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Correction and Weight Distribution of Periodic Random Errors

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ABSTRACT

In data transmission and processing, error-correcting codes are used to protect the data from errors during communication. As the transmission channels are accompanied with predominant error patterns, we need to apply appropriate codes for an appropriate channel. In this paper, we consider a particular type of error "periodic random error" and obtain necessary and sufficient conditions for the existence of linear codes correcting such errors. We also obtain the Hamming weight distribution of such error pattern and derive an upper bound on the total Hamming weight of all codewords on such error correcting codes.

Keywords: Error correction; Parity check matrix; Periodic random error; Syndromes; Weight distribution

1. Introducion

Error pattern plays an important role in the efficiency of the communication channels. Codes are constructed keeping in mind the nature of error pattern that occur in a channel used for communication. Majority of error patterns in communication channels are either random error or burst error type, while there are some channels where these error patternsrepeat periodically. Such errors are observed in
(i) power lines, data channels in close distance to electronically controlled power supply units or inverters, car electric, compact discs, and CD-ROM [1],
(ii) lithographic stages for semiconductor fabrication, transducer calibration, and axis position feedback for precision cutting and measuring machines, mixing between the two heterodyne frequencies in differential-path interferometry [2].

When random error repeats periodically,theyare called "periodic random error". Detection of such errors was studied in [3]. Such error can be defined as follows.

Definition 1.1 A s-periodic random error of length b is an n-tuple whose non-zero components are confined to distinct sets of b

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consecutive positions such that the sets are separated by s positions.

For example, 4-periodic random errors of length 3 in a vector of length 16 are 1010000110000010, 101000011000001, 10100001100000, etc.

In [3], necessary and sufficient conditions for existence of linear code **detecting** s-periodic random error of length b as well as Reiger's type [4] of bound for codes correcting such errors are studied. In this paper, we derive the necessary and sufficient conditions for existence of linear code **correcting** such errors. We also obtain the Hamming weight distribution of such error pattern. The result is equivalent to Plotkin bound [5] (also Lemma 4.1 of [6]). Similar works can be found in [7-9, 10]. Finally, we give an upper bound on the total Hamming weight of all codewords on the periodic random error correcting linear codes.

Rest of the paper is written as follows. Section 2 provides necessary and sufficient conditions for existence of linear code correcting s-periodic random error of length b. In Section 3, we give an upper bound on the total Hamming weight of all codewords on the periodic random error correcting linear codes along with Hamming weight distribution of the error pattern.

2. Codes Correcting Periodic Random Errors

In this section, we give necessary and sufficient conditions for existence of linear codes correcting periodic random errors. They are equivalent to Fire bound [11] and Campopiano Bound [12] (also Theorem 4.16 and Theorem 4.17 of [6]) respectively. We provide examples also.

Theorem 2.1 For given non-negative integers n,b and $s\left(n\geq b+s\right)$, let $n=\lambda\left(b+s\right)+l$ for some non-negative integers λ and l, where $0\leq l<b+s$. Then the necessary number of check digits for an $\left(n,k\right)$ linear code over $GF\left(q\right)$ that can correct all s-periodic random errors of length b satisfies is given by

$$q^{n-k}\geq q^{\rho}+\sum_{i=1}^{s+b+1}\left(q^{\beta_i}-1\right)q^{(b-1)\beta_i+l'i} -\sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}}, \tag{2.1}$$

where $\beta_i=\left\lceil\dfrac{n-i-b+1}{s+b}\right\rceil$ and

(i) when $l=0,\rho=b\lambda$ and

$$l'_i=\begin{cases}0 & \text{for } 1\leq i\leq s,\\ s+b-i & \text{for } s+1\leq i\leq s+b,\end{cases}$$

(ii) when $1\leq l\leq b,\rho=b\lambda+l$ and

$$l'_i=\begin{cases}l-i & \text{for } 1\leq i\leq l,\\ 0 & \text{for } l+1\leq i\leq s+b,\end{cases}$$

(iii) when $b<l<s+b,\rho=b\left(\lambda+1\right)$ and

$$l'_i=\begin{cases}0 & \text{for } 1\leq i\leq l-b, 1<i<s+b,\\ l-i & \text{for } l-b+1\leq i\leq l.\end{cases}$$

(Here $\sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}}=0$ for $b=1$.)

Proof. For the proof, we calculate the total number of s-periodic random errors of length b and compare with the available number of cosets.

