} \end{bmatrix}$$

and

$$\phi = -2P_{12}+P_{13}+P_{13}^T.$$

From the Leibinz-Newton formula, for any real constant matrices $C_i, i = 1,2,...,6$ with appropriate dimensions, the following equations are true

$$0 = 2r_2[y^T(t)C_4 + y^T(t-r(t))C_5$$

$$+\int_{t-r(t)}^{t} z^T(s)dsC_6] \\ [y(t) - y(t-r(t)) \\ -\int_{t-r(t)}^{t} z(s)ds], \quad (3.14)$$

$$0 = 2r_2[y^T(t)C_1 + y^T(t-r_2)C_2 \\ +\int_{t-r_2}^{t} z^T(s)dsC_3] \\ [y(t) - y(t-r_2) \\ -\int_{t-r_2}^{t} z(s)ds]. \quad (3.15)$$

From $Eq.$ (2.6) $-$ $Eq.$ (2.8), for any scalars ϵ_1, ϵ_2 and ϵ_3 are positive real constants, it can investigated that the following inequalities hold:

$$\begin{aligned} 0 &\leq \epsilon_1 l_1^2 y^T(t)y(t) \\ &\quad -\epsilon_1 g_1^T(t,y(t))g_1(t,y(t)), \quad (3.16) \\ 0 &\leq \epsilon_2 l_2^2 y^T(t-r(t))y(t-r(t)) \\ &\quad -\epsilon_2 g_2^T(t,y(t-r(t))) \\ &\quad g_2(t,y(t-r(t))), \quad (3.17) \\ 0 &\leq \epsilon_3 l_3^2 \dot{y}^T(t-k(t))\dot{y}(t-k(t)) \\ &\quad -\epsilon_3 g_3^T(t,\dot{y}(t-k(t))) \\ &\quad g_3(t,\dot{y}(t-k(t))). \quad (3.18) \end{aligned}$$

From $Eq.$ (3.11), we have

$$\begin{aligned} 0 &= 2\dot{y}(t)J_1[-\dot{y}(t) + Ay(t) + By(t-r(t)) \\ &\quad +C\dot{y}(t-k(t)) + g_1(t,y(t)) \\ &\quad +g_2(t,y(t-h(t))) \\ &\quad +g_3(t,\dot{y}(t-\tau(t)))], \quad (3.19) \\ 0 &= 2\dot{y}(t-k(t))J_2[-\dot{y}(t) + Ay(t) \\ &\quad +By(t-r(t)) + C\dot{y}(t-k(t)) \\ &\quad +g_1(t,y(t)) + g_2(t,y(t-h(t))) \\ &\quad +g_3(t,\dot{y}(t-\tau(t)))]. \quad (3.20) \end{aligned}$$

From $Eq.$ (3.10), we obtain

$$0 = [\dot{y}^T M_1 + z^T(t)M_2][\dot{y}(t) - z(t)]. \quad (3.21)$$

According to Eq. (3.13) - Eq. (3.21), we can conclude that

$$\dot{V}(t) + 2\alpha V(t) - 2\tilde{z}(t)\omega(t) \leq \delta^T(t)\sum\delta(t),$$

where $\delta(t) = \big[z(t), y(t), \dot{y}(t), y(t-r(t)), y(t-r_2), \dot{y}(t-k(t)), \int_{t-r(t)}^{t} y(s)ds, \int_{t-r_2}^{t-r(t)} y(s)ds, \int_{t-r(t)}^{t} z(s)ds, \int_{t-r_2}^{t-r(t)} z(s)ds, g_1(t,y(t)), g_2(t,y(t-r(t))), g_3(t,\dot{y}(t-k(t))), \frac{1}{r_2}\int_{t-r_2}^{t} y(\theta)d\theta, \int_{-r_2}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)dsd\theta, \omega(t)\big]$ and $\sum$ is defined in Eq. (3.1). If conditions Eq. (3.2)-Eq. (3.6) hold, then

$$\dot{V}(t) + 2\alpha V(t) \leq 2\tilde{z}(t)\omega(t), \quad \forall t \in \mathbb{R}^+. \quad (3.22)$$

