P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia* Science & Technology Asia
Vol.27 No.3 July - September 2022 Page: [109-120]
Original research article

An Inertial Projection-Like Method for Solving a Generalized Nash Equilibrium Problem

Premyuda Dechboon[1,2], Poom Kumam[1,2,3,*], Parin Chaipunya[1,3]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand*
[2] *KMUTTFixed Point Research Laboratory, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand*
[3] *Center of Excellence in Theoretical and Computational Science, Science Laboratory Building, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand*

Received 1 November 2021; Received in revised form 8 January 2022
Accepted 9 January 2022; Available online 29 September 2022

ABSTRACT

In this paper, we propose an algorithm for solving the generalized Nash equilibrium for noncooperative games by means of the quasi-variational inequality. Incorporating the inertial steps to a projection-like method, we show the convergence of the generated sequence to the solution of a quasi-variational inequality, and hence the Nash equilibrium. We also implement the algorithm to some test problems, where the numerical experiment portrays that the convergence of our proposed algorithm is about twice as fast compared to the known projection-like method without the inertial steps.

Keywords: Convergence; Inertial method; Nash equilibrium problem; Projection-like method; Quasi-variational inequality

1. Introduction

Game theory is the study of competitions and how to equilibrate them, formulated in the theoretical framework. A noncooperative game depicts a situation where each of the involved competitors (called players) is not allowed to speak to other competitors and have control over their own decision. Of course, each of the player's losses is also affected by the choice of other players. The basic setup for a noncooperative game consists of the set of players, usually denoted by $N := \{1, 2, \cdots, n\}$, and the strategy space X_i of each player $i \in N$. The concept of a Nash equilibrium is the most common equilibrium concept for

***Corresponding author:** poom.kum@kmutt.ac.th
doi: 10.14456/scitechasia.2022.49

a noncooperative game, and we shall call the problem of seeking such equilibrium the Nash equilibrium problem. To solve a Nash equilibrium problem, each player is required to solve an optimization problem, and therefore, one may wish to exploit the method of convex analysis and optimization [1]. We may also impose the moving constraints into the model to capture the limitation of resources and the like. For this, each player is required to solve an optimization whose constraints also depend on the decisions of other players. The difficulty arises as each player's optimization problems are prescribed to be solved simultaneously and blindly. This gives a strong motivation that this generalized Nash equilibrium problem should be solved as a quasi-variational inequality problem (see [2]). There have been various results in this direction (see e.g. [3–7]).

Let us recall that the generalized Nash equilibrium can be formally stated as follows. Let $N = \{1, 2, \cdots, n\}$ be the set of players and let $X_i \subseteq H_i$ be the strategy set of player i, for $i \in N$. Here H_i denotes a Hilbert space. Let us define $X := \prod_{i=1}^{n} X_i$ and $X_{-i} := \prod_{j\in N\setminus\{i\}} X_i$ for each $i \in N$. If $x \in X$ and $i \in N$, we adopt the notations $x_i := (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$ and $(x_{-i}, x_i) := x$. Each player $i \in N$ is also equipped with the constraint map $K_i : X_{-i} \longrightarrow X_i$ and a cost function $f_i : X \longrightarrow (-\infty, +\infty]$. A generalized Nash equilibrium problem then amounts to each player $i \in N$ solve the following optimization problem:

$$\left.\begin{array}{ll} \min & f_i(x_{-i}, x_i) \\ \text{s.t.} & x_i \in K_i(x_{-i}). \end{array}\right\} \tag{1.1}$$

If all f_i's are convex and continuously differentiable and all K_i's have closed convex values, then $\overline{x}$ is a generalized Nash equilibrium if and only if $\overline{x} \in K(\overline{x})$ and

$$\langle F(\overline{x}), y - \overline{x}\rangle \geq 0 \text{ for all } y \in K(\overline{x}), \tag{1.2}$$

where $F(x) = (\nabla_{x_i} f_i(x))_{i\in N}$ for $x \in X$. The inequality Eq. (1.2) is known as a quasi-variational inequality problem (QVI). This approach provides an efficient computation method for solving a generalized Nash equilibrium through the QVI formulation. In particular, the class of projection methods has been studied for solving both convex optimization problems and monotone variational inequality problems (see [8, 9]). These methods require only a small storage, take advantage of any separable structure in constrained sets of the problems, and many constraints can be attached or removed from the active set at each iteration. In this way, many projection type methods were introduced, for example, the extragradient algorithm (see [10, 11]), the cutting hyperplane method [12], half-space projection method [13], and many more. In 2010, Zhang et al. [14] introduced a projection-like method for solving the generalized Nash equilibrium which involves initiating constants $\vartheta \in (0, 1)$, $\mu \in (0, 1)$, $\rho \in (0, 2)$ and generating from any $x_0 \in X$ the following sequence

