P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : https://tci-thaijo.org/index.php/SciTechAsia
Science & Technology Asia
Vol. 26 No.4 October - December 2021
Page: [74-83]
Original research article

A Computational Approach for Solving Fractional Model of Malignant Tumour Growth based on Dynamics of Cell Cycle

Jagdev Singh*, Rashmi Agrawal

Department of Mathematics, JECRC University, Jaipur 303905, India

Received 05 August 2021; Received in revised form 09 November 2021
Accepted 26 November 2021; Available online 30 December 2021

ABSTRACT

We have studied the fractional order mathematical model of malignant tumour growth based on cell cycle dynamics in this work. The model describes on three different tumour cell dynamics of the population; quiescent cells, interphase cells, and mitotic cells. The studied model is based on fractional order differential equations. A computational approach has been implemented to give approximate solution of this fractional model. The model can be used to describe the graphical behavior of tumour cells. The computational results have been presented graphically to show the advantage and the efficiency of the scheme for fractional order malignant tumour growth model.

Keywords: Atangana-Baleanu fractional derivative; Analytical method; Cell Cycle; Fractional differential equation; Malignant tumour

1. Introduction

Fractional calculus is a branch of applied mathematics, including integrals and derivatives of real, complex and arbitaray order. In the past decade, fractional calculus has been used in different science and engineering disciplines. Fractional differential equations are increasingly being used to solve and evaluate problems in the field of fluid mechanics, biology, electromagnetic, acoustics, diffusion, signal processing, and many other physical processes [1-10]. Fractional calculus is the advancement of ordinary calculus by introducing derivatives in addition integrals of fractional order. Up to the middle of the 20th century many works have been done in the area of special branch namely fractional calculus by famous mathematicians such as Caputo [11], Lutzen and Liouville [12], Miller and Ross [8] ect. The applications of fractional calculus have been studied by many famous scientist and mathematician. So to name a few, Bagley and Torvik [13] discussed a model of arbitrary order of viscoelastic

***Corresponding author:** jagdevsinghrathore@gmail.com **doi:** 10.14456/scitechasia.2021.67

nature, cancer model is given by Ahmed et.al [14], Singh et al. [15] examined of epidemiological model for computer viruses. The Drinfeld-Sokolov-Wilson (DSW) equation is investigated by Singh et.al [16], and diabetes models of fractional order have been analyzed by Singh et. al [17] with the exponential law. Cancer is one of the most dangerous and serious disease in the history of pathology. Cancer cells grow out of control and grow abnormally in the body. Cells are the basic units of the body building and cancer grows from normal cells. More recently, mathematical biology has been contributing to the development of theoretical models for the description and understanding of tumour development. Almost all these models focus on the categorization of tumour cells in three groups: necrotic cells (which are dead and located in the most internal part of the solid tumour), quiescent cells (are not dead, but do not have sufficient nutrients for cell growth or division), proliferating cells (the active part is that it undergo mitosis). A large class of cancer development models focuses on the proliferation dynamics of cells involved in the development of the tumour mass. When examining tumour growth, the process of cell senescence and cell division can not be ignored. All eukaryotic cells undergo a cell cycle, which is a four-step sequence [18-20]. Before DNA replicates in S phase, G1 phase is required for cell growth. This is followed by a second developmental phase (G2), the mitotic phase (M) ending the cycle involving the cell nucleus and cytoplasmic division. As a result of sequence completion, two subunits enter the G1 cycle. The first three stages are usually collectively referred to as the "intermediate stage." Cells can also enter the so-called G0 state, in which they will not grow, divide or live in a quiescent state. Usually, cells lacking growth factors will stop at the checkpoint and, move from G1 to G0, and the resume circulation after a period of time. The main focus here is the cell cycle events that occur when growth factors stimulate G0 cell proliferation.

In this article we inspired and motivated by the ongoing research in this area, we will discuss the approximate method for solving nonlinear equations by homotopy analysis sumudu transform method (HASTM).The HASTM is a novel and efficient mixture of the homotopy analysis method (HAM) [21-23] and sumudu transform method (ST) [24]. Research study of the linear fractional malignant tumour equation explained by the homotopy analysis sumudu transforms method [25-27]. It provides the solutions in terms of convergent series with easily computable components in a direct way without using, perturbation, linearization or restrictive assumptions. The advantage of this method is that it combines two powerful methods to obtain accurate and approximate analytical solutions of nonlinear equations. The HASTM solution contains auxiliary parameters $\hbar$ that can be used to manage convergence of solution.

The advancement of this paper is given as follows: In section 2, the integrals and sumudu transformation method of Atangana -Baleanu derivative and fractional order are discussed. In section 3, the fundamental plan of HASTM is presented. In section 4, fractional order for malignant tumour assoicated with non-singular kernel is discussed. In section 5, stability is verified . In section 6, solution of HASTM solution for malignant tumour associated with AB derivative is investigated. In section 7, numerical results and discussion are elaborated . In section 8,closing remarks are discussed.

