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Original research article

The Number of Integers Divisible by a^k Except $a^{k+l}b$

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ABSTRACT

For positive integers $a,b,k,l,n,$, let $A_{(k,l)}^{(a,b)}(n)$ be the set of all integers divisible by a^k except $a^{k+l}b$ between 1 and n. Under certain condition on these integers, we obtain the explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$. We also provide bounds on $\left|A_{(k,l)}^{(a,b)}(n)\right|$ and show that these are the best bounds by examples.

Keywords: Bounds; Ceiling function; Divisibility; Floor function; Integer

1. Introduction

First, we describe the basic problem of sets in Mathematics as follows: How many integers between 1 and 100 both inclusive are divisible by 2 and 3? By the property of the floor function $\lfloor \cdot \rfloor$ defined by [1, 2].

$\lfloor x \rfloor$ is the greatest integer less thanor equal to a real number x, the answer is

$$\left\lfloor \frac{100}{6} \right\rfloor = 16.$$

What happens if the question is changed from ‘ and’ to ‘ or’ ? That is “ how many integers between 1 and 100 both inclusive that are divisible by 2 or 3 ?”

Let

$$A = \{1 \le n \le 100 : 2|n\},$$

and

$$B = \{1 \le n \le 100 : 3|n\}.$$

We need to find $|A \cup B|$, where $|X|$ denotes the number of elements in a finite set X . This problem can be solved by the Inclusion-Exclusion Principle [3, 4] which stated that for arbitrary finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. Again, we have

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$$\begin{aligned}|A\cup B| &= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{6} \right\rfloor \\ &= 50+33-16 \\ &= 67.\end{aligned}$$

Indeed, we can always find the number of all integers divisible by an integer α or/and an integer β between 1 and a positive integer n using the above concepts.

In this work, we are interested in dealing with the similar problem on indivisibility. For positive integers $a,b,k,l,n,$ define a finite set of positive integers $A_{(k,l)}^{(a,b)}(n)$ by

$$A_{(k,l)}^{(a,b)}(n) = \left\{ x \le n : a^k \,|\, x, a^{k+l}b \nmid x \right\}.$$

We see that $\left|A_{(k,l)}^{(a,b)}(n)\right| = 0$ if $n < a^k$ or $a^l b = 1$. Thus, only the case $n \ge a^k$ and $a^l b \ne 1$ is considered. We can find the explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$ under some condition on $a,b,k,l,n.$ Moreover, bounds on $\left|A_{(k,l)}^{(a,b)}(n)\right|$ are provided in terms of the floor function and the ceiling function $\lceil \cdot \rceil$ denoted by

$\lceil x \rceil$ is the smallest integer greater than or equal to a real number x.

Some examples to confirm that these are the best bounds are also given.

2. Some explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$

In this section, we find the explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$ under certain condition on $a,b,k,l,n.$ The following theorem is an important tool for establishing the explicit formula.

Theorem 2.1. Let a,b,k,l,n be positive integers such that $n \ge a^k$ and $a^l b \ne 1$. Then

$$\begin{aligned}\left|A_{(k,l)}^{(a,b)}(n)\right| - 1 + \frac{2}{a^l b} &\le \frac{\left(a^l b - 1\right)n + a^k}{a^{k+l} b} \\ &< \left|A_{(k,l)}^{(a,b)}(n)\right| + 1.\end{aligned}$$

Proof. By the property of the floor function, we have

$$a^k \left\lfloor \frac{n}{a^k} \right\rfloor \le n < a^k \left(\left\lfloor \frac{n}{a^k} \right\rfloor + 1 \right). \qquad (2.1)$$

Assume that $\lfloor n/a^k \rfloor \equiv i \left(\operatorname{mod} a^l b \right)$ for some $0 \le i < a^l b$. Write

$$\left\lfloor \frac{n}{a^k} \right\rfloor = i + m a^l b \qquad (2.2)$$

for some positive integer m. Since

$$\begin{aligned}A_{(k,l)}^{(a,b)}(n) = \{&a^k(1), a^k(2), \ldots, a^k\left(a^l b - 1\right), \\ &a^k\left(a^l b + 1\right), a^k\left(a^l b + 2\right), \ldots, a^k\left(2a^l b - 1\right), \\ &a^k\left(2a^l b + 1\right), \ldots, a^k\left(3a^l b - 1\right), \\ &\vdots \\ &a^k\left((m-1)a^l b + 1\right), \ldots, a^k\left(m a^l b - 1\right), \\ &a^k\left(m a^l b + 1\right), \ldots, a^k\left(m a^l b + i\right)\},\end{aligned}$$

we get

$$\begin{aligned}\left|A_{(k,l)}^{(a,b)}(n)\right| &= m a^l b - m + i \\ &= \left(a^l b - 1\right)m + i. \qquad (2.3)\end{aligned}$$