Let $S_i\left(1\leq i\leq s+b\right)$ represents the collection of all s-periodic random errors of length b starting from the i^{th} position. Then the total number of distinct s-periodic random errors of length b is

$$\left|S_1\cup S_3\cup S_3\cup\cdots\cup S_{s+b}\right| =\left|S_1\right|+\left|S_2\setminus S_1\right|+\left|S_3\setminus\left(S_1\cup S_2\right)\right|+\cdots +\left|S_{s+b}\setminus\left(S_1\cup S_2\cup\cdots\cup S_{s+b-1}\right)\right|.$$

In calculating $\left|S_{i+1} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_i\right)\right|$, we note that the last position of each set of b consecutive positions of $S_{i+1}\left(i=1,2,...,s\right)$ is not in S_i and from $S_{s+i}\left(i=s+1,\mathrm{s}+2,...,s+b-1\right)$, the last position of each set is already considered in the set S_i.

Case-(i): When $l=0$, we compute

$$\left|S_1\right| = q^{b\lambda},$$

$$\left|S_2 \setminus \mathrm{S}_1\right| = \left(q^{\beta_1}-1\right)q^{(b-1)\beta_1},$$

where $\beta_1 = \left\lceil \dfrac{n-1-b+1}{s+b} \right\rceil$,

$$\vdots$$

$$\left|S_{s+1} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_s\right)\right| = \left(q^{\beta_s}-1\right)q^{(b-1)\beta_s},$$

where $\beta_s = \left\lceil \dfrac{n-s-b+1}{s+b} \right\rceil$,

$$\left|S_{s+2} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+1}\right)\right| = \left(q^{\beta_{s+1}}-1\right)q^{(b-1)\beta_{s+1}+(b-1)} - \left(q^{\beta_{s+1}}-1\right),$$

where $\beta_{s+1} = \left\lceil \dfrac{n-(s+1)-b+1}{s+b} \right\rceil$,

$$\left|S_{s+3} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+2}\right)\right| = \left(q^{\beta_{s+2}}-1\right)q^{(b-1)\beta_{s+2}+(b-2)}\left(q^{\beta_{s+2}}-1\right),$$

where $\beta_{s+2} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil$.

Therefore,

$$\left|S_1 \cup S_3 \cup S_3 \cup \cdots \cup S_{s+b}\right| = q^{b\lambda} + \sum_{i=1}^{s+b+1}\left(q^{\beta_i}-1\right)q^{(b-1)\beta_i+l'i} - \sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}},$$

where $\beta_1 = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$,

$$l'_i = \begin{cases} 0 & \text{for } 1 \le i \le s, \\ s+b-i & \text{for } s+1 \le i \le s+b, \end{cases}$$

Case-(ii): When $1 \le l \le b$, we consider two sub-cases.

Subcase (i): If $s+2 < l$, then

$$\left|S_1\right| = q^{b\lambda+l},$$

$$\left|S_2 \setminus \mathrm{S}_1\right| = \left(q^{\beta_1}-1\right)q^{(b-1)\beta_1+(l-1)},$$

where $\beta_1 = \left\lceil \dfrac{n-1-b+1}{s+b} \right\rceil$,

$$\vdots$$

$$\left|S_{s+2} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+1}\right)\right| = \left(q^{\beta_{s+1}}-1\right)q^{(b-1)\beta_{s+1}+(l-s-1)} - \left(q^{\beta_{s+1}}-1\right),$$

where $\beta_{s+1} = \left\lceil \dfrac{n-(s+1)-b+1}{s+b} \right\rceil$,

$$\left|S_{s+3} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+2}\right)\right| = \left(q^{\beta_{s+2}}-1\right)q^{(b-1)\beta_{s+2}+(l-s-2)} - \left(q^{\beta_{s+2}}-1\right)q^{\beta_{s+2}},$$

where $\beta_{s+2} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil$.

$$\vdots$$

$$\left|S_l \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{l-1}\right)\right| = \left(q^{\beta_{l-1}}-1\right)q^{(b-1)\beta_{l-1}+1} - \left(q^{\beta_{l-1}}-1\right)q^{(l-s-2)\beta_{l-1}},$$

where $\beta_{l-1} = \left\lceil \dfrac{n-(l-1)-b+1}{s+b} \right\rceil$.

$$\left|S_{l+1} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_l\right)\right| = \left(q^{\beta_l}-1\right)q^{(b-1)\beta_l} - \left(q^{\beta_l}-1\right)q^{(l-s-2)\beta_l},$$

where $\beta_l = \left\lceil \dfrac{n-l-b+1}{s+b} \right\rceil$.

$$\vdots$$

$$\left|S_{s+b} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+b-1}\right)\right| = \left(q^{\beta_{s+b-1}}-1\right)q^{(b-1)\beta_{s+b-1}} - \left(q^{\beta_{s+b-1}}-1\right)q^{(b-2)\beta_{s+b-1}},$$

where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil$.