2. Preliminaries

Definition 1. Let $f \in H^1(a,b), b > a,$ then for $\lambda \in (0,1]$ the Atangana-Baleanu fractional (AB) derivative [28] in Caputo sense is given below

$$ {}^{ABC}_{a}D_t^{\lambda}\left(\varphi(\tau)\right)=\frac{B(\lambda)}{1-\lambda}\int_a^{\tau}\varphi'(\delta)E_{\lambda}\left[-\lambda\frac{(\tau-\delta)^{\lambda}}{1-\lambda}\right]d\delta \tag{2.1}$$

where $B(\lambda)$ is a normalization function under the conditions $B(0)=B(1)=1$. The related fractional integral is defined by

$$I^{\lambda}\left(\varphi(\tau)\right)=\frac{1-\lambda}{B(\lambda)}\varphi(\tau)+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\tau}\varphi(\delta)(\tau-\delta)^{\lambda-1}d\delta \tag{2.2}$$

Definition 2. The sumudu transform [24] is obtained over the set of function

$$A=\{\varphi(\tau):\exists\, N,\rho_1,\rho_2>0,|\varphi(\tau)|<N\exp\left(\frac{|\tau|}{\rho_i}\right),$$

if $\tau\in(-1)^i\times[0,\infty)\}$,

as

$$S\left[\varphi(\tau)\right]=G(u)=\int_0^{\infty}\varphi(u\tau)\exp(-\tau)dt,\quad u\in(-\rho_1,\rho_2) \tag{2.3}$$

Definition 3. The sumudu transform of AB fractional derivative in Caputo sense [29] is given as follows

$$S\left({}^{ABC}_{0}D_t^{\lambda}\left(\varphi(\tau)\right)\right)=\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}\left(G(u)-\varphi(0)\right). \tag{2.4}$$

3. Fundamental plan of method

To discuss the initial idea of HASTM, let us take a non-linear partial differential equation of fractional order,

$$ {}^{ABC}_{0}D_{\vartheta}^{\eta}\upsilon(\varsigma,\vartheta)+R\upsilon(\varsigma,\vartheta)+N\upsilon(\varsigma,\vartheta)=\varphi(\varsigma,\vartheta),\quad 0<\eta\le 1, \tag{3.1}$$

where $\upsilon(\varsigma,\vartheta)$ is a function of two variable ς and ϑ, ${}^{ABC}_{0}D_{\vartheta}^{\eta}$ denotes a fractional operator of order η defined in terms of AB fractional derivative , $k\in N,R$ represents a linear operator bounded in two variables ς and ϑ i.e. we can have the number $\alpha>0$ s.t. $\|R\upsilon\|\le\alpha\|\upsilon\|$ and N denote the general nonlinear differential operator in ς and ϑ, in Lipschitz continous with $\delta>0$ holding the condition $|N\upsilon-N\varphi|\le\tau|\upsilon-\varphi|$ and $\varphi(\varsigma,\vartheta)$ is the source term. Applying ST on Eq . (3.1), we have

$$S\left[{}^{ABC}_{0}D_{\vartheta}^{\eta}\upsilon\right]+S\left[R\upsilon+N\upsilon\right]=\mathrm{S}\left[\varphi(\varsigma,\vartheta)\right]. \tag{3.2}$$

By explaining the differential characteristics of ST on Eq. (3.2), we obtain the following result

$$\frac{B(\eta)}{1-\eta+\eta u^{\eta}}S\left[\upsilon(u)-\upsilon(0)\right]+S\left[R\upsilon+N\upsilon\right]=S\left[\varphi(\varsigma,\vartheta)\right]. \tag{3.3}$$

According to the Eq. (3.3), the nonlinear operator is expressed as follows

$$N\left[\rho(\varsigma,\vartheta;p)\right]=S\left[\rho(\varsigma,\vartheta;p)\right]-\upsilon(0)+\frac{1-\eta+\eta u^{n}}{B(\eta)}\left\{S\left[R\rho(\varsigma,\vartheta;p)+N\rho(\varsigma,\vartheta;p)\right]-S\left[\varphi(\varsigma,\vartheta;p)\right]\right\}, \tag{3.4}$$

where $0\le p\le 1$ is a parameter known as the embedding parameter and $\rho(\varsigma,\vartheta;p)$ is denoting a function ς,ϑ and p.

