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Fekete-Szegö Inequalities for New Classes of Analytic Functions Associated with Fractional q-Differintegral Operator

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ABSTRACT

The fractional q-differintegral operator defines two new subclasses of analytic functions in the open unit disc in this article. Fekete-Szegö inequalities are also derived for these newly defined subclasses.

Keywords: Convex functions; Starlike functions; Fractional q-calculus operators; Subordination; Univalent functions

1. Introduction and Preliminaries

The theory of fractional calculus has risen to prominence and acclaim in recent years, owing to its demonstrable breakthroughs in a variety of fields of science and engineering. Because of the widespread usage of the q-calculus in mathematics and physics, there has been a huge expansion in the quantity of articles written in this area recently. Inquisitive readers might consult the articles on the issue by [1–6] for further details.

Indeed, the first author of this paper examined some classes of analytic functions related with fractional q-calculus operators with Choi and Purohit in [7]. Several scholars have recently introduced new classes of analytic functions that are defined utilizing quantum calculus operators. We recommend that the reader consult the articles [8–14] and the references included therein for some new studies on the classes of analytic functions defined in combination with quantum calculus operators and related points. The goal of this paper is to

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introduce two new classes of analytic functions defined using q-calculus operators and the notion of subordination in the open unit disc. Fekete-Szegö inequalities for functions belonging to these classes are also obtained.

Let $\mathcal{H}(a,n)$ represents the class of functions $f(\omega)$ of the type

$$f(\omega) = a + \sum_{p=0}^{\infty} a_{n+p}\omega^{n+p}, \quad (\omega \in \mathbb{D}), \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{D} = \{\omega \in \mathbb{C} : |\omega| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(0,1)$ having functions in the type

$$f(\omega) = \omega + \sum_{n=2}^{\infty} a_n\omega^n. \tag{1.2}$$

The classes of functions in $\mathcal{A}$ that are univalent, convex, starlike and close-to-convex in $\mathbb{D}$ are represented respectively by $\mathcal{S}$, $\mathcal{S}^*$, C and $\mathcal{K}$. Let $f_1(\omega)$ and $f_2(\omega)$ are functions analytic in $\mathbb{D}$. A function f_1 is subordinate to f_2 in $\mathbb{D}$, for the existence of an analytic function $\varphi(\omega)$ in $\mathbb{D}$ with the condition

$$\varphi(0) = 0, \quad |\varphi(\omega)| < 1 \quad (\omega \in \mathbb{D}),$$

such that

$$f_1(\omega) = f_2(\varphi(\omega)) \qquad (\omega \in \mathbb{D}).$$

This subordination is denoted by $f_1(\omega) \prec f_2(\omega)$. Further, if the function $f_2(\omega)$ is univalent in $\mathbb{D}$, then $f_1(\omega) \prec f_2(\omega) \quad (\omega \in \mathbb{D}) \Longleftrightarrow f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$.

Using the extended fractional q-derivative operator $D^{\tau}_{q,z}$, which was recently investigated in [9], a fractional q-differintegral operator $\Omega^{\tau}_{q,z} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as below:

$$\begin{aligned}\Omega^{\tau}_{q,z} f(\omega) &= \frac{\Gamma_q(2-\tau)}{\Gamma_q(2)}\,\omega^{\tau}\, D^{\tau}_{q,z}\, f(\omega)\\ &= \omega + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\tau)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\tau)}\, a_n\omega^n,\end{aligned} \tag{1.3}$$

$$(\tau < 2; \quad 0 < q < 1; \; \omega \in \mathbb{D}),$$

where $D^{\tau}_{q,z} f(\omega)$ in (1.3) denotes, a fractional q-integral of the function $f(\omega)$ of order τ when $-\infty < \tau < 0$ (Refer: [18]) and a fractional q-derivative of the function $f(\omega)$ of order τ when $0 \le \tau < 2$. Note $\Omega^{0}_{q,z} f(\omega) = f(\omega)$.

