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Difference Equations and Gaussian Distribution for the Digital Signal's Noise Reduction

D. Dorathy Prema Kavitha, L. Francis Raj*

Department of Mathematics, Voorhees College, Affiliated to Thiruvalluvar University Vellore, Tamil Nadu 632115, India

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ABSTRACT

This paper presents the dynamical approach of Kalman Filter (KF) and Extended Kalman Filter (EKF) along with the Gaussian distribution for the digital signal's noise reduction. It is observed that the discrete time varying finite dimensional EKF evaluates the state estimation constraints and this reduces the mean square error and termed it as noise. In addition to the state dynamics EKF converges, to its steady state in the presence of Kalman gain. In this process state estimation constraints are measured and updated. Then the EKF dynamics is transformed in terms of Gaussian distribution and conditional probability function. Utilizing the Bayesian concepts and conditional probability, the filter propagates the probability density function of the expected quantities, i.e., the claimed value of the filter propagates to its conditional probability density function. This function is defined like $p\left(x(k)\,|\,y(1),\ldots,y(k),u(0),\ldots,u(k-1)\right)$ for the increasing values of k. At each time instant k, the absolute error leads to propagate the probability. This probability density function calculates the amount of noise free value of $x(k)$. Hence the predicted state estimation of the n^{th} state at time instant $l+1$ and $\bar{x}_n(l+1|l)$ is taking into an account since the observation dynamics and the noise has holding zero mean. Simultaneously, a complete probability space on which a scalar white noise has also zero mean. So, it is well known that the predicted measurement of noise free environment follows the same observation dynamics. Subsequently the mean, of this conditional probability density functions are coincide in a particular state. The Extended Kalman filter, that propagates the conditional probability density functions and obtains the state estimate by optimizing the system of equations, which provides the best estimate among all the possible inputs.

***Corresponding author:** francisraj73@yahoo.co.in **doi:** 10.14456/scitechasia.2021.75

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1. Introduction

In the digital era, difference equations characterize various productive conceptual algorithms to the digital signal processing [1]. This can be expressed in several traditional ways. In the field of digital signal processing, in particular, in any digital filters, one of the way of predicting noise level is Gaussian probability density function and Kalman gain [2]. The system analyses of the structures of discrete systems are periodically performed through difference equations [3]. As a complement to the discrete time case many researchers contributed to get the solvability of the difference equations [4]. Moreover, majority of the applications from the communication and technology are governed by various difference equations [5]. This significant application motivates many researchers, to study the concept of difference equation in a distinct manner [6].

In the Digital Signal Processing systems, communication protocols are introduced to evaluate filtering problems [7]. Filtering problem is very much needs to investigate the behaviour of the signal and hence it is a fundamental one in signal controlling and signal processing [8]. The transmission error and noises are controlled by various difference equations [9]. These difference equations focus the concept of stability analysis, signal control, parameter and coordinator identification, state estimation, noise diagnosis and so on [10-11]. At the same time the recursive filtering problems are specifically used in the time varying systems and delayed systems [12]. This recursive filtering problems includes Kalman filter, Extended Kalman filter, Weiner filter, unscented filter and so on [13]. These filter based difference equations are widely used in many wireless communications, especially in control systems, related to state space estimation problems [14-15]. Various mathematical modelling are resolved to reduce the delay of state space estimation problems.

The Gaussian and conditional probability functions are developed in this paper which is used to estimate the error covariance with the dynamic characteristics of the digital signals. Along with this Kalman gain, filter gain is computed by solving the system of difference equations [16]. The deterministic values are used to reach the convergence and zero mean stability of the filter. If the input values are non Gaussian then it leads to high computation [17]. In the distributed networks, it is a great challenge to get the noise level information through an optimal distributed filter. The fundamental natures of digital filters are classified with their sampled systems. The input and output coefficients in the signals are classified according to its oscillation behaviour.

This paper is structured as follows: Section 2 focused the problem formulation. The best estimate of $x_i(l)$ is arrived through the main techniques of difference equations and Gaussian probability density function. Section 3 dealt with optimal control problem of discrete time systems with multiplicative noise. The simulation results are established in the section 3. Finally, section 4 concludes the paper.

2. Problem Formulation

The generalized state dynamics of a nonlinear time varying system filter problem is formulated like

$$x_i(l+1) = f(x(l), j(l), k(l)),$$
$$y_i(l) = g(x(l), v(l)),$$

be thebe the state dynamics of a general nonlinear time varying system. Here,

$x \in R^n$ is the system state vector,

$f(.,.,.)$ refers the system's dynamics (noise characterization),

$j \in R^m$ is the control vector,

k is the vector that conveys the system error sources,

$j \in R^r$ is the observation vector,

g refers the measurement function (noise characterization) and

v is the vector that represents the measurement error sources.

Now consider the initial value $u(0),...,u(k-1)$, which are used to control the noise, and the set of measurements can be written as $y(1), y(1), y(2),..., y(k)$. The extended Kalman filter is used to attain the optimal state estimate of the desired output of the systems state (even if the environment is noisy). The perception of optimality is showed by the best state estimate which corresponds to the minimization of the error.