Next ,we propose the homotopy represented by Eq. (3.5)

$$(1-p)S\left[\rho(\varsigma,\vartheta;p)-\upsilon_0\right]=\bar{h}N\left[\upsilon(\varsigma,\vartheta)\right], \tag{3.5}$$

where S denotes the sumudu transform operator, $\bar{h}\neq 0$ is an auxiliary parameter, $\upsilon_0(\varsigma,\vartheta)$ is an initial approximation of $\upsilon(\varsigma,\vartheta)$ and $\rho(\varsigma,\vartheta;p)$ is the unknown function. If the embedded parameters are $p=0$ and $p=1$, it is easy to explain

$$\rho(\varsigma,\vartheta;0)=\upsilon_0(\varsigma,\vartheta),\rho(\varsigma,\vartheta;1)=\upsilon(\varsigma,\vartheta). \tag{3.6}$$

Therefore, when p values from 0 to 1, the solution $\rho(\varsigma,\vartheta;p)$ changes from the initial approximation $\upsilon_0(\varsigma,\vartheta)$ to the solution $\upsilon(\varsigma,\vartheta)$. Enlarging the function $\rho(\varsigma,\vartheta;p)$ in the form of Taylor series, we get the subsequent equation about p

$$\rho(\varsigma,\vartheta;p)=\upsilon_0(\varsigma,\vartheta)+\sum_{n=1}^{\infty}\upsilon_n(\varsigma,\vartheta)p^n, \tag{3.7}$$

where,

$$\upsilon_n(\varsigma,\vartheta)=\frac{1}{n!}\frac{\partial^n}{\partial q^n}\{\rho(\varsigma,\vartheta;p)\}\Big|_{p=0}. \tag{3.8}$$

If the initial approximation $\upsilon_0(\varsigma,\vartheta)$ and the convergence control parameter $\overline{h}$ are suitably specified, then Eq.(3.7) converges at $p=1$, we have

$$\upsilon(\varsigma,\vartheta)=\upsilon_0(\varsigma,\vartheta)+\sum_{n=1}^{\infty}\upsilon_n(\varsigma,\vartheta). \tag{3.9}$$

The result expressed by the Eq. (3.9) must be one of the solutions of the given sequence of nonlinear differential equations. By using the result (3.9), then the governing equation can be obtained by using the Eq. (3.5). We express the vector as follows

$$\vec{\upsilon}_n=\{\upsilon_0(\varsigma,\vartheta),\upsilon_1(\varsigma,\vartheta),\upsilon_2(\varsigma,\vartheta),\ldots,\upsilon_n(\varsigma,\vartheta)\}. \tag{3.10}$$

Now, we differentiate the Eq. (3.5) n-times w.r.t. p , then divide by n! Finally, in the case of $p=0$, we get the following equation

$$S\left[\upsilon_n(\varsigma,\vartheta)-\chi_n\upsilon_{n-1}(\varsigma,\vartheta)\right]=\overline{h}\Re_n(\vec{\upsilon}_{n-1}). \tag{3.11}$$

Now using the inverse of sumudu transform on Eq. (3.11), we arrived at the subsequent result

$$\upsilon_n(\varsigma,\vartheta)=\chi_n\upsilon_{n-1}(\varsigma,\vartheta)+\overline{h}S^{-1}\left[\Re_n(\vec{\upsilon}_{n-1})\right], \tag{3.12}$$

where χ_n is given as

$$\chi_n=\begin{cases}0, & n\le 1,\\ 1, & n>1.\end{cases} \tag{3.13}$$

And we define the value of $\Re_n(\upsilon_{n-1})$ in an improved manner as

$$\Re_n(\vec{\upsilon}_{n-1})=S\left[\upsilon_{n-1}(\varsigma,\vartheta)\right]-\left\{\upsilon(0)+\frac{1-\eta+\eta u^n}{B(\eta)}S\left[\varphi(\varsigma,\vartheta)\right]\right\}(1-\chi_n)+\frac{1-\eta+\eta u^n}{B(\eta)}S\left[R\upsilon_{n-1}+p_{n-1}\right]. \tag{3.14}$$

In Eq. (3.14),

P_n is the homotopy polynomial [30] and is presented as

$$P_n=\frac{1}{\Gamma(\eta)}\left[\frac{\partial^n}{\partial q^n}N\rho(\varsigma,\vartheta;p)\right]_{p=0}, \tag{3.15}$$

and,

$$\rho(\varsigma,\vartheta;p)=\rho_0+p\rho_1+p^2\rho_2+\cdots. \tag{3.16}$$

Using Eq. (3.14) in Eq. (3.12), we have

$$\upsilon_n(\varsigma,\vartheta)=(\chi_n+\overline{h})\upsilon_{n-1}(\varsigma,\vartheta)-\overline{h}(1-\chi_n)S^{-1}\left\{\upsilon(0)+\frac{1-\eta+\eta u^n}{B(\eta)}S\left[\varphi(\varsigma,\vartheta)\right]\right\}+\frac{1-\eta+\eta u^n}{B(\eta)}\overline{h}S^{-1}\left[S\left[R\upsilon_{n-1}+p_{n-1}\right]\right]. \tag{3.17}$$