The linear multiplier fractional q-differintegral operator $\mathcal{D}^{\tau,m}_{q,p,\lambda}$ determined recently by Selvakumaran et al. [23] as follows:

$$\begin{aligned}\mathcal{D}^{\tau,0}_{q,\lambda} f(\omega) &= f(\omega),\\ \mathcal{D}^{\tau,1}_{q,\lambda} f(\omega) &= (1-\lambda)\Omega^{\tau}_{q,z} f(\omega)\\ &\quad + \lambda\,\omega\, D_q(\Omega^{\tau}_{q,z} f(\omega)),\ (\lambda \ge 0),\\ \mathcal{D}^{\tau,2}_{q,\lambda} f(\omega) &= \mathcal{D}^{\tau,1}_{q,\lambda}(\mathcal{D}^{\tau,1}_{q,\lambda} f(\omega)),\\ &\;\;\vdots\\ \mathcal{D}^{\tau,m}_{q,\lambda} f(\omega) &= \mathcal{D}^{\tau,1}_{q,\lambda}(\mathcal{D}^{\tau,m-1}_{q,\lambda} f(\omega)), \quad m \in \mathbb{N}.\end{aligned} \tag{1.4}$$

If $f(\omega)$ is of the form (1.2), by (1.4) we have

$$D^{\tau,m}_{q,\lambda} f(\omega) = \omega + \sum_{n=2}^{\infty} \left(\frac{\Gamma_q(2-\tau)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\tau)} \left[1-\lambda+[n]_q\lambda\right] \right)^m a_n\omega^n.$$

More simply,

$$\mathcal{D}^{\tau,m}_{q,\lambda} f(\omega) = \omega + \sum_{n=2}^{\infty} \Xi^{\tau,m}_{q,\lambda}(n) a_n\omega^n,$$

where $\Xi^{\tau,m}_{q,\lambda}(n)$
$= \left(\frac{\Gamma_q(2-\tau)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\tau)}\left[1-\lambda+[n]_q\lambda\right]\right)^m$.
The operator $\mathcal{D}^{\tau,m}_{q,\lambda}$ reduces to many new and known differential and integral operators. Particularly, when $q \to 1^-$ and $\tau = 0$ the operator $\mathcal{D}^{\tau,m}_{q,\lambda}$ becomes an operator introduced in [19] and if $q \to 1^-$, $\tau = 0$ and $\lambda = 1$ it becomes an operator introduced by Sălăgean [22].

Let the subclass of functions ϱ be $\mathcal{P}$ that are univalent and analytic in $\mathbb{D}$ and for which $\varrho(\mathbb{D})$ is convex and $\varrho(0) = 1$, $\Re(\varrho(\omega)) > 0$ for $\omega \in \mathbb{D}$.

Definition 1.1. A function $f \in \mathcal{A}$ is called the class $\mathcal{S}^{\tau,m}_{q,\lambda,\gamma}(\varrho)$ provided that

$$1 + \frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}^{\tau,m}_{q,\lambda} f(\omega))}{\mathcal{D}^{\tau,m}_{q,\lambda} f(\omega)} - 1\right) \prec \varrho(\omega),$$
$$(\gamma \in \mathbb{C} \setminus \{0\}, \quad \varrho \in \mathcal{P}). \tag{1.5}$$

Definition 1.2. A function $f \in \mathcal{A}$ is called the class $C^{\tau,m}_{q,\lambda,\gamma}(\varrho)$ provided that

$$1 + \frac{1}{\gamma}\left(\frac{D_q\big(\omega D_q(\mathcal{D}^{\tau,m}_{q,\lambda} f(\omega))\big)}{D_q(\mathcal{D}^{\tau,m}_{q,\lambda} f(\omega))} - 1\right) \prec \varrho(\omega),$$
$$(\gamma \in \mathbb{C} \setminus \{0\}, \quad \varrho \in \mathcal{P}). \tag{1.6}$$

We can easily verify the following:

(i) If $m = 0$, then $\mathcal{S}^{\tau,0}_{q,\lambda,\gamma}(\varrho) = \mathcal{S}_{q,\gamma}(\varrho)$ and $C^{\tau,0}_{q,\lambda,\gamma}(\varrho) = C_{q,\gamma}(\varrho)$, $(\gamma \in \mathbb{C} \setminus \{0\})$ (Seoudy and Aouf [24]).