Subcase (ii): If $s+2 \ge l$, we have

$$\left|S_1\right| = q^{b\lambda+l},$$

$$\left|S_2 \setminus S_1\right| = \left(q^{\beta_1} - 1\right) q^{(b-1)\beta_1 + (l-1)},$$

where $\beta_1 = \left\lceil \dfrac{n-1-b+1}{s+b} \right\rceil,$

$$\vdots$$

$$\left|S_{s+2} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+1}\right)\right|$$
$$= \left(q^{\beta_{s+1}} - 1\right) q^{(b-1)\beta_{s+1}} - \left(q^{\beta_{s+1}} - 1\right),$$

where $\beta_{s+1} = \left\lceil \dfrac{n-(s+1)-b+1}{s+b} \right\rceil,$

$$\left|S_{s+3} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+2}\right)\right|$$
$$= \left(q^{\beta_{s+2}} - 1\right) q^{(b-1)\beta_{s+2}} - \left(q^{\beta_{s+2}} - 1\right) q^{\beta_{s+2}},$$

where $\beta_{s+2} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil.$

$$\vdots$$

$$\left|S_{s+b} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_{s+b-1}\right)\right|$$
$$= \left(q^{\beta_{s+b-1}} - 1\right) q^{(b-1)\beta_{s+b-1}} - \left(q^{\beta_{s+b-1}} - 1\right) q^{(b-2)\beta_{s+b-1}},$$

where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil.$

In either subcase, we have

$$\left|S_1 \cup S_3 \cup S_3 \cup \cdots \cup S_{s+b}\right|$$
$$= q^{b\lambda + l} + \sum_{i=1}^{s+b+1} \left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'i} -$$
$$\sum_{i'=1}^{b-1} \left(q^{\beta_{s+i'}} - 1\right) q^{(i'-1)\beta_{s+i'}},$$

where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$ and

$$l'_i = \begin{cases} l-i & \text{for } 1 \le i \le l, \\ 0 & \text{for } l+1 \le i \le s+b. \end{cases}$$

Case-(iii): When $b < l \le s+b-1,$ let $l = b+t$ and then we can compute as

$$\left|S_1\right| = q^{b(\lambda+1)},$$
$$\left|S_{i+1} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_i\right)\right| = \left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'_i},$$

for $1 \le i \le s,$ and

$$\left|S_{i+1} \setminus \left(S_1 \cup S_2 \cup \cdots \cup S_i\right)\right|$$
$$= \left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'_i} - \left(q^{\beta_i} - 1\right) q^{(i-s-1)\beta_i},$$

for $s+1 \le i \le s+b-1,$ where

$$\beta_i = \left\lceil \frac{n-i-b+1}{s+b} \right\rceil$$

and

$$l'_i = \begin{cases} 0 & \text{for } 1 \le i \le t, b+t < i < s+b, \\ t+b-i & \text{for } t+1 \le i \le t+b. \end{cases}$$

Hence,

$$\left|S_1 \cup S_3 \cup S_3 \cup \cdots \cup S_{s+b}\right|$$
$$= q^{b(\lambda+1)} + \sum_{i=1}^{s+b-1} \left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'i} -$$
$$\sum_{i'=1}^{b-1} \left(q^{\beta_{s+i'}} - 1\right) q^{(i'-1)\beta_{s+i'}},$$

where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$ and

$$l'_i = \begin{cases} 0 & \text{for } 1 \le i \le t, b+t < i < s+b, \\ t+b-i & \text{for } t+1 \le i \le t+b. \end{cases}$$

Therefore, combining all the three cases, for an $\left(n = \lambda(b+s) + l, k\right)$ code correcting s-periodic random errors of length b, we must have

$$q^{n-k} \ge q^{\rho} + \sum_{i=1}^{s+b+1} \left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'i}$$
$$- \sum_{i'=1}^{b-1} \left(q^{\beta_{s+i'}} - 1\right) q^{(i'-1)\beta_{s+i'}},$$

where the values of ρ, β_i and l'_i are given according to case (i), (ii) and (iii).