An improvement in the scheme used is that a novel correction function (3.17) has been introduced with the aid of homotopy polynomials. Therefore, using Eq. (3.17), we can determine $\upsilon_n(\varsigma,\vartheta)$ for $n\ge 1$ different components and the solution is represented by the following equation

$$\upsilon(\varsigma,\vartheta)=\sum_{n=0}^{\infty}\upsilon_n(\varsigma,\vartheta). \tag{3.18}$$

4. Fractional Model for Malignant Tumour Growth Assoicated with Non-Singular Kernel

When $q_h(t)$ be the quiescent cells number (G0) at a time t, $u_h(t)$ be the mitotic cells number at a time t and $\upsilon_h(t)$ be the interphase cells number (G1;s and G2-phase) at a time t, then the mathematical model describing the tumour cell cycle as follows [31]

$$\begin{aligned}\frac{dq_h}{dt} &= \alpha_Q \upsilon_h(t) - \left(\beta_Q + \gamma_{G_0}\right) q_h(t),\\ \frac{du_h}{dt} &= 2\sigma_1 u_h(t) + \beta_Q q_h(t) - \delta_1 u_h \frac{d\upsilon_h}{dt}\\ &= -\left(\omega_0 + \alpha_Q\right)\upsilon_h(t).\end{aligned} \tag{4.1}$$

With the following intial condition

$$q_h(0) = c_1, u_h(0) = c_2, \upsilon_h(0) = c_3,$$

where $\gamma_{G_0}, \alpha_Q, \beta_Q, \sigma_1, \omega_0$ and δ_1 represent death rate G0 cells, transition rate from G1 to G0, transition rate from G0 to G1, M cells divison rate , death rate G1 cells and death rate M cells respectively.This mathematical model for tumour cell cycle assocated with intger order derivative was studied by using homotopy perturbation scheme [32] and modified differential transform technique [33].

The classical derivatives especially Riemann and Caputo fractional derivatives, have restrictions beacuase their kernel is singular. Because the kernel is used by the physical system to explain the memory effect. It is clear that because of this weakness, mutually derived memory may not accurately reflect the full effect. We use the fractional derivative given by Atangana and Baleanu [28] to study the malignant tumour Growth model and to explain in a better and more efficient manner. The main advantages of this kind of opreator is that the singular power-law kernel is now replaced by non singular kernel ,which is easier to use in theoretical analysis, numerical calculations and real world applications.

We modify the model (4.1) by empolying the Atangana-Baleanu fractional derivative as follows

$$\begin{aligned}{}^{ABC}_{0}D^{\lambda} q_h(t) &= \alpha_Q \upsilon_h(t) - \left(\beta_Q + \gamma_{G_0}\right) q_h(t),\\ {}^{ABC}_{0}D^{\lambda}_t u_h(t) &= 2\sigma_1 u_h(t) + \beta_Q q_h(t) - \delta_1 u_h(t),\\ {}^{ABC}_{0}D^{\lambda}_t \upsilon_h(t) &= -\left(\omega_0 + \alpha_Q\right)\upsilon_h(t).\end{aligned} \tag{4.2}$$

With the following intial condition

$$q_h(0) = c_1, u_h(0) = c_2, \upsilon_h(0) = c_3, \tag{4.3}$$

5. Stability analysis

In this section we discuss the steady-state stability analysis without cancer and establish the conditions of model parameters for tumour growth or eradication. The fractional order mathematical model of malignant tumour growth based on cell cycle dynamics (4.2) gives a linear system with the only stationary state $R^* = (0,0,0)$. Calculate the eigen values of the coefficient matrix to analyze the stability.

$$A = \begin{vmatrix} -\left(\beta_Q + \gamma_{G_0}\right) & 0 & 0\\ \beta_Q & 2\sigma_1 - \delta_1 & 0\\ 0 & 0 & -\left(\omega_0 + \alpha_Q\right)\end{vmatrix}.$$

Characteristic equation is given as

$$\lambda^3 + b_1\lambda^2 + b_2\lambda - b_3 = 0,$$

where

$$\begin{aligned}b_1 &= \delta_1 - 2\sigma_1 - \left(\beta_Q + \gamma_{G_0} + \omega_0 + \alpha_Q\right),\\ b_2 &= \left(\omega_0 + \alpha_Q\right)\left(\beta_Q + \gamma_{G_0}\right) + \left(\omega_0 + \alpha_Q\right)\left(2\sigma_1 - \delta_1\right) +\\ &\left(\beta_Q + \gamma_{G_0}\right)\left(2\sigma_1 - \delta_1\right),\\ b_3 &= \left(2\sigma_1 - \delta_1\right)\left(\beta_Q + \gamma_{G_0}\right)\left(\omega_0 + \alpha_Q\right).\end{aligned} \tag{5.1}$$

The charactersitic values are ,

$$\lambda_1 = -\left(\beta_Q + \gamma_{G_0}\right), \lambda_2 = \left(2\sigma_1 - \delta_1\right),$$
$$\lambda_3 = -\left(\omega_0 + \alpha_Q\right).$$

All roots are real ,

$$\lambda_1 < 0, \lambda_3 < 0.$$

The stationary state R^* is locally asymptotically stable, if $\sigma_1 < \frac{\delta_1}{2}$.