(ii) If $m = 0$ and $q \to 1^-$, then $\mathcal{S}^{\tau,0}_{q,\lambda,\gamma}(\varrho) = \mathcal{S}_\gamma(\varrho)$ and $C^{\tau,0}_{q,\lambda,\gamma}(\varrho) = C_\gamma(\varrho)$, $(\gamma \in \mathbb{C}\setminus\{0\})$ (Ravichandran et al. [21]).

(iii) If $m = 0$, $\gamma = 1$ and $q \to 1^-$, then $\mathcal{S}^{\tau,0}_{q,\lambda,1}(\varrho) = \mathcal{S}^*(\varrho)$ and $C^{\tau,0}_{q,\lambda,1}(\varrho) = C(\varrho)$ (Ma and Minda [15]).

(iv) If $m = 0$, $\varrho(\omega) = \frac{1+(1-2\alpha)z}{1-z}$ and $q \to 1^-$, then $\mathcal{S}^{\tau,0}_{q,\lambda,\gamma}(\varrho) = \mathcal{S}^*_\alpha(\gamma)$ and $C^{\tau,0}_{q,\lambda,\gamma}(\varrho) = C_\alpha(\gamma)$, $(\gamma \in \mathbb{C} \setminus \{0\}, 0 \le \alpha < 1)$(Frasin [20]).

(v) If $m = 0$, $\varrho(\omega) = \frac{1+z}{1-z}$ and $q \to 1^-$, then $\mathcal{S}^{\tau,0}_{q,\lambda,\gamma}(\varrho) = \mathcal{S}^*(\gamma)$ and $C^{\tau,0}_{q,\lambda,\gamma}(\varrho) = C(\gamma)$, $(\gamma \in \mathbb{C} \setminus \{0\})$ (Nasr and Aouf [16, 17]).

By making use of the following lemmas we are establishing our primary outcomes.

Lemma 1.3. *[15] If $g_1(\xi) = 1+u_1\xi+u_2\xi^2+\cdots$ is a function having positive real part in $\mathbb{D}$ and μ is a complex number, then we get*

$$|u_2 - \mu u_1^2| \le 2\max\{1; |2\mu - 1|\}.$$

For the functions $g_1(\omega) = (1+\xi^2)/(1-\xi^2)$ and $g_1(\omega) = (1+\xi)/(1-\xi)$ the result is sharp.

Lemma 1.4. *[15] If $g_1(\xi) = 1+u_1\xi+u_2\xi^2+\cdots$ is an analytic function having positive real part in $\mathbb{D}$, thus*

$$|u_2 - \alpha u_1^2| \le \begin{cases} -4\alpha + 2 & \text{if} \quad \alpha \le 0, \\ 2 & \text{if} \quad 0 \le \alpha \le 1, \\ 4\alpha - 2 & \text{if} \quad \alpha \ge 1. \end{cases}$$

When $\alpha < 0$ or $\alpha > 1$, the equality exists iff $g_1(\xi)$ is $(1+\xi)/(1-\xi)$ or one of its rotations. If $0 < \alpha < 1$, the equality exists iff $g_1(\xi)$ is $(1+\xi^2)/(1-\xi^2)$ or one of its rotations. If $\alpha = 0$, the equality exists iff

$$g_1(\xi) = \left(\frac{1+\gamma}{2}\right)\frac{1+\xi}{1-\xi} + \left(\frac{1-\gamma}{2}\right)\frac{1-\xi}{1+\xi}$$
$$(0 \le \gamma \le 1)$$

or one of its rotations. If $\alpha = 1$, the equality exists iff $g_1(\omega)$ is the reciprocal of one of the functions such that the equality holds for $\alpha = 0$. The upper bound mentioned above *is also sharp, and it can be enhanced as follows: when* $0 < \alpha < 1$*,*

$$|u_2 - \alpha u_1^2| + \alpha |u_1|^2 \le 2 \quad (0 < \alpha \le 1/2)$$

and

$$|u_2 - \alpha u_1^2| + (1-\alpha)|u_1|^2 \le 2 \quad (1/2 < \alpha \le 1).$$

2. Main Results

Theorem 2.1. *Suppose* $\varrho(\omega) = 1 + H_1\omega + H_2\omega^2 + H_3\omega^3 + \cdots$ *with* $H_1 \neq 0$*. If the function* $f(\omega)$ *determined by (1.2) belongs to* $\mathcal{S}_{q,\lambda,\gamma}^{\tau,m}(\varrho)$*, then we get*