Example 2.1 Taking $n = 12, b = 2, s = 3$ and $q = 2$ in Theorem 2.1, we have $l = 2 = b$ and $\lambda = 2$. Then

$$\rho = b\lambda + l = 6, \beta_1 = \left\lceil \frac{12-1-2+1}{5} \right\rceil = 2,$$

$$\beta_2 = \left\lceil \frac{12-2-2+1}{5} \right\rceil = 2, \beta_3 = \left\lceil \frac{12-3-2+1}{5} \right\rceil = 2,$$

$$\beta_4 = \left\lceil \frac{12-4-2+1}{5} \right\rceil = 2, l'_1 = 1,, l'_2 = 0, l'_3 = 0, l'_4 = 0.$$

The total number of 3-periodic errors of length 2 (including zero vector) is

$$q^{\rho}+\sum_{i=1}^{s+b-1}\left(q^{\beta_i}-1\right)q^{(b-1)\beta_i+l'i}-\sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}}=121.$$

These 120 error patterns (excluding zero vector) are given below. 000000000001, 000000000010, 000000000011, 000000100000, 000001000000, 000001100000, 000000100001, 000000100010, 000000100011, 000001000001, 000001000010, 000001000011, 000001100001, 000001100010, 000001100011, 010000000000, 010000000001, 010000000010, 010000000011, 010000100000, 010001000000, 010001100000, 010000100001, 010000100010, 010000100011, 010001000001, 010001000010, 010001000011, 010001100001, 010001100010, 010001100011, 100000000000, 100000000001, 100000000010, 100000000011, 100000100000, 100001000000, 100001100000, 100000100001, 100000100010, 100000100011, 100001000001, 100001000010, 100001000011, 100001100001, 100001100010, 100001100011, 110000000000, 110000000001, 110000000010, 110000000011, 110000100000, 110001000000, 110001100000, 110000100001, 110000100010, 110000100011, 110001000001, 110001000010, 110001000011, 110001100001, 110001100010, 110001100011, 000000010000, 000000110000, 000000010001, 000000110001, 001000000000, 001000000001, 001000010000, 001000100000, 001000110000, 001000010001, 001000100001, 001000110001, 010000010000, 010000110000, 010000010001, 010000110001, 011000000000, 011000000001, 011000010000, 011000100000, 011000110000, 011000010001, 011000100001, 011000110001, 000000001000, 000000011000, 000100000000, 000100001000, 000100010000, 000100011000, 001000001000, 001000011000, 001100000000, 001100001000, 001100010000, 001100011000, 000000000100, 000000001100, 000010000000, 000010000100, 000010001000, 000010001100, 000100000100, 000100001100, 000110000000, 000110000100, 000110001000, 000110001100, 000000000110, 000001000100, 000001000110, 000010000010, 000010000110, 000011000000, 000011000010, 000011000100, 000011000110.

Theorem 2.2 For given non-negative integers n,b and $s\left(n\ge b+s\right)$, let $n=\lambda\left(b+s\right)+1$ and $n-b=\lambda'\left(b+s\right)+L$ for some non-negative integers λ, λ′, l and L, where $0\le l,L<b+s$. Then the sufficient condition for the existence of an $\left(n,k\right)$ linear code over $GF\left(q\right)$ that can correct all s-periodic random errors of length b is given below:

$$q^{n-k}>q^{\delta}\left[q^{\eta}+\sum_{i=1}^{s+b-1}\left(q^{\beta_i}-1\right)q^{(b-1)\beta_i+l'i}-\sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}},\right. \tag{2.2}$$

where

$$\beta_i=\left\lceil\frac{n-i-2b+1}{s+b}\right\rceil,$$

$$\delta=\begin{cases}\lambda b-1 & \text{for } l=0,\\ \lambda b-1+l & \text{for } 1\le l\le b,\\ \lambda b-1+b & \text{for } b<l<s+b,\end{cases}$$

$$\eta=\begin{cases}b\lambda' & \text{for } L=0,\\ b\lambda'+L & \text{for } 1\le L\le b,\\ b\left(\lambda'+1\right) & \text{for } b+1\le L<s+b,\end{cases}$$

$$l_i'=\begin{cases}0 & \text{for } 1\le i\le s,\\ s+b-i & \text{for } s<i\le s+b,\end{cases}\quad \text{if } L=0,$$

$$l_i'=\begin{cases}L-i & \text{for } 1\le i\le L,\\ 0 & \text{for } \mathrm{L}<i\le s+b,\end{cases}\quad \text{if } 1\le L\le b,$$

$$l_i'=\begin{cases}0 & \text{for } 1\le i\le L-b, L<i\le s+b,\\ L-i & \text{for } L-b+1\le i\le L,\end{cases}\quad \text{if } b<L<s+b.$$

(Here $\sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}}-1\right)q^{(i'-1)\beta_{s+i'}}=0$ for $b=1$.)