$\sigma_1 < \frac{\delta_1}{2}$ means that the death rate of mitotic cells exceeds the division rate, so cancer cells will not grow and the tumour will disappear at one point [31].

6. HASTM Solution for Malignant Tumour Growth Associated with AB Fractional Derivative

This section represents HASTM solution to analyze the numerical results of malignant tumour related to AB derivative. Firstly ,we apply the Sumudu transform method to malignant tumour model (4.2) ,it yields

$$\begin{aligned}
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}\left[S\left(q_h(t)\right)-c_1\right]\\
&=S\left[\alpha_Q \upsilon_h(t)-\left(\beta_Q+\gamma_{G_0}\right)q_h(t)\right],\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}\left[S\left(u_h(t)\right)-c_2\right]\\
&=S\left[2\sigma_1 u_h(t)+\beta_Q q_h(t)-\delta_1 u_h(t)\right],\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}\left[S\left(\upsilon_h(t)\right)-c_3\right]\\
&=S\left[-\left(\omega_0+\alpha_Q\right)\upsilon_h(t)\right].
\end{aligned} \tag{6.1}$$

On simplification we have,

$$\begin{aligned}
&S\left(q_h(t)\right)-c_1-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[\alpha_Q \upsilon_h(t)-\left(\beta_Q+\gamma_{G_0}\right)q_h(t)\right]=0,\\
&S\left(u_h(t)\right)-c_2-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[2\sigma_1 u_h(t)+\beta_Q q_h(t)-\delta_1 u_h(t)\right]=0,\\
&S\left(\upsilon_h(t)\right)-c_3-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[-\left(\omega_0+\alpha_Q\right)\upsilon_h(t)\right]=0.
\end{aligned} \tag{6.2}$$

We present the nonlinear operators as

$$\begin{aligned}
&N_1\left[\xi_1(t;x)\right]=S\left[\xi_1(t;x)\right]-c_1-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[\alpha_Q\xi_3(t;x)-\left(\beta_Q+\gamma_{G_0}\right)\xi_1(t;x)\right]=0,\\
&N_2\left[\xi_2(t;x)\right]=S\left[\xi_2(t;x)\right]-c_2-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[2\sigma_1\xi_2(t;x)+\beta_Q\xi_1(t;x)-\delta_1\xi_2(t;x)\right]=0,\\
&N_3\left[\xi_3(t;x)\right]=S\left[\xi_3(t;x)\right]-c_3-\\
&\frac{B(\lambda)}{1-\lambda+\lambda u^{\lambda}}S\left[-\left(\omega_0+\alpha_Q\right)\xi_3(t;x)\right]=0.
\end{aligned} \tag{6.3}$$

Hence we have,

$$\begin{aligned}
&\Re_{1,n}\left(q_{h(n-1)}\right)=S\left[q_{h(n-1)}\right]-c_1\left(1-\chi_n\right)-\\
&\frac{1-\lambda+\lambda u^{\lambda}}{B(\lambda)}S\left[\alpha_Q\upsilon_{h(n-1)}-\left(\beta_Q+\gamma_{G_0}\right)q_{h(n-1)}\right],\\
&\Re_{2,n}\left(u_{h(n-1)}\right)=S\left[u_{h(n-1)}\right]-c_2\left(1-\chi_n\right)-\\
&\frac{1-\lambda+\lambda u^{\lambda}}{B(\lambda)}S\left[2\sigma_1 u_{h(n-1)}+\beta_Q q_{h(n-1)}-\delta_1 u_{h(n-1)}\right],\\
&\Re_{3,n}\left(\upsilon_{h(n-1)}\right)=S\left[\upsilon_{h(n-1)}\right]-c_3\left(1-\chi_n\right)-\\
&\frac{1-\lambda+\lambda u^{\lambda}}{B(\lambda)}S\left[-\left(\omega_0+\alpha_Q\right)\upsilon_{h(n-1)}\right].
\end{aligned} \tag{6.4}$$

The n^{th} order deformation equation is expressed as follows

$$\begin{aligned}
&S\left[q_{hn}(t)-\chi_n\left(q_{h(n-1)}\right)(t)\right]=\bar{h}\Re_{1,n}\left(q_{h(n-1)}\right),\\
&S\left[u_{hn}(t)-\chi_n\left(u_{h(n-1)}\right)(t)\right]=\bar{h}\Re_{2,n}\left(u_{h(n-1)}\right),\\
&S\left[\upsilon_{hn}(t)-\chi_n\left(\upsilon_{h(n-1)}\right)(t)\right]=\bar{h}\Re_{3,n}\left(\upsilon_{h(n-1)}\right).
\end{aligned} \tag{6.5}$$