$$|a_3 - \mu a_2^2| \le \frac{|\gamma H_1|}{q\,(1+q)|\Xi_{q,\lambda}^{\tau,m}(3)|} \max\left(1; \left|\frac{H_2}{H_1} + \frac{\gamma H_1}{q}\left\{1 - \frac{(1+q)\Xi_{q,\lambda}^{\tau,m}(3)}{(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right\}\right|\right).$$

The result is sharp.

Proof. Let

$$u(\omega) := 1 + \frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega))}{\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega)} - 1\right) = 1 + b_1\omega + b_2\omega^2 + \cdots. \tag{2.1}$$

If $f \in \mathcal{S}_{q,\lambda,\gamma}^{\tau,m}(\varrho)$, there exist a Schwarz function $\xi(\omega)$, that is analytic in $\mathbb{D}$ with $|\xi(\omega)| < 1$ and $\xi(0) = 0$ in $\mathbb{D}$, in such a way that

$$1 + \frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega))}{\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega)} - 1\right) = \varrho(\xi(\omega)).$$

Since the function $\varrho(\omega)$ is univalent and $p \prec \varrho$, then the function

$$u_1(\omega) = \frac{1 + \varrho^{-1}(u(\omega))}{1 - \varrho^{-1}(u(\omega))} = \frac{1 + \xi(\omega)}{1 - \xi(\omega)} = 1 + c_1\omega + c_2\omega^2 + \cdots$$

is the analytic function having positive real part in $\mathbb{D}$. Hence we have,

$$u(\omega) = \varrho(\xi(\omega)) = \varrho\left(\frac{u_1(\omega)-1}{u_1(\omega)+1}\right) = 1 + H_1\left(\frac{u_1(\omega)-1}{u_1(\omega)+1}\right) + H_2\left(\frac{u_1(\omega)-1}{u_1(\omega)+1}\right)^2 + \cdots$$
$$= 1 + \frac{H_1 c_1}{2}\omega + \left[\frac{H_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{H_2 c_1^2}{4}\right]\omega^2 + \cdots. \tag{2.2}$$

From equations (2.1) and (2.2), we get $b_1 = \frac{H_1 c_1}{2}$ and $b_2 = \frac{H_1}{2}(c_2 - \frac{c_1^2}{2}) + \frac{H_2 c_1^2}{4}$. On the other hand, since

$$\frac{\omega D_q(\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega))}{\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega)} = 1 + q\,\Xi_{q,\lambda}^{\tau,m}(2)a_2\omega + [q\,(1+q)\Xi_{q,\lambda}^{\tau,m}(3)a_3 - q\,(\Xi_{q,\lambda}^{\tau,m}(2))^2 a_2^2]\omega^2 + \cdots,$$

from equation (2.1), we have

$$\gamma b_1 = q\,\Xi_{q,\lambda}^{\tau,m}(2)a_2$$
$$\gamma b_2 = q\,(1+q)\Xi_{q,\lambda}^{\tau,m}(3)a_3 - q\,(\Xi_{q,\lambda}^{\tau,m}(2))^2 a_2^2,$$

or equivalently,

$$a_2 = \frac{\gamma b_1}{q\,\Xi_{q,\lambda}^{\tau,m}(2)} = \frac{\gamma H_1 c_1}{2q\,\Xi_{q,\lambda}^{\tau,m}(2)},$$
$$a_3 = \frac{1}{q\,(1+q)\Xi_{q,\lambda}^{\tau,m}(3)}\left[\gamma b_2 + \frac{\gamma^2 b_1^2}{q}\right] = \frac{\gamma H_1}{2q\,(1+q)\Xi_{q,\lambda}^{\tau,m}(3)}\left[c_2 - \frac{1}{2}\left(1 - \frac{H_2}{H_1} - \frac{\gamma H_1}{q}\right)c_1^2\right].$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{\gamma H_1}{2q\,(1+q)\Xi_{q,\lambda}^{\tau,m}(3)}[c_2 - \upsilon c_1^2], \tag{2.3}$$

where

$$\upsilon = \frac{1}{2}\left[1 - \frac{H_2}{H_1} - \frac{\gamma H_1}{q} + \frac{\gamma H_1 (1+q)\Xi_{q,\lambda}^{\tau,m}(3)}{q\,(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right]. \tag{2.4}$$