$$\begin{cases} z_k = P_{K(x_k)}\left(x_k - F\left(x_k\right)\right) \\ y_k = \left(1 - \beta_k\right) x_k + \beta_k z_k \\ x_{k+1} = P_{K(x_k)}\left(x_k - \alpha_k d_k\right) \end{cases}$$

where α_k and d_k are given by

$$\alpha_k = \rho(1-\mu)\frac{\|x_k - z_k\|^2}{\|d_k\|^2}$$

and

$$d_k = x_k - z_k - \frac{F\left(y_k\right)}{\beta_k}.$$

Here, the parameter $\beta_k = \vartheta^{m_k}$ is computed by the following line search: find m_k is the smallest nonnegative integer m such that

$$\begin{aligned} \langle F\left(x_k\right) - \ F\left(\left(1 - \vartheta^m\right) x_k + \vartheta^m z_k\right), x_k - z_k\rangle \\ \leq \mu \left\|x_k - z_k\right\|^2. \end{aligned} \tag{1.3}$$

On the other hand, in 1964, Polyak [15] studied and developed an algorithm with the idea of increasing the speed of the convergence. The iterative method with an additional inertial step was then introduced. The inertial step improves the successive speed of the two-step algorithm by taking into account the memory of the previous two steps of each iteration. The algorithms with an inertial step are known as inertial-type algorithms. This concept has been applied in several methods such as the inertial proximal point algorithm [16], inertial extragradient algorithms [17], inertial forward-backward splitting methods [18], etc. Recently, in 2020, Shehu et al. [19] also established strong convergent results using an inertial projection-type method for solving quasi-variational inequalities in real Hilbert spaces.

In this paper, we consider an inertial projection-type method for solving the quasi-variational inequality formulated the generalized Nash equilibrium problem by taken into account the idea of Zhang et al. [14] in the setting of a Hilbert space. Our results extend the projection-like algorithms of [14] to an infinite-dimensional setting and the convergence analysis shows that the inertial modification still grants a strong convergence even though the dimension can be infinite. Our numerical experiment illustrates a significant improvement, showing that only half the number of iterations, compared to the known projection-like algorithm, is required to converge with the same tolerance.

The remaining parts of the paper are organized as follows: Section 2 consists of some definitions and tools utilized to prove the main results. After that, Section 3 provides the proof of convergence theorem with the proposed algorithm which its examples of the implementation as numerical results are included in Section 4. Lastly, Section 5, the summary of this work is briefly written.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $\mathcal{K}$ be a closed convex set in $\mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ be a continuous mapping. Given a set-valued mapping K defined by $u \mapsto K(u)$, which associates a closed convex set $K(u)$ of $\mathcal{H}$ with any element of $\mathcal{H}$. Recall that a problem finding $u \in \mathcal{H}$ such that $u \in K(u)$ and

$$\langle F(u), v - u \rangle \geq 0 \qquad (2.1)$$

for all $v \in K(u)$, where (2.1) is called quasi-variational inequality (QVI). Moreover, it can be observed that the unique nearest point in $\mathcal{K}$ from each element in $\mathcal{H}$ known as the orthogonal projection from $\mathcal{H}$ onto $\mathcal{K}$ is defined by

$$P_{\mathcal{K}}(x) = \operatorname{argmin} \{\|x - y\| \mid y \in \mathcal{K}\}$$

for any $x \in \mathcal{H}$. Then, it is acquired that

$$\|x - P_{\mathcal{K}}(x)\| \leq \|x - y\|$$

for any $y \in \mathcal{K}$. Furthermore, the projection mapping is nonexpansive, that is,

$$\|P_{\mathcal{K}}(x) - P_{\mathcal{K}}(y)\| \leq \|x - y\|$$

for all $x, y \in \mathcal{H}$.

Proposition 2.1 ([20]). *Let $\mathcal{H}$ be a real Hilbert space. The following properties hold, for any $x, y, z \in \mathcal{H}$ and $\lambda \in \mathbb{R}$,*

(a) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$

(b) $2\langle x - y, x - z \rangle = \|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2$

(c) $\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2$

Lemma 2.2 ([21]). *Let $\mathcal{K}$ be a nonempty closed convex subset in $\mathbb{R}^n$. Then a vector $z := P_{\mathcal{K}}(x)$ if and only if*

$$\langle x - z, z - y\rangle \geq 0$$

for all $y \in \mathcal{K}$.