It follows from Eq. (2.1) and Eq. (2.2) together with Eq. (2.3) that

$$
\begin{aligned}
a^k\left(a^l bm+i\right) \le & \quad n & & < a^k\left(a^l bm+i+1\right) \\
m+\frac{i}{a^l b} \le & \quad \frac{n}{a^{k+l}b} & & < m+\frac{i}{a^l b}+m+\frac{1}{a^l b} \\
\left(a^l b-1\right)m+i-\frac{i}{a^l b} \le & \quad \frac{\left(a^l b-1\right)n}{a^{k+l}b} & & < \left(a^l b-1\right)m+i-\frac{i}{a^l b}+1-\frac{1}{a^l b} \\
\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b} \le & \quad \frac{\left(a^l b-1\right)n}{a^{k+l}b} & & < \left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+1-\frac{1}{a^l b} \\
\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+\frac{1}{a^l b} \le & \quad \frac{\left(a^l b-1\right)n+a^k}{a^{k+l}b} & & < \left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+1.
\end{aligned} \tag{2.4}
$$

Since

$$
\begin{aligned}
&\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+\frac{1}{a^l b} \\
&\ge \left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{a^l b-1}{a^l b}+\frac{1}{a^l b} \\
&= \left|A_{(k,l)}^{(a,b)}(n)\right|-1+\frac{2}{a^l b},
\end{aligned}
$$

and

$$\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+1 \le \left|A_{(k,l)}^{(a,b)}(n)\right|+1,$$

the result follows from Eq. (2.4). ■

The explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$ under some condition on positive integers a,b,k,l,n follows from the inequality Eq. (2.4) in the proof of Theorem 2.1 as follows:

Corollary 2.1. Let a,b,k,l,n be positive integers such that $n \ge a^k$ and $a^l b \ne 1$. Then

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lfloor \frac{\left(a^l b-1\right)n+a^k}{a^{k+l}b} \right\rfloor,$$

whenever $\lfloor n/a^k \rfloor \equiv i \left(\operatorname{mod} a^l b\right)$ with $0 \le i \le 1$.

Proof. If $0 \le i \le 1$, then we have

$$\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+\frac{1}{a^l b} \ge \left|A_{(k,l)}^{(a,b)}(n)\right|$$

and

$$\left|A_{(k,l)}^{(a,b)}(n)\right|-\frac{i}{a^l b}+1 \le \left|A_{(k,l)}^{(a,b)}(n)\right|+1.$$

It follows from Eq. (2.4) that

$$
\begin{aligned}
\left|A_{(k,l)}^{(a,b)}(n)\right| &\le \frac{\left(a^l b-1\right)n+a^k}{a^{k+l}b} \\
&< \left|A_{(k,l)}^{(a,b)}(n)\right|+1.
\end{aligned}
$$

The proof is complete by the property of the floor function. ■

The next corollary follows from Corollary 2.1 and the fact that $a^l b \le 2$ implies $\lfloor n/a^k \rfloor \equiv i \left(\operatorname{mod} a^l b\right)$ such that $i = 0,1$.

Corollary 2.2. Let a,b,k,l,n be positive integers such that $n \ge a^k$ and $a^l b \ne 1$. If $a^l b \le 2$, then

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lfloor \frac{\left(a^l b-1\right)n+a^k}{a^{k+l}b} \right\rfloor.$$

The number of integers divisible by 2^k except 2^{k+1} between 1 and n can be established from Corollary 2.2 as the following example:

Example 2.3. Let n, k be positive integers, $a=2, b=1$ and $l=1$. Since $a^l b \le 2$, Corollary 2.2 implies

$$\left|A_{(k,1)}^{(2,1)}(n)\right| = \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor.$$

Observe that for positive integers n and k, if $a=1, b=2$ and $l=1$, then we have from Corollary 2.2 that $\left|A_{(k,1)}^{(1,2)}(n)\right|$ is the number of odd integers between 1 and n, i.e.,

$$\left|A_{(k,1)}^{(1,2)}(n)\right| = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

3. Bounds on $\left|A_{(k,l)}^{(a,b)}(n)\right|$

In this section, bounds on $\left|A_{(k,l)}^{(a,b)}(n)\right|$ are given by using Theorem 2.1 as follows:

Theorem 3.1. Let a,b,k,l,n be positive integers such that $n \ge a^k$ and $a^l b \ne 1$. Then

$$\begin{aligned} \left\lceil \frac{(a^l b-1)n+a^k}{a^{k+l}b} - 1 \right\rceil &\le \left|A_{(k,l)}^{(a,b)}(n)\right| \\ &\le \left\lfloor \frac{(a^l b-1)n+a^k}{a^{k+l}b} + 1 - \frac{2}{a^l b} \right\rfloor. \end{aligned}$$

Proof. From Theorem 2.1, we have

$$\left|A_{(k,l)}^{(a,b)}(n)\right| \le \frac{(a^l b-1)n+a^k}{a^{k+l}b} + 1 - \frac{2}{a^l b}$$

and

$$\left|A_{(k,l)}^{(a,b)}(n)\right| > \frac{(a^l b-1)n+a^k}{a^{k+l}b} - 1.$$

These complete the proof by the property of the floor function and the ceiling function. ■

Finally, we show that Theorem 3.1 contains the perfect bounds. The following examples show that the equality holds in Theorem 3.1 for some integer inputs.