Applying the inverse sumudu transform to Eq. (6.5), it yields

$$q_{hn}(t)=\chi_n\left(q_{h(n-1)}\right)(t)+\bar{h}S^{-1}\left[\Re_{1,n}\left(q_{h(n-1)}\right)\right],$$
$$u_{hn}(t)=\chi_n\left(u_{h(n-1)}\right)(t)=\bar{h}S^{-1}\left[\Re_{2,n}\left(u_{h(n-1)}\right)\right],$$
$$\upsilon_{hn}(t)=\chi_n\left(\upsilon_{h(n-1)}\right)(t)=\bar{h}S^{-1}\left[\Re_{3,n}\left(\upsilon_{h(n-1)}\right)\right]. \quad (6.6)$$

Taking the initial condition

$q_h(0)=2\times10^5, u_h(0)=10^5, \upsilon_h(0)=4\times10^5,$ and solving Eq.(6.6) for $n=1,2,3\ldots.$ We obtain the values of $q_{hn}(t), u_{hn}(t), \upsilon_{hn}(t)$. So, the solution of fractional order malignant tumour cell model (4.2) is given as,

$$q_h(t)=q_{h0}+q_{h1}+q_{h3}+\cdots,$$
$$u_h(t)=u_{h0}+u_{h1}+u_{h3}+\cdots,$$
$$\upsilon_h(t)=\upsilon_{h0}+\upsilon_{h1}+\upsilon_{h3}+\cdots,$$

7. Numerical Results and Discussion

This section elaborate the numerical simulation of malignant tumour. The numerical results for the model (4.1) are calculated using HASTM. We have taken the values of parameter $\alpha_Q=0.02, \beta_Q=0.2,$ $\gamma_{G_0}=0.1/10^4, \sigma_1=1, \delta_1=0.28, \omega_0=0.11$ and the initial conditions are given as $q_h(0)=c_1=2\times10^5, \mathrm{u}_h(0)=c_2=10^5,$ $\upsilon_h(0)=c_3=4\times10^5$. In this model, the dynamics of three different tumour cell population (quiescent cells, interphase cells, and mitotic cells) are being studied. From Fig. 1(a), it is easy to see that the number of quiescent cells increases as the value of λ increases after some times is nature is opposite. Similarly, the effect of λ on mitotic cells and interphase cells is shown in Figs. 1(b)-(c) . From Figs. 2(a)-(b), it can be seen that as values of α_Q (transition rate G1 to G0) increases, the number of quiescent cells increases and as a values of α_Q increase the number of interphase cells decrease . It can be seen that from Fig. 3 (a)-(b) as the the value of β_Q (transition rate from G0 to G1) increases, the number of quiescent cells decreases and as the value of β_Q (transition rate from G0 to G1) increases, the number of mitotic cells increases. Fig. 4 shows that as the values of death rate of M cells (δ_1) increases the number of mitotic cells decreases.

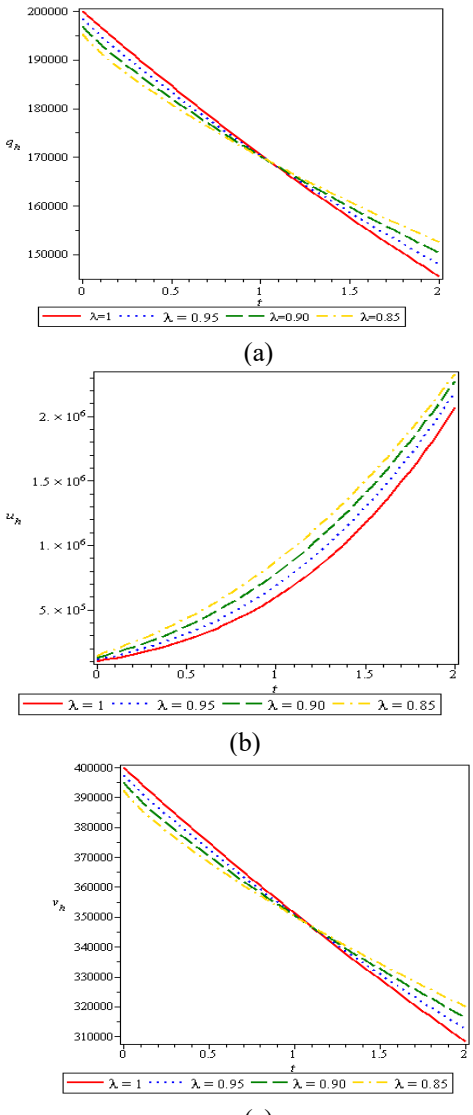
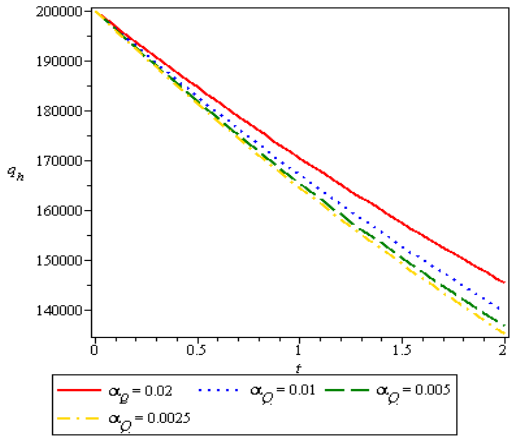
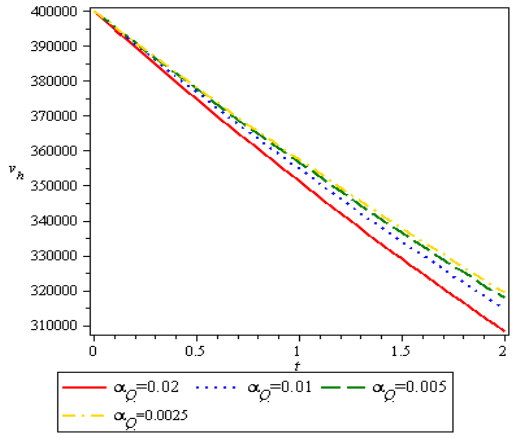