Thus, the result following the application of Lemma 1.3 and we observe that the functions defined by

$$1+\frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega))}{\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega)}-1\right)=\varrho(\omega^2)$$

and

$$1+\frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega))}{\mathcal{D}_{q,\lambda}^{\tau,m} f(\omega)}-1\right)=\varrho(\omega).$$

This completes the proof. □

For the function class $C_{q,\lambda,\gamma}^{\tau,m}(\varrho)$ we have the accompanying outcome. The proof of the Theorem 2.2 is neglected as it is same to that of Theorem 2.1.

Theorem 2.2. *Suppose $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1\neq 0$. If $f(\omega)$ defined by (1.2) is in $C_{q,\lambda,\gamma}^{\tau,m}(\varrho)$, then we get*

$$|a_3-\mu a_2^2|\leq \frac{|\gamma H_1|}{q\,[3]_q!|\Xi_{q,\lambda}^{\tau,m}(3)|}\times$$
$$\max\left\{1;\left|\ \frac{H_2}{H_1}+\frac{\gamma H_1}{q}\left(1-\frac{(1+q+q^2)\Xi_{q,\lambda}^{\tau,m}(3)}{(1+q)(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right)\ \right|\right\}.$$

The result is sharp.

Substituting $m=0$ in Theorem 2.1, we get the inequality for the function class $\mathcal{S}_{q,\gamma}(\varrho)$.

Corollary 2.3. *[24] Let the function $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1\neq 0$. If $f(\omega)\in\mathcal{S}_{q,\gamma}(\varrho)$, then*

$$|a_3-\mu a_2^2|\leq \frac{|\gamma H_1|}{q\,(1+q)}\times$$
$$\max\left\{1;\left|\frac{H_2}{H_1}+\frac{\gamma H_1}{q}\big(1-(1+q)\mu\big)\right|\right\}.$$

The inequality is sharp.

Taking $q\to 1^-$ in the Corollary 2.3, we get the inequality for the function class $\mathcal{S}_\gamma(\varrho)$ which enhances the result of ([21], Theorem 4.1).

Corollary 2.4. *Suppose $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1\neq 0$. If $f(\omega)\in\mathcal{S}_\gamma(\varrho)$, then we get*

$$|a_3-\mu a_2^2|$$
$$\leq \frac{|\gamma H_1|}{2}\max\left\{1;\left|\frac{H_2}{H_1}+(1-2\mu)\,\gamma H_1\right|\right\}.$$

The inequality is sharp.

Substituting $m=0$ in Theorem 2.2, we get the accompanying corollary.

Corollary 2.5. *[24] Suppose $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1\neq 0$. If $f(\omega)\in C_{q,\gamma}(\varrho)$, then we get*

$$|a_3-\mu a_2^2|\leq \frac{|\gamma H_1|}{q\,[3]_q!}$$
$$\times\max\left\{1;\left|\frac{H_2}{H_1}+\frac{\gamma H_1}{q}\left(1-\frac{(1+q+q^2)}{(1+q)}\mu\right)\right|\right\}.$$

The inequality is sharp.

Letting $q\to 1^-$ in Corollary 2.5, we get the result for the function class $C_\gamma(\varrho)$.

Corollary 2.6. *Let the function $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1\neq 0$. If $f(\omega)\in C_\gamma(\varrho)$, then we get*

$$|a_3-\mu a_2^2|$$
$$\leq \frac{|\gamma H_1|}{6}\max\left\{1;\left|\ \frac{H_2}{H_1}+\left(1-\tfrac{3}{2}\mu\right)\gamma H_1\ \right|\right\}.$$

The inequality is sharp.