It means that systems Eq. (2.1)-Eq. (2.3) are exponentially passive. □

Now the exponential stability criteria of equation $Eq.$ (2.1) when $D_1 = 0$ is demonstrated. We define a new parameter

$$\tilde{\sum} = [\tilde{\Xi}_{i,j}]_{15\times 15}, \quad (3.23)$$

where $\tilde{\Xi}_{i,j} = \Xi_{j,i}^T$, $\quad i,j = 1,2,3,...,15.$

Corollary 3.2. *For $||C|| + l_3 < 1$, the system $Eq.$ (2.1) where $D_1 = 0$ is exponentially stable with a decay rate $\alpha > 0$ if there exist positive definite symmetric matrices P_m, $m = 1,2,...,13$, any appropriate dimensional matrices $S_l, l = 1,2,...,16$, and positive real constants $\epsilon_n, n = 1,2,3$, such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0, \quad (3.24)$$

$$\begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} > 0, \quad (3.25)$$

$$\begin{bmatrix} e^{-2\alpha r_2}P_7 & S_{11} & S_{12} \\ * & S_{13} & S_{14} \\ * & * & S_{15} \end{bmatrix} \geq 0, \quad (3.26)$$

$$\begin{bmatrix} P_{12} & P_{13} \\ * & P_{12} \end{bmatrix} > 0, \quad (3.27)$$

$$\tilde{\sum} < 0. \quad (3.28)$$

4. Numerical Examples

We give two numerical examples to present the improvement and performance of our stability criteria by comparing the least upper bounds of the parameter λ_2 and considering the rate of convergence α for guaranteeing exponential stability.

Example 4.1. Consider system $Eq.$ (2.1) – $Eq.$ (2.3) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix},$$
$$C = 0.1I, \quad \tilde{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

For $l_1 = 0.1, l_2 = 0.05, l_3 = 0.05$, and the different k_d and α, the maximum value of r_2 with convergance index ϵ for exponential passivity of the system $Eq.$ (2.1) is listed in Table 1. Moreover, we compute an upper bound α with the different k_2 and r_2, obtained from Theorem 1 are listed in Table 2 for different r_2 and k_2.

Table 1. Maximum allowable bound r_2

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
$k_d = 0$	0.171	0.117	0.030
$k_d = 0.05$	0.168	0.114	0.026
$k_d = 0.1$	0.165	0.111	0.0220

Table 2. Maximum allowable bound α

	$k_2 = 0.5$	$k_2 = 0.9$
$r_2 = 0.01$	0.804	0.710
$r_2 = 0.05$	0.691	0.603
$r_2 = 0.1$	0.486	0.419

Example 4.2. Consider system $Eq.$ (2.1) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix},$$
$$C = 0.1I, \quad D_1 = 0.$$

For $l_1 = 0.1, l_2 = 0.05, l_3 = 0.05$, the maximum value of r_2 with a convergance index ϵ for exponential stability of the system $Eq.$ (2.1) is listed in Table 3.

Table 3. Maximum allowable bound r_2

	$r_d = 0.5$	unknown
[18] $\alpha = 0.5$	0.731	-
[6] $\alpha = 0.5$	1.603	-
Corollary 11	-	1.151
[18] $\alpha = 0.7$	0.358	-
[6] $\alpha = 0.7$	1.169	-
Corollary 11	-	0.912
[18] $\alpha = 0.9$	0.103	-
[6] $\alpha = 0.9$	0.691	-
Corollary 11	-	0.711

5. Conclusion

From this paper, the problems of the delay-dependent exponential passivity and stability criteria for neutral system with time-varying delays and nonlinear perturbations have been investigated. Applying new class of augmented Lyapunov-Krasovskii functional, we acquire new exponential passivity criteria in terms of LMIs. The improvement and effectiveness of the proposed results have been guaranteed by some numerical examples.

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References

[1] Kwon OM, Park JH. Exponential stability of uncertain dynamic systems including state delay. Applied Mathematics Letters 2006;19(9):901-7.