Fig. 1. Effect of order of AB fractional derivative on population (a) Quiescent cells (b) Mitotic cells (c) Interphase cells.

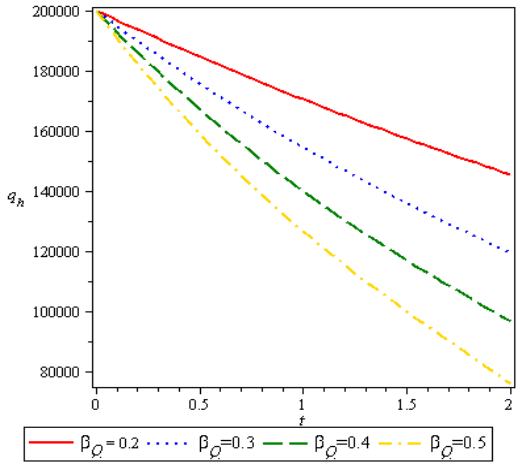


(a)

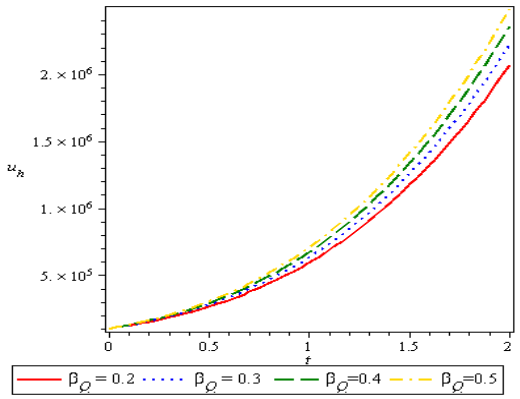


(b)

Fig. 2. Effect of transition rate from G1 to G0 on population when $\lambda = 1$ (a) Quiescent cells (b) Interphase cells.



(a)



(b)

Fig. 3. Effect of transition rate from G0 to G1 on populationn when $\lambda = 1$ (a) Quiescent cells (b) Mitotic cells.

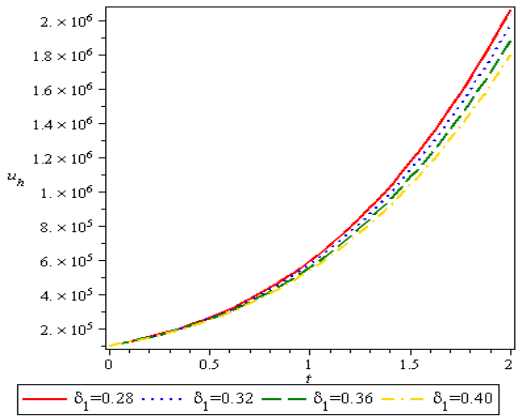


Fig. 4. Effect of death rate of M cells on populationn when $\lambda = 1$ for Mitotic cells.

8. Conclusion

In this article, we studied the cell cycle model of malignant tumour associated with non-singular kernel. It can be seen that the growth of cancer mainly depends on the death rate of mitotic cells and the rate at which mitotic cells enter G1 without treatment. When the rate of mitotic is greater than the death rate of mitotic cells, cancer begins to grow.We use homotopy analysis sumudu transform method to study the fractional order mathematical model of malignant tumour growth based on cell cycle dynamics. The main benefit of this method is that it provides an analytical

approximate solution. The effects of order of fractional derivative and other parameters on populations of tumour cells have been described graphically.

References

[1] H. Beyer and S. Kempfle, Definition of physically consistent damping laws with fractional derivatives, Zeitschrift für Ange- wandte Mathematik und Mechanik, 1995; 75(8): 623-35.