Theorem 2.7. *Let the function $\varrho(\omega)=1+H_1\omega+H_2\omega^2+H_3\omega^3+\cdots$ with $H_1>0$ and $H_2\geq 0$. Consider*

$$\sigma_1=\frac{(\Xi_{q,\lambda}^{\tau,m}(2))^2}{\Xi_{q,\lambda}^{\tau,m}(3)}\left[\frac{q(H_2-H_1)+\gamma H_1^2}{(1+q)\gamma H_1^2}\right],$$

$$\sigma_2 = \frac{(\Xi^{\tau,m}_{q,\lambda}(2))^2}{\Xi^{\tau,m}_{q,\lambda}(3)}\left[\frac{q(H_2+H_1)+\gamma H_1^2}{(1+q)\gamma H_1^2}\right],$$
$$\sigma_3 = \frac{(\Xi^{\tau,m}_{q,\lambda}(2))^2}{\Xi^{\tau,m}_{q,\lambda}(3)}\left[\frac{qH_2+\gamma H_1^2}{(1+q)\gamma H_1^2}\right].$$

If $f(\omega)$ given by (1.2) belongs to $\mathcal{S}^{\tau,m}_{q,\lambda,\gamma}(\varrho)$ with $\gamma > 0$, then

$$|a_3-\mu a_2^2| \le \frac{\gamma H_1}{q(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}\times \begin{cases} \frac{H_2}{H_1}+\frac{\gamma H_1}{q}\left(1-\frac{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}{(\Xi^{\tau,m}_{q,\lambda}(2))^2}\mu\right), & \text{if } \mu\le\sigma_1 \\ 1, & \text{if } \sigma_1\le\mu\le\sigma_2 \\ -\frac{H_2}{H_1}-\frac{\gamma H_1}{q}\left(1-\frac{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}{(\Xi^{\tau,m}_{q,\lambda}(2))^2}\mu\right), & \text{if } \mu\ge\sigma_2 \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{q(\Xi^{\tau,m}_{q,\lambda}(2))^2}{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)\gamma H_1^2}\times\left[H_1 - H_2 - \frac{\gamma H_1^2}{q}\left(1-\frac{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}{(\Xi^{\tau,m}_{q,\lambda}(2))^2}\mu\right)\right]|a_2|^2 \le \frac{\gamma H_1}{q(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}$$

and if $\sigma_3 \le \mu \le \sigma_2$, then we get

$$|a_3-\mu a_2^2| + \frac{q(\Xi^{\tau,m}_{q,\lambda}(2))^2}{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)\gamma H_1^2}\times\left[H_1 + H_2 + \frac{\gamma H_1^2}{q}\left(1-\frac{(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}{(\Xi^{\tau,m}_{q,\lambda}(2))^2}\mu\right)\right]|a_2|^2 \le \frac{\gamma H_1}{q(1+q)\Xi^{\tau,m}_{q,\lambda}(3)}.$$

The result is sharp.

Proof. Using Lemma 1.4 to the equations (2.3) and (2.4), we can demonstrate our outcomes. To show the sharpness of the bounds, we define $\mathcal{F}_{\varrho,n}$ ($n = 2, 3, \ldots$) by

$$1+\frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{F}_{\varrho,n}(\omega))}{\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{F}_{\varrho,n}(\omega)}-1\right) = \varrho(\omega^{n-1}),$$
$$\mathcal{F}_{\varrho,n}(0) = 0 = \mathcal{F}'_{\varrho,n}(0)-1$$

and the functions $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda$ ($0 \le \lambda \le 1$) by

$$1+\frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{G}_\lambda(\omega))}{\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{G}_\lambda(\omega)}-1\right) = \varrho\left(\frac{\omega(\omega+\lambda)}{1+\lambda z}\right),$$
$$\mathcal{G}_\lambda(0) = 0 = \mathcal{G}'_\lambda(0)-1$$

and $1+\frac{1}{\gamma}\left(\frac{\omega D_q(\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{H}_\lambda(\omega))}{\mathcal{D}^{\tau,m}_{q,\lambda}\mathcal{H}_\lambda(\omega)}-1\right) = \varrho\left(-\frac{1+\lambda z}{\omega(\omega+\lambda)}\right)$,

$$\mathcal{H}_\lambda(0) = 0 = \mathcal{H}'_\lambda(0)-1.$$

Clearly, $\mathcal{F}_{\varrho,n}$, $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda \in \mathcal{S}^{\tau,m}_{q,\lambda,\gamma}(\varrho)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality exists iff f is $\mathcal{F}_{\varrho,2}$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality exists iff f is $\mathcal{F}_{\varrho,3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality exists iff f is $\mathcal{G}_\lambda$ or one of its rotations. If $\mu = \sigma_2$, then the equality exists iff f is $\mathcal{H}_\lambda$ or one of its rotations. □