[2] Kwon OM, Park MJ, Park JH, Lee SM, Cha EJ. New delay-partitioning ap-

proaches to stability criteria for uncertain neutral systems with time-varying delays. SciVerse ScienceDirect 2012;349:2799-823.

[3] Liu PL. A delay decomposition approach to stability analysis of neutral systems with time-varying delay. Applied Mathematical Modelling 2013;37:5013-26.

[4] Mohajerpoor R, Lakshamanan S, Abdi H, Rakkiyappan R, Nahavandi S. New delay-range-dependent stability criteria for interval-time-varying delay system via Wirtinger-based inequalities. Int J. Robust Nonlinear Control 2016.

[5] Krishnasamy R, Balasubramaniam P. A descriptor system approach to the delay-dependent exponential stability analysis for switch neutral systems with nonlinear perturbations. Nonlinear Analysis 2015;15:23-36.

[6] Liu Y, Ma W, Mahmoud MS, Lee SM. Improved delay-dependent exponential stability criteria for neutral-delay systems with nonlinear uncertainties. Applied Mathematical Modelling 2015;39:3164-74.

[7] Mohajerpoor R, Shanmugam L, Abdi H, Rakkiyappan R, Nahavandi S, et al. Improved delay-dependent stability criteria for neutral systems with mixed interval time-varying delays and nonlinear disturbances. Journal of the Franklin Institute 2017;354:1169-94.

[8] Cheng J, Zhu H, Zhong S, Li G. Novel delay-dependent robust stability criteria for neutral systems with mixed time-varying delays and nonlinear perturbations. Applied Mathematics and Computation 2013;219(14):7741-53.

[9] Liu PL. Improved results on delay-interval-dependent robust stability criteria for uncertain neutral-type systems with time-varying delays. ISA Transactions 2016;60:53-66.

[10] Singkibud P, Mukdasai K. On robust stability for uncertain neutral systems with non-differentiable interval time-varying discrete delay and nonlinear perturbations. Asian-European Journal of Mathematics 2018;11(01):1-29.

[11] Singkibud P, Niamsup P, Mukdasai K. Improved results on delay-range-dependent robust stability criteria of uncertain neutral systems with mixed interval time-varying delays. IAENG Int. J. Appl. Math. 2017;47(2):209-22.

[12] Zhu S, Shen Y, Chen G. Exponential passivity of neural networks with time-varying delay and uncertainty. Physics Letters A 2010;375:136-42.

[13] Park PG, Ko JW, Jeong CK. Reciprocally convex approach to stability of systems with time-varying delays. Automatica 2011;47:235-8.

[14] Seuret A, Gouaisbaut F. Wirtinger-based integral inequality: application to time-delay system. Automatica 2013;49(9):2860-6.

[15] Chanthorn P, Rajchakit G, Ramalingam S, Lim CP and Ramachandran R, Robust dissipativity analysis of hopfield-type complex-valued neural networks with time-varying delays and linear fractional uncertainties, Mathematics 2020; 8 (595): 22 pages.

[16] Humphries U, Rajchakit G, Sriraman R, Kaewmesri P, Chanthorn P, Lim CP and Samidurai R, An extended analysis on robust dissipativity of uncertain stochastic generalized neural networks with markovian jumping parameters, Symmetry 2020; 12 (1035): 20 pages.

[17] Peng C, Fei MR, An improved result on the stability of uncertain T-S fuzzy systems with interval time-varying delay. Fuzzy Sets Syst 2013;212:97-109.

[18] Ali MS. On exponential stability of neutral delay differential system with nonlinear uncertainties. Commum. Nonlinear Sci. Numer. Simul. 2012;17:2595-601.

[19] Fang M, Park JH. A multiple integral approach to stability of neutral times-delay systems. Applied Mathematics and Computation 2013;224:714-8.

[20] Fridman E. Lypunov-based stability analysis, Introduction to time-delay system. Spinger 2014.

[21] Gomaz MA, Egorov AV, Mondie S. A new stability criterion for neutral-type systems with one delay. IFAC PapersOnline 2018;51-54:714-8.

[22] Zhao N, Lin C, Chen B, Wang QG. A new double integral inequality and application to stability test for time-delay systems. Appl. Math. Lett. 2017;65:26-31.