[2] J.H. He, Some applications of nonlinear fractional differential equations and their approximations, Bulletin of Science, Tech- nology & Society, 1999; 15(2): 86-90.

[3] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering, 1998; 167(1-2): 57-68.

[4] R. Hilfer, Fractional time evolution, In applications of fractional calculus in physics, World Scientific, river edge, nj, USA, 2000; 87-130.

[5] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999; 198.

[6] L. Debnath, Fractional integral and fractional differential equations in fluid mechanics, Fractional Calculus & Applied Analysis, 2003; 6(2): 119-55.

[7] M. Caputo, Elasticita e dissipazione, zani-chelli, bologna, Italy, 1969.

[8] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley & Sons, New York, Ny, USA, 1993.

[9] K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to arbitrary order, Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1974; 111.

[10] A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, The Netherlands, 2006; 204.

[11] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II. J Roy Austral Soc, 1967;13: 529-39.

[12] J. Lutzen, and J. Liouville, Contribution to the theory of integral equations, 1982; 9: 373-91.

[13] R. L. Bagley, and P. J. Torvik, On the fractional calculus model of viscoelastic behaviour, Journal of Rheology, 1986; 30:133-5.

[14] E. Ahmed, A. H. Hashis and F. A. Rihan, On fractional order cancer model, Journal of Fractional Calculus and Applications, 2012; 3:1-6.

[15] J. Singh, D. Kumar, Z. Hammouch, and A. Atangan, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, Applied Mathematics and Computation, 2018; 316: 504-15.

[16] J. Singh, D. Kumar, D. Baleanu and S. Rathore, An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation, Applied Mathematics and computation, 2018; 335:12-24.

[17] J. Singh, D. Kumar, and D. Baleanu, On the analysis of fractional diabetes model with exponential law, Advances in Difference Equations, 2018; 1-15. doi.org/10.1186/s13662- 018-1680-1.

[18] B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts, and P. Walter, Molecular biology of the cell, 5th edition

Biochemistry and Molecular Biology Education, 2008; 36(4): 317-8.

[19] Geoffrey M. Cooper and R. E. Hausman, The cell: A Molecular approach, Press Washington, 1997.

[20] H. Lodish, A. Berk, A. Kaiser, M. Krieger, A. Bretscher, H. Ploegh, A. Amon and M. P. Scott, Molecular cell biology, Wiley Online Library, New York, 1995.

[21] S.J. Liao, Beyond perturbation, Introduction to homotopy analysis method, Chapman and hall/crc press, Boca Raton, 2003.

[22] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Compute. 2004; 147:499-513.

[23] S.J. Liao, An approximate solution technique not depending on small parameters: a special example, Int. J. Non-Linear Mech. 1995; 30: 371-80.

[24] G. K. Watugala, Sumudu transform – a new integral transform to solve differential equation and control engineering problems, Math. Eng. Ind. 1998; 6(4): 319-29.

[25] D. Kumar, J. Singh and D. Baleanu, Numerical computation of a fractional model of differential-difference equation, J. Comput. Nonlinear Dyn. 2016;11(6). 061004. doi:10.1115/1.4033899.

[26] S. Rathore, D. Kumar, J. Singh and S. Gupta, Homotopy analysis sumudu transform method for nonlinear equations, Int. J. Ind. Math. 2012; 4(4):13. Article id ijim-00204.

[27] D. Kumar and R.P. Sharma, Numerical approximation of newell–Whitehead–Segel equation of fractional order, Nonlinear eng. 2012; 5(2): 81-6.

[28] A. Atangana and D. Baleanu, New fractional derivative with nonlocal and nonsingular kernel: Theory and application to heat transfer model, Thermal Sci. 2016; 20(2): 763-9.

[29] A. Bokharia, D. Baleanu and R. Belgacem, Application of Shehu transform to Atangana-Baleanu derivatives, J. Math. Computer Sci. 2020; 20: 101-10.

[30] Z. Odibat and S. A. Bataineh, An adaptation of homotopy analysis method for reliable treatment of strongly nonlinear problems: construction of homotopy polynomials, Math. Methods Appl. Sci. 2014; 38(5): 991-1000.

[31] M.V. Barbarossa, C. Kuttler, J. Zinsl, Delay equations modeling the effects of phase-specific drugs and immunotherapy on proliferating tumor cells. Math Biosci Eng. 2012; 9(2): 241-57.

[32] N. S. Ravindran, M. Mohamed Sheri and P. Krishnapriya, Homotopy perturbation method for solving cell cycle of tumoural Cells, British Journal of Mathematics & Computer Science2014; 4(23):3271-86.

[33] Khushbu D. Patel and D.C. Joshi,Study of malignant tumour growth based on dynamics of cell cycle by modified differential transform method, The international journal of analytical and experimental modal analysis, 2019;11(10):2673.