In the same way, we can get the inequality for $C^{\tau,m}_{q,\lambda,\gamma}(\varrho)$.

Theorem 2.8. *Let the function $\varrho(\omega) = 1 + H_1\omega + H_2\omega^2 + H_3\omega^3 + \cdots$ with $H_1 > 0$ and $H_2 \ge 0$. Let*

$$\varepsilon_1 = \frac{(\Xi^{\tau,m}_{q,\lambda}(2))^2(1+q)^2}{\Xi^{\tau,m}_{q,\lambda}(3)}\left[\frac{q(H_2-H_1)+\gamma H_1^2}{[3]_q!\gamma H_1^2}\right],$$

$$\varepsilon_2 = \frac{(\Xi^{\tau,m}_{q,\lambda}(2))^2(1+q)^2}{\Xi^{\tau,m}_{q,\lambda}(3)}\left[\frac{q(H_2+H_1)+\gamma H_1^2}{[3]_q!\gamma H_1^2}\right], \textit{ and } .$$

$$\varepsilon_3 = \frac{(\Xi^{\tau,m}_{q,\lambda}(2))^2(1+q)^2}{\Xi^{\tau,m}_{q,\lambda}(3)}\left[\frac{qH_2+\gamma H_1^2}{[3]_q!\gamma H_1^2}\right]$$

If $f(\omega)$ given by (1.2) belongs to $C_{q,\lambda,\gamma}^{\tau,m}(\varrho)$ with $\gamma > 0$, then

$$|a_3 - \mu a_2^2| \leq \frac{\gamma H_1}{q[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)} \times \begin{cases} \frac{H_2}{H_1} + \frac{\gamma H_1}{q}\left(1 - \frac{[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)}{(1+q)^2(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right), & \text{if } \mu \leq \varepsilon_1 \\ 1, & \text{if } \varepsilon_1 \leq \mu \leq \varepsilon_2 \\ -\frac{H_2}{H_1} - \frac{\gamma H_1}{q}\left(1 - \frac{[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)}{(1+q)^2(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right), & \text{if } \mu \geq \varepsilon_2 \end{cases}$$

In addition, if $\varepsilon_1 \leq \mu \leq \varepsilon_3$, then we get

$$|a_3 - \mu a_2^2| + \frac{q(1+q)^2(\Xi_{q,\lambda}^{\tau,m}(2))^2}{[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)\gamma H_1^2}\left[H_1 - H_2 - \frac{\gamma H_1^2}{q}\left(1 - \frac{(1+q+q^2)\Xi_{q,\lambda}^{\tau,m}(3)}{(1+q)(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right)\right]|a_2|^2 \leq \frac{\gamma H_1}{q[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)}$$

and if $\varepsilon_3 \leq \mu \leq \varepsilon_2$, then we get

$$|a_3 - \mu a_2^2| + \frac{q(1+q)^2(\Xi_{q,\lambda}^{\tau,m}(2))^2}{[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)\gamma H_1^2}\left[H_1 + H_2 + \frac{\gamma H_1^2}{q}\left(1 - \frac{(1+q+q^2)\Xi_{q,\lambda}^{\tau,m}(3)}{(1+q)(\Xi_{q,\lambda}^{\tau,m}(2))^2}\mu\right)\right]|a_2|^2 \leq \frac{\gamma H_1}{q[3]_q!\Xi_{q,\lambda}^{\tau,m}(3)}.$$

The result is sharp.

Letting $q \to 1^-$ and $m = 0$ in Theorem 2.7, we get the accompanying outcome.