Proof. The existence of this type of linear code is shown by constructing an appropriate $\left(n-k\right)\times n$ parity-check matrix H. Take the first column h_1 as any nonzero $\left(n-k\right)$-tuple and the remaining columns $h_2,h_3,...,h_n$ are added to H one after another by the condition given below:

$$h_n \neq \left(\sum_{i=1}^{b-1} a_{i1} h_{n-i} + \sum_{i=0}^{b-1} b_{i1} h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2} h_{n-2(s+b)-i} + \cdots + \sum_{i=0}^{g_1-1} b_{i\lambda} h_{n-\lambda(s+b)-i}\right) + \left(\sum_{i=1}^{b-1} \alpha_{i1} h_{j'-i} + \sum_{i=0}^{b-1} \beta_{i1} h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} \beta_{i2} h_{j'-2(s+b)-i} + \cdots + \sum_{i=0}^{g_2-1} \beta_{i\lambda} h_{j'-\lambda(s+b)-i}\right), \quad (2.3)$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in GF(q), j' \le n-b$,

$$g_1 = \begin{cases} 0 & \text{if } l = 0, \\ l & \text{if } 1 \le l \le b, \\ b & \text{if } b < l < s+b, \end{cases}$$

and

$$g_2 = \begin{cases} 0 & \text{if } L = 0, \\ L & \text{if } 1 \le L \le b, \\ b & \text{if } b < L < s+b. \end{cases}$$

(We take $\sum_{i=0}^{g_i-1} b_{i\lambda} h_{n-\lambda(s+b)-i} = 0$ for $g_i = 0$)

The number of linear combinations in first bracket on R.H.S. of (2.3) is

$$\begin{cases} q^{\lambda b-1} & \text{if } l = 0, \\ q^{\lambda b-1+l} & \text{if } 1 \le l \le b, \\ q^{\lambda b-1+b} & \text{if } b < l < s+b, \end{cases}$$

The second bracket of (2.3) gives the number of s-periodic random error of length b in a vector of length $n-b$. This is given by Theorem 2.1 as

$$q^{\lambda} + \sum_{i=1}^{s+b-1}\left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'i} - \sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}} - 1\right) q^{(i'-1)\beta_{s+i'}},$$

where $\beta_i = \left\lceil \dfrac{n-i-2b+1}{s+b} \right\rceil$,

(i) $l'_i = \begin{cases} 0 & \text{for } 1 \le i \le s, \\ sb - i & \text{for } s < i \le s+b, \end{cases}$ if $L = 0$,

(ii) $l'_i = \begin{cases} L - i & \text{for } 1 \le i \le L, \\ 0 & \text{for } L < i \le s+b, \end{cases}$ if $1 \le L \le b$,

(iii) $l'_i = \begin{cases} 0 & \text{for } 1 \le i \le L-b, L < i \le s+b, \\ L - i & \text{for } L-b+1 \le i \le L, \end{cases}$ if $b < L < s+b$.

Therefore, the total number of all possible linear combinations on R.H.S. of (2.3) is

$$q^{\delta}\left[q^{\eta} + \sum_{i=1}^{s+b-1}\left(q^{\beta_i} - 1\right) q^{(b-1)\beta_i + l'i} - \sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}} - 1\right) q^{(i'-1)\beta_{s+i'}},\right. \quad (2.4)$$

where

$$\delta = \begin{cases} \lambda b - 1 & \text{for } l = 0, \\ \lambda b - 1 + l & \text{for } 1 \le l \le b, \\ \lambda b - 1 + b & \text{for } b < l < s+b, \end{cases}$$

$$\eta = \begin{cases} b\lambda' & \text{for } L = 0, \\ b\lambda' + L & \text{for } 1 \le L \le b, \\ b(\lambda' + 1) & \text{for } b+1 \le L < s+b, \end{cases}$$

and

$l'_i = \begin{cases} 0 & \text{for } 1 \le i \le s, \\ s+b-i & \text{for } s < i \le s+b, \end{cases}$ if $L = 0$,

$l'_i = \begin{cases} L - i & \text{for } 1 \le i \le L, \\ 0 & \text{for } L < i \le s+b, \end{cases}$ if $1 \le L \le b$,

$l'_i = \begin{cases} 0 & \text{for } 1 \le i \le L-b, L < i \le s+b, \\ L - i & \text{for } L-b+1 \le i \le L, \end{cases}$ if $b < L < s+b$.