The following statement gives a necessary and sufficient condition to be a solution of the QVI problem (2.1).

Lemma 2.3 ([14]). *A point x is a solution of the QVI problem (2.1) if and only if*

$$r_{K(x)}(x) := \left\| x - P_{K(x)}(x - F(x))\right\| = 0.$$

Lemma 2.4 ([22]). *Let $\{\phi_k\} \subseteq [0,\infty)$ and $\{\delta_k\} \subseteq [0,\infty)$. If the following conditions are satisfied:*

(a) $\phi_{k+1} - \phi_k \leq \theta_k (\phi_k - \phi_{k-1}) + \delta_k$

(b) $\sum_{k=1}^{\infty} \delta_k < +\infty$

(c) $\{\theta_k\} \subseteq [0,\theta]$ *and* $\theta \in [0,1)$

where $k \in \mathbb{N}$. Then $\{\phi_k\}$ is a convergent sequence and $\sum_{k=1}^{\infty}[\phi_{k+1} - \phi_k]_+ < +\infty$ where $[t]_+ := \max\{t,0\}$ for all $t \in \mathbb{R}$.

Now we give some concepts for the continuity of a set-valued mapping.

Definition 2.5. Let X be a Hilbert space and K a set-valued map from X to itself. The mapping K is said to be

(a) **weakly upper semicontinuous** at x_0 if for any $\{x_k\} \subseteq X$ such that $x_k \rightharpoonup x_0$ and for any $\{y_k\} \subseteq K(x_k)$ such that $y_k \rightharpoonup y_0$ imply that $y_0 \in K(x_0)$.

(b) **weakly lower semicontinuous** at x_0 if for any $\{x_k\} \subseteq X$ such that $x_k \rightharpoonup x_0$ implies that for each $y_0 \in K(x_0)$, there exists a sequence $\{y_k\} \subseteq K(x_k)$ such that $y_k \rightharpoonup y_0$.

(c) **weakly continuous at** x_0 if it is both weakly upper and weakly lower semicontinuous at x_0.

(d) **weakly continuous on** X if it is weakly continuous at every point of X.

Definition 2.6. Let $X \subseteq \mathcal{H}$ be a Hilbert space and $F : X \to X$ be a mapping. Then F is said to be

(a) **pseudo monotone on** X if for any $x, y \in X$, $\langle F(y), x - y\rangle \geq 0$ implies $\langle F(x), x - y\rangle \geq 0$.

(b) **monotone on** X if for any $x, y \in X$, $\langle F(x) - F(y), x - y\rangle \geq 0$.

Definition 2.7. Let $X \subseteq \mathcal{H}$ be a Hilbert space. For any $x \in X$, a mapping $F : X \to X$ is said to be

(a) **monotone at** x, if for any $y \in X$, $\langle F(y) - F(x), y - x\rangle \geq 0$.

(b) **strictly monotone at** x, if for any $y \in X$, $\langle F(y) - F(x), y - x\rangle > 0$, whenever $x \neq y$.

Lemma 2.8 ([14]). *Let $x \in X$ be arbitrary. For any $\alpha \in (0,1)$, define*

$$z = P_{K(x)}(x - F(x)),\ y(\alpha) = (1-\alpha)x + \alpha z.$$

Then for any given $\mu \in (0,1)$, when $\alpha > 0$ is sufficiently small, we have

$$\langle F(x) - F(y(\alpha)), x - z\rangle \leq \mu \|x - z\|^2 .$$

3. Main results

In this section, the convergence result is provided through some lemmas. By the way, some assumptions are required to prove the main theorem. First we denote the notation

$$S^* := \left\{ x \in \bigcap_{x \in X} K(x) \mid \langle F(x), y - x\rangle \geq 0 \right.$$

$$\left.\text{for all } y \in \bigcup_{x\in X} K(x)\right\}.$$

We suppose the following assumptions:

(A1) S^* is nonempty.

(A2) $F(\cdot)$ is pseudo monotone on X.

(A3) For any $x \in X$, $x \in K(x)$.

(A4) $K(\cdot)$ is weakly continuous on X.

By the previous motivation of an inertial step with projection-like method for solving the generalized Nash equilibrium, we propose the following algorithm.