Corollary 2.9. *Let the function $\varrho(\omega) = 1 + H_1\omega + H_2\omega^2 + H_3\omega^3 + \cdots$ with $H_1 > 0$ and $H_2 \geq 0$. Consider*

$$\sigma_4 = \frac{(H_2 - H_1) + \gamma H_1^2}{2\gamma H_1^2},$$

$$\sigma_5 = \frac{(H_2 + H_1) + \gamma H_1^2}{2\gamma H_1^2}, \ \sigma_6 = \frac{H_2 + \gamma H_1^2}{2\gamma H_1^2}.$$

If $f(\omega) \in \mathcal{S}_\gamma(\varrho)$ with $\gamma > 0$, then we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma H_2}{2} + \frac{\gamma^2 H_1^2}{2}(1 - 2\mu), & \text{if } \mu \leq \sigma_4 \\ \frac{\gamma H_1}{2}, & \text{if } \sigma_4 \leq \mu \leq \sigma_5 \\ -\frac{\gamma H_2}{2} - \frac{\gamma^2 H_1^2}{2}(1 - 2\mu), & \text{if } \mu \geq \sigma_5 \end{cases}$$

In addition, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$|a_3 - \mu a_2^2| + \frac{1}{2\gamma H_1^2}\left[H_1 - H_2 - \gamma H_1^2(1 - 2\mu)\right]|a_2|^2 \leq \frac{\gamma H_1}{2}$$

and if $\sigma_6 \leq \mu \leq \sigma_5$, then

$$|a_3 - \mu a_2^2| + \frac{1}{2\gamma H_1^2}\left[H_1 + H_2 + \gamma H_1^2(1 - 2\mu)\right]|a_2|^2 \leq \frac{\gamma H_1}{2}.$$

The inequality is sharp.

Letting $q \to 1^-$ and $m = 0$ in Theorem 2.8, we get the accompanying outcome.

Corollary 2.10. *Suppose $\varrho(\omega) = 1 + H_1\omega + H_2\omega^2 + H_3\omega^3 + \cdots$ with $H_1 > 0$ and $H_2 \geq 0$. Let $\varepsilon_4 = \frac{2[(H_2 - H_1) + \gamma H_1^2]}{3\gamma H_1^2}$, $\varepsilon_5 = \frac{2[(H_2 + H_1) + \gamma H_1^2]}{3\gamma H_1^2}$, $\varepsilon_6 = \frac{2[H_2 + \gamma H_1^2]}{3\gamma H_1^2}$. If $f(\omega) \in C_\gamma(\varrho)$ with $\gamma > 0$, then we get*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma H_2}{6} + \frac{\gamma^2 H_1^2}{6}\left(1 - \frac{3\mu}{2}\right), & \text{if } \mu \leq \varepsilon_4 \\ \frac{\gamma H_1}{6}, & \text{if } \varepsilon_4 \leq \mu \leq \varepsilon_5 \\ -\frac{\gamma H_2}{6} - \frac{\gamma^2 H_1^2}{6}\left(1 - \frac{3\mu}{2}\right), & \text{if } \mu \geq \varepsilon_5 \end{cases}$$

In addition, if $\varepsilon_4 \leq \mu \leq \varepsilon_6$*, then we get*

$$|a_3 - \mu a_2^2| + \frac{2}{3\gamma H_1^2}\big[H_1 - H_2 - \gamma H_1^2\big(1 - \frac{3\mu}{2}\big)\big]|a_2|^2 \leq \frac{\gamma H_1}{6}$$

and if $\varepsilon_6 \leq \mu \leq \varepsilon_5$*, then we get*

$$|a_3 - \mu a_2^2| + \frac{2}{3\gamma H_1^2}\big[H_1 + H_2 + \gamma H_1^2\big(1 - \frac{3\mu}{2}\big)\big]|a_2|^2 \leq \frac{\gamma H_1}{6}.$$

The inequality is sharp.

3. Conclusion

By suitably choosing the parameters, one can further easily obtain a number of coefficient inequalities from the main results. Moreover, the classes $\mathcal{S}_{q,\lambda,\gamma}^{\tau,m}(\varrho)$, $C_{q,\lambda,\gamma}^{\tau,m}(\varrho)$ defined in this article can also be used in the investigation of various geometric properties in the unit disk.

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