Utilizing the Bayesian concepts and conditional probability, the filter propagates the probability density function of the expected quantities, i.e., the claimed value of the filter propagates to its conditional probability density function. This function is defined like $p(x(k) | y(1),..., y(k), u(0),..., u(k-1))$ for the increasing values of k. At each time instant k, the absolute error leads to propagate the probability. This probability density function conveys the amount of noise free value of $x(k)$. Now consider that, for a given time instant k, the sequence of past inputs and the sequence of past measurements are denoted by the following best state estimates. Now this best estimated filters propagates the conditional probability density function for increasing values of k, and for each k, this obtains the state estimate as,

$$\begin{aligned}
Y_0 &= P(x_0) | Y_0^1 \\
Y_1 &= P(x_1) | Y_0^1 \cdot Y_1^1 \\
Y_2 &= P(x_2) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \\
Y_3 &= P(x_3) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \cdot Y_3^1 \\
&\vdots \\
Y_n &= P(x_n) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \cdot Y_3^1 \ldots \cdot Y_n^1
\end{aligned} \tag{2.1}$$

Higher order optimization criteria may be chosen to proceed further and, its leads to the different estimates of the system's state vector. Now define the minimum mean square state estimation error in terms of difference equation as,

$$\overline{x}_0(l+1|l) = \overline{x}_1(l+1) - \overline{x}_0(l+1),$$

and proceeding in this expression towards the values of $\overline{x}_0(l+1)$ and $\overline{x}_k(l+1)$ which yields that

$$\begin{aligned}
\overline{x}_n(l+1|l) &= A_l x_l + B_l j_l + C_l v_l - \\
&A_l \overline{x}_n(l|l) - B_l \overline{x}_n(l|l) - C_l \overline{x}_n(l|l),
\end{aligned}$$

and the filtering error is defined again in terms of the difference equation like

$$\overline{x}_n(l+1|l) = \overline{x}_{n-2}(l+1|l) - \overline{x}_{n-1}(l+1|l),$$

and thus, the Gaussian probability density function of the predicted measurement is written in terms of difference equation is

$$\overline{y}_n(l+1|l) = E\left[\overline{x}_{n-2}(l+1|l) - \overline{x}_{n-1}(l+1|l)\right],$$
$$\overline{y}_n(l+1|l) = E\left[\overline{y}_l(l+1|l) - \overline{y}_{l-1}(l+1|l)\right].$$

The measurement prediction error can be defined like,

$$\overline{y}_n(l+1|l) = E\left[\overline{y}_n(l+1|l)\right] - E\left[\overline{y}_{n-1}(l+1|l)\right].$$

Hence the predicted state estimation of the n^{th} state at time instant $l+1$ and $\overline{x}_n(l+1|l)$ are taking into an account since the observation dynamics and the noise has zero mean. A complete probability space on which a scalar white noise has also zero mean. So, it is well known that the predicted measurement of (2.1) follows the same observation dynamics of the real-life scenario to arrive the optimality.

3. Difference Equations with Multiplicative Noises

The general principle is proposed for the optimal control problems driven by jointly Gaussian random vectors along with the conditional probability distribution function. The solvability of (1) has wide applications in the difference equations based on optimal control problem of discrete time systems with multiplicative noise.

Theorem: If

$$Y_0 = P(x_0) | Y_0^1$$
$$Y_1 = P(x_1) | Y_0^1 \cdot Y_1^1$$
$$Y_2 = P(x_2) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1$$
$$Y_3 = P(x_3) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \cdot Y_3^1$$
$$\vdots$$
$$Y_n = P(x_n) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \cdot Y_3^1 \ldots \cdot Y_n^1$$

are jointly Gaussian random vectors then,

$$E[x_0] | Y_0^1 = E[x_0] + R(x_0)\left[Y_0^1 - E\left[Y_0^1\right]\right],$$
$$E[x_1] | Y_1^2 = E[x_1] + R(x_1)\left[Y_1^2 - E\left[Y_1^2\right]\right],$$
$$E[x_2] | Y_2^3 = E[x_2] + R(x_2)\left[Y_2^3 - E\left[Y_2^3\right]\right],$$
$$\vdots$$
$$E[x_{n-1}] | Y_{n-1}^n = E[x_{n-1}] + R(x_{n-1})\left[Y_{n-1}^n - E\left[Y_{n-1}^n\right]\right],$$
$$E[x_n] | Y_n^{n+1} = E[x_n] + R(x_n)\left[Y_n^{n+1} - E\left[Y_n^{n+1}\right]\right], \tag{3.1}$$

where R stand for the random vectors.

The initial state x_0, is a Gaussian random vector whose mean is, $E[x_0] = \bar{x}_0$ and its corresponding covariance matrix will be

$$E\left[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right] = \sum_{i=0}^n u_i$$

where u_i is a deterministicsequence. For the system of equations (3.1), the Kalman filter attains its minimum means quare state error estimate. In fact, when (x_0) is a Gaussian vector, the state and observations noises function $f(x(l), j(l), k(l))$ and $g(x(l), v(l))$ are white and Gaussian noises and the state observation dynamics becomes linear and well recognized. Then the conditional probability density functions $P(x_n) | Y_0^1 \cdot Y_1^1 \cdot Y_2^1 \cdot Y_3^1 \ldots \cdot Y_n^1$ are Gaussian for any n. This helps us to attain the Kalman gain through mean square state error estimate.