Example 3.2. Taking $a=2, b=5, k=1, l=1$, and $n=100$; we obtain

$$\begin{aligned} A_{(k,l)}^{(a,b)}(n) &= \{1 \le x \le 100 : 2 \mid x, 20 \nmid x\} \\ &= \{2,\ 4,\ 6,\ \ldots,\ 16,\ 18, \\ &\quad 22,\ 24,\ 26,\ \ldots,\ 36,\ 38, \\ &\quad 42,\ 44,\ 46,\ \ldots,\ 56,\ 58, \\ &\quad 62,\ 64,\ 66,\ \ldots,\ 76,\ 78, \\ &\quad 82,\ 84,\ 86,\ \ldots,\ 96,\ 98\}. \end{aligned}$$

Hence, $\left|A_{(k,l)}^{(a,b)}(n)\right| = 45$. Since

$$\begin{aligned} \frac{(a^l b-1)n+a^k}{a^{k+l}b} - 1 &= \frac{(10-1)100+2}{20} - 1 \\ &= 44.1, \end{aligned}$$

it follows that

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lceil \frac{(a^l b-1)n+a^k}{a^{k+l}b} - 1 \right\rceil.$$

Similarly, it is easy to see that

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lfloor \frac{(a^l b-1)n+a^k}{a^{k+l}b} + 1 - \frac{2}{a^l b} \right\rfloor.$$

Example 3.3. Taking $a=3, b=2, k=1, l=2$, and $n=120$; we obtain

$$\begin{aligned} A_{(k,l)}^{(a,b)}(n) &= \{1 \le x \le 120 : 3 \mid x, 54 \nmid x\} \\ &= \{3,\ 6,\ 9,\ \ldots,\ 24,\ 27, \\ &\quad 30,\ 33,\ 36,\ \ldots,\ 51,\ 57, \\ &\quad 60,\ 63,\ 78,\ \ldots,\ 81,\ 84, \\ &\quad 87,\ 90,\ 105,\ \ldots,\ 111,\ 114, \\ &\quad 117,\ 120\}. \end{aligned}$$

Hence, $\left|A_{(k,l)}^{(a,b)}(n)\right| = 38$. Since

$$\begin{aligned} \frac{(a^l b-1)n+a^k}{a^{k+l}b} + 1 - \frac{2}{a^l b} &= \frac{(18-1)120+3}{54} + 1 - \frac{2}{18} \\ &= 38.72, \end{aligned}$$

it follows that

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lfloor \frac{(a^l b - 1)n + a^k}{a^{k+l} b} + 1 - \frac{2}{a^l b} \right\rfloor.$$

4. Conclusion

Let a,b,k,l,n be positive integers such that $n \geq a^k$ and $a^l b \neq 1$. Denote a finite set of positive integers $A_{(k,l)}^{(a,b)}(n)$ by

$$A_{(k,l)}^{(a,b)}(n) = \left\{ x \leq n : a^k \middle| x, a^{k+l} b \nmid x \right\}.$$

From Corollary 2.1, we have some explicit formula for $\left|A_{(k,l)}^{(a,b)}(n)\right|$ as follows:

$$\left|A_{(k,l)}^{(a,b)}(n)\right| = \left\lfloor \frac{(a^l b - 1)n + a^k}{a^{k+l} b} \right\rfloor,$$

whenever $\lfloor n/a^k \rfloor \equiv i \pmod{a^l b}$ with $0 \leq i \leq 1$. Notice that $a^l b \leq 2$ implies $\lfloor n/a^k \rfloor \equiv i \pmod{a^l b}$ such that $i = 0,1$. The best bounds on $\left|A_{(k,l)}^{(a,b)}(n)\right|$ are provided in Theorem 3.1, i.e.,

$$\begin{aligned} \left\lceil \frac{(a^l b - 1)n + a^k}{a^{k+l} b} - 1 \right\rceil &\leq \left|A_{(k,l)}^{(a,b)}(n)\right| \\ &\leq \left\lfloor \frac{(a^l b - 1)n + a^k}{a^{k+l} b} + 1 - \frac{2}{a^l b} \right\rfloor \end{aligned}$$

Our future work is to generalize these results, for example, the number of integers divisible by a except ab.

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