Since we can have at most q^{n-k} columns, so taking q^{n-k} greater than the term computed in (2.4) with different values of δ, η and l'_i gives the sufficient condition for the existence of the required code. This proves the theorem.

Example 2.2 Consider $n = 12, s = 4, b = 2$ and $q = 2$ in Theorem 2.2. Here $\lambda = 2, l = 0$, $\lambda' = 1, L = 4$. Then $\delta = 3, \eta = 4$, $\beta_1 = \left\lceil \dfrac{12-1-4+1}{6} \right\rceil = \left\lceil \dfrac{8}{6} \right\rceil = 2, \beta_2 = \left\lceil \dfrac{7}{6} \right\rceil = 2,$

$\beta_3 = \left\lceil \frac{6}{6} \right\rceil = 1, \beta_4 = \left\lceil \frac{5}{6} \right\rceil = 1, l'_1 = 0, l'_2 = 0, l'_3 = 1,$

$l'_4 = 0, l'_5 = 0.$

By the inequality (2.2) in the proof of Theorem 2.2, we have

$$2^{n-k} > 2^{\delta}\left[q^{\eta} + \sum_{i=1}^{s+b-1}\left(q^{\beta_i} - 1\right)q^{(b-1)\beta_i + l'i} - \sum_{i'=1}^{b-1}\left(q^{\beta_{s+i'}} - 1\right)q^{(i'-1)\beta_{s+i'}} = 376.\right.$$

This gives $n-k=9$. Therefore, $n=12$, $s=4, b=2$ and $q=2$ gives rise to a (12, 3) linear code whose parity check matrix H, constructed by the procedure mentioned in the proof of Theorem 2.2, is given below.

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}_{9\times 12}$$

The null space of this matrix can correct all 4-periodic random error of length $b=2$. From the Error Pattern-Syndrome Table, the syndromes of all 66 periodic random errors are found to be nonzero and distinct.

3. Total Weight of Periodic Error Correcting Codes

In this section, we first give Hamming weight distribution of periodic random errors and then give an upper bound on the total weight on the periodic random error correcting codes.

In the calculation of the number of s-periodic random errors of length b in an n-tuple having Hamming weight t, we take note of the situation that s-periodic random errors of length b that start after $(s+1)^{th}$ position, there are some vectors which are already counted in the number of errors that start from the initial $b-1$ positions. Excluding them, we calculate the number of s-periodic random errors of length b having Hamming weight t and the numberis given in the following lemma.

Lemma 3.1 Let $n = \lambda(b+s) + l$ (where $0 \le l < b+s$) and $W_{s,b}(t)$ be the total number of s-periodic random errors of length b in an n-tuple having Hamming weight t. Then

$$W_{s,b}(1) = \sum_{i=1}^{s+1}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1}\right](q-1) + \sum_{i=s+2}^{s+b}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1}\right](q-1),$$

and

$$W_{s,b}(t) = \sum_{i=1}^{s+1}\left[\binom{m_i}{t} - \binom{k_{i-1}}{t}\right](q-1)^t + \sum_{i=s+2}^{s+b}\left[\binom{m_i}{t} - \binom{k_{i-1}}{t} - \binom{p_{i-s-1}}{t} + \binom{p_{i-s-2}}{t}\right](q-1)^t,$$

for $t \ne 1$,

where $k_0 = 0, p_0 = 1, k_i = m_{i+1} - \beta_i$,

$$\beta_i = \left\lceil \frac{n-i-b+1}{s+b} \right\rceil,$$

$$m_i = \begin{cases} \lambda b & \text{for } 1 \le i \le s+1, \\ \lambda b + s - i + 1 & \text{for } s+2 \le i \le s+b, \end{cases} \quad \text{if } l = 0,$$

$$m_i = \begin{cases} b\lambda + l - i + 1 & \text{for } 1 \le i \le l, \\ b\lambda & \text{for } l+1 \le L \le s+l+1, \\ b\lambda + s + l - i + 1 & \text{for } s+l+2 \le i < s+b, \end{cases}$$

if $1 \le l < b$,

$$m_i = \begin{cases} b(\lambda+1) & \text{for } 1 \le i \le l-b+1, \\ b(\lambda+1) + l - b - i + 1 & \text{for } l-b+1 < i \le l, \\ b\lambda & \text{for } 1 + l < i \le s+b, \end{cases}$$

if $b \le l < s+b$,

$p_i = i\beta_{i+s}$ if $l = 0, b \le l < s+b$,

$$p_i = \begin{cases} i\beta_{i+s} & \text{for } 1 \le i \le l, \\ L - i & \text{for } l < i \le b-1, \end{cases} \quad \text{if } 1 \le l < b.$$