Subsequently the mean, of this conditional probability density functions are coincide in a particular state. The Extended Kalman filter, that propagates the conditional probability density functions and obtains the state estimate by optimizing the criteria (system of equations), which provides the best estimate among all the possible inputs.

The tables 1-3 shows the values of the Extended Kalman filter's minimum mean square state estimate of order 2^n to 4^n.

Table 1. Extended Kalman Filter's minimum mean square state estimate of order 2^n.

No. Steps	Error	Error/h
2	2.5000e-01	0.500
4	1.2500e-01	0.500
8	6.2500e-02	0.500
16	3.1250e-02	0.500
32	1.5625e-02	0.500
64	7.8125e-03	0.500

Table 2. Extended Kalman Filter's minimum mean square state estimate of order 3^n.

No. Steps	Error	Error/h
3	-8.3333e-01	-1.250
9	-1.2778e+00	-5.750
27	-1.4259e+00	-19.250
81	-1.4753e+00	-59.750
243	-1.4918e+00	-181.250
729	-1.4973e+00	-545.750

Table 3. Extended Kalman Filter's minimum mean square state estimate of order 4^n.

No. Steps	Error	Error/h
4	-2.8750e+00	-3.833
16	-3.7188e+00	-19.833
64	-3.9297e+00	-83.833
256	-3.9824e+00	-339.833
1024	-3.9956e+00	-1363.833
4096	-3.9989e+00	-5459.833

The corresponding noise level for mean square state estimate of order 2^n to 4^n are illustrated in the Figs. 1-3. On the other hand, all the Gaussian probability density function which involved in the noise reduction processes through Extended Kalman Filter are associated with the locus of equal probability. The filter dynamics around the predicted and measured state estimated values constitute to get the mean value of the conditional probability density function that propagates the Kalman gain.

The Kalman gain as $K_y(k+1) = E(k+1|k)$. However, the stationary conditions along with the initial conditions are not required to evaluate the Kalman gain. Obviously, the initial state prediction and its measurement has arrived its unique value when it attains the order $2^n, 3^n, 4^n$ and so on. Hence the best approximation is coincides when the measurement values of order $2^n, 3^n, 4^n$ and so on.

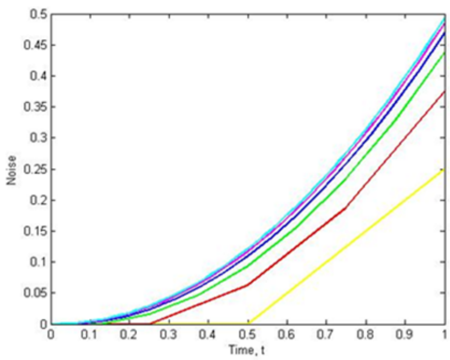

Fig. 1. Noise level for mean square state estimate of order 2^n.

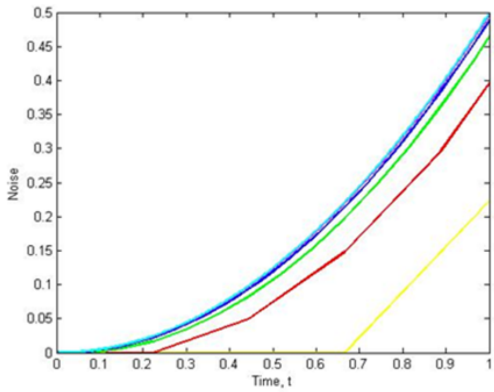

Fig. 2. Noise level for mean square state estimate of order 3^n.

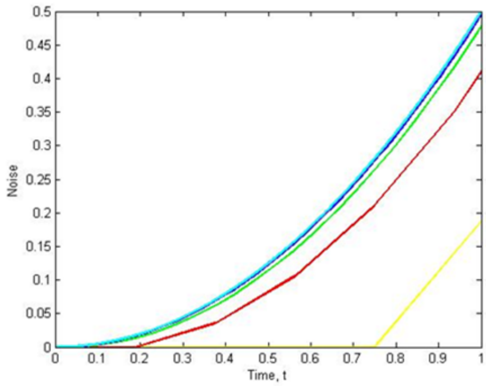

Fig. 3. Noise level for mean square state estimate of order 4^n.

4. Conclusion

Discrete time varying finite dimensional EKF values are evaluated through the state estimation constraints. We identified that this reduces the mean square error and termed its multiplicative noise at various levels like $2^n, 3^n, 4^n$ and so on. In addition to the state dynamics EKF converges to its steady state in the presence of Kalman gain. Then the EKF dynamics is transformed in terms of Gaussian distribution and conditional probability function. In this process state estimation is measured through Gaussian probability density function and updated its values periodically.

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