Note that for $n=\lambda(b+s)+l\ (0\le l<b+s)$, the maximum Hamming weight of a s-periodic random error of length b, corresponding to the three cases arises: $l=0, 1\le l<b$ and $b\le l<b+s$, is given by

$$\begin{cases}\lambda b & \text{for } l=0,\\ \lambda b+l & \text{for } 1\le l<b,\\ b(\lambda+1) & \text{for } b\le l<s+b.\end{cases}$$

Example 3.1 Taking $n=12, s=3, b=2$ in Lemma 3.1, we have $\lambda=2$ and $l=2$. Then $m_1=2(2+1)=6, m_2=2(2+1)+0-2+1=5$, $m_3=m_4=m_5=2\times2=4$,

$\beta_1=\left\lceil\frac{12-1-2+1}{5}\right\rceil=\left\lceil\frac{10}{5}\right\rceil=2$, similarly, $\beta_2=\beta_3=\beta_4=2$.

Also, $p_1=\beta_4=2, k_1=m_2-\beta_1=5-2=3$ and similarly $k_2=k_3=k_4=2$.

$$W_{3,2}(1)=\sum_{i=1}^{4}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}\right](2-1)+\sum_{i=5}^{5}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}-\binom{\beta_{i-1}}{1}\right](2-1)=12,$$

$$W_{3,2}(2)=\sum_{i=1}^{5}\left[\binom{m_i}{2}-\binom{k_{i-1}}{2}\right](2-1)^2+\sum_{i=5}^{5}\left[\binom{m_i}{2}-\binom{k_{i-1}}{2}-\binom{p_{i-s-1}}{2}+\binom{p_{i-s-2}}{2}\right](2-1)^2$$
$$=36.$$

Similarly, we can find $W_{3,2}(3)=41$, $W_{3,2}(4)=23, W_{3,2}(5)=7$ and $W_{3,2}(6)=1$. [The maximum weight of a 3-periodic random error of length 2 is $b(\lambda+1)=6$.] These weight distributions can be verified from Example 2.1.

Example 3.2 Let $n=12, s=4, b=2$, (so $l=0$) in Lemma 3.1. Then $m_1=m_2=m_3=m_4=m_5=2\times2=4, m_6=(2\times2)+4-6+1=3$, $\beta_1=\left\lceil\frac{10}{6}\right\rceil=2$, similarly, $\beta_2=\beta_3=\beta_4=2$ and $\beta_5=1$. Also, $p_1=\beta_5=1, k_1=m_2-\beta_1=2$ and similarly $k_2=k_3=k_4=k_5=2$. We can calculate $W_{4,2}(1)=12, W_{4,2}(2)=28$, $W_{4,2}(3)=21$ and $W_{4,2}(4)=5$.

Theorem 3.1 The total Hamming weight of all codewords of an $(n=\lambda(b+s)+l, k)$ linear code over $GF(q)$ that corrects any s-periodic random error of length b is at most

$$\begin{cases}\sum_{i=1}^{n}\binom{n}{i}i(q-1)^i-\sum_{t_1}t_1W_{s-b,2b}(t_1) & \text{if } s>b,\\ \sum_{i=1}^{n}\binom{n}{i}i(q-1)^i-\sum_{t_2}t_1W_{1,b+s-1}(t_2) & \text{if } s\le b,\end{cases}$$

where $W_{s-b,2b}(t_1)$ and $W_{1,b+s-1}(t_2)$ are given by Lemma 3.1, t_1 varies from 1 to

$$\begin{cases}2b\lambda+l & \text{if } 0\le l\le 2b-1,\\ 2b(\lambda+1) & \text{if } 2b\le l<s+b,\end{cases}$$

and t_2 varies from 1 to

$$\begin{cases}(b+s-1)\lambda+l & \text{if } 0\le l\le b+s-2,\\ (b+s-1)\lambda & \text{if } b+s-1\le l<s+b,\end{cases}$$

Case (i): If $s>b$, then any $(s-b)$-periodic random error of length $2b$ can be expressed as the sum (difference) of two s-periodic random errors of length b. So, the code can detect any $(s-b)$-periodic random error of length b. Any $(s-b)$-periodic random error of length $2b$ can not be a codevector. Therefore, total weight of all codevectors of a linear code over $GF(q)$ that corrects any s-periodic random error of length b is at most

[*Total weight on all* n*- tuples -total weight of all* $(s-b)$*-periodic random errors of length* $2b$]

$$\sum_{i=1}^{n}\binom{n}{i}i(q-1)^{i}-\sum_{t_1}t_1W_{s-b,2b}(t_1),$$

where t_1 varies from 1 to

$$\begin{cases}2b\lambda+l & \text{if } 0\le l\le 2b-1,\\ 2b(\lambda+1) & \text{if } 2b\le l<s+b,\end{cases}$$

Case (ii): If $s\le b$, then any 1-periodic random error of length $b+s-1$ can be expressed as the sum (difference) of two s-periodic random errors of length b. So, any 1-periodic random error of length $b+s-1$ can not be a codevector. Therefore, total weight of all codevectors of the linear code is at most

[*Total weight on all* n*- tuples -total weight of all 1-periodic random errors of length* $b+s-1$]

$$\sum_{i=1}^{n}\binom{n}{i}i(q-1)^{i}-\sum_{t_2}t_1W_{1,b+s-1}(t_2),$$

where t_2 varies from 1 to

$$\begin{cases}(b+s-1)\lambda+l & \text{if } 0\le l\le b+s-2,\\ (b+s-1)\lambda & \text{if } b+s-1\le l<s+b,\end{cases}$$

4. Conclusion

This paper derived necessary and sufficient conditions for the existence of periodic random error correcting linear codes andan upper bound on the total Hamming weight of all codewords of such a code.Further, wederived Hamming weight distribution of such error pattern. The work of the paper can be extended to the situation when the error patternoccurs with lowdensity, a concept introduced by Wyner [13].

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References

[1] Lange, L. Error correcting codes on periodically disturbed data channels.Proceeding of 1994 IEEE International Symposium on Information Theory1994; 33.

[2] Schmitz, TL, Chu D and Houck III, L. First-order periodic error correction: validation for constant and non-constant velocities with variable error magnitudes. Measurement Science and Technology2006; 17: 3195-203.

[3] Das, PK. Codes on s- periodic random error of length b .Palestine Journal of Mathematics 2014; 3(2):168-74.

[4] Reiger, S. Codes for correction of clustered errors.IRE Transactions on Information Theory1960; 6(1):16-21. Reiger, S. Codes for correction of clustered errors.IRE Transactions on Information Theory1960; 6(1):16-21.

[5] Ding, C., Li, C., Li, N. and Zhou, Z. Three-weight cyclic codes and their weight distributions. Discrete Mathematics 2016; 339(2):415-27.

[6] Peterson,WWand Weldon, EJ. Error Correcting Codes. 2nd Edition, MIT Press, Cambridge, Massachusetts, 1972.

[7] Das, PK and Kumar, S. Location and weight distribution of key errors. MathematickiVesnik 2021;73(1): 43-54.

[8] Reiger, S. Codes for correction of clustered errors.IRE Transactions on Information Theory1960; 6(1):16-21.

[9] Plotkin, M. Binary code with specified minimum weight.IRE Transactions on Information Theory1960; IT-6: 445-50.

[10] Sharma, BD andRohtagi, B. Some results on weights of vectors having m-repeated bursts. Cybernetics and Information Technologies 2011;11(3): 3-11.

[11] Fire, P. A class of multiple-error-correction binary codes for non-independent errors. Sylvania Report RSL-E-2, Sylvania Reconnaissance Systems Laboratory, Mountain View, California, 1959.

[12] Campopiano,CN. Bounds on burst error correcting codes.IRE Transactions on Information Theory1962; IT-8: 257-9.

[13] Wyner, AD. Low-density-burst-correcting codes. IEEE Transactions on Information Theory 1963;IT-9: 124.

[14] Prange, E. Cyclic error-correcting codes in two symbols. Air Force Cambridge Research Center, AFCRC-TN-57, 1957; 103.

[15] flow due to magnetic field inside a porous media considering Brownian motion. J Mol Liq [Internet]. 2018 Jan 1 [cited 2021 Mar 18];249:921-9. Available from: https://linkinghub.elsevier.com/retrieve/pii/S0167732217344057