Algorithm 1 An Inertial Projection-Like Method

procedure Find: x_{k+1}.
Initialization: $\mu \in (0,1)$, $\vartheta \in (0,1)$, $\rho \in (0,2)$, Tol $\to 0$.
Take: $x_0, x_1 \in X$ and $k=1$.
repeat Set $w_k = x_k + \gamma_k (x_k - x_{k-1})$ where $0 \le \gamma_k \le \overline{\gamma}_k$ such that

$$\overline{\gamma}_k = \begin{cases} \min\left\{c, \frac{\xi_k}{\|x_k - x_{k-1}\|^2}\right\} & \text{if } x_k \neq x_{k-1}, \\ c & \text{if } x_k = x_{k-1} \end{cases}$$

when $\xi_k \in [0,\infty)$ such that $\sum_{k=1}^{\infty} \xi_k < +\infty$ and $c \in [0,1)$.
if $r_{K(w_k)}(w_k) = 0$ **then** Stop
end if Set

$$z_k = P_{K(w_k)}(w_k - F(w_k))$$

and

$$y_k = (1-\beta_k) w_k + \beta_k z_k$$

where $\beta_k = \vartheta^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\langle F(w_k) - F((1-\vartheta^m) w_k + \vartheta^m z_k), w_k - z_k\rangle \le \mu \|w_k - z_k\|^2. \quad (3.1)$$

Set $x_{k+1} = P_{K(w_k)}(w_k - \alpha_k d_k)$ where α_k and d_k are given by

$$\alpha_k = \rho(1-\mu)\frac{\|w_k - z_k\|^2}{\|d_k\|^2} \quad \text{and} \quad d_k = w_k - z_k - \frac{F(y_k)}{\beta_k}.$$

until $r_{K(x_k)}(x_k) <$ Tol
end procedure

By Lemma 2.8, it is easy to show that the sequences generated by the Algorithm 1 are satisfied this lemma also. With the purpose of proving the feasibility of Algorithm 1, it is enough to prove the following lemma.

Lemma 3.1. *Suppose that the assumptions (A1)-(A3) hold. If $r_{K(w_k)}(w_k) \neq 0$ then $d_k \neq 0$.*

Proof. By the assumption (A1), we can let $x^* \in S^*$. Moreover, by the assumption (A3), we have $w_k \in K(w_k)$ and $y_k \in K(y_k)$ which implies that

$$\langle F(x^*), w_k - x^*\rangle \ge 0$$

and

$$\langle F(x^*), y_k - x^*\rangle \ge 0.$$

By the pseudo monotonicity, we obtain that

$$\langle F(w_k), w_k - x^*\rangle \ge 0 \quad (3.2)$$

and

$$\langle F(y_k), y_k - x^*\rangle \ge 0. \quad (3.3)$$

Since $x^* \in K(w_k)$ and from the fact that $z_k = P_{K(w_k)}(w_k - F(w_k))$, by Lemma 2.2, then we have

$$\langle w_k - z_k - F(w_k), z_k - x^*\rangle \ge 0. \quad (3.4)$$

Observe that, since $y_k = (1-\beta_k) w_k + \beta_k z_k$, we can simplify to be

$$w_k - y_k = \beta_k (w_k - z_k) \quad (3.5)$$

From (3.2), (3.3), (3.15) and (3.5), it follows that

$$\begin{aligned} &\langle d_k, w_k - x^*\rangle \\ &\quad = \langle w_k - z_k, w_k - x^*\rangle + \left\langle \frac{F(y_k)}{\beta_k}, w_k - x^*\right\rangle \\ &\quad \ge \langle w_k - z_k - F(w_k), w_k - x^*\rangle \end{aligned}$$

$$
\begin{aligned}
&+\left\langle \frac{F(y_k)}{\beta_k}, w_k - x^* \right\rangle \\
\geq\ & \langle w_k - z_k - F(w_k), w_k - z_k \rangle \\
&+\left\langle \frac{F(y_k)}{\beta_k}, w_k - x^* \right\rangle \\
\geq\ & \langle w_k - z_k - F(w_k), w_k - z_k \rangle \\
&+\frac{1}{\beta_k} \langle F(y_k), w_k - y_k \rangle \\
=\ & \langle w_k - z_k - F(w_k), w_k - z_k \rangle \\
&+\langle F(y_k), w_k - z_k \rangle \\
\geq\ & \|w_k - z_k\|^2 - \mu \|w_k - z_k\|^2 \\
=\ & (1-\mu) \|w_k - z_k\|^2 .
\end{aligned}
$$

That is,

$$\langle d_k, w_k - x^* \rangle \geq (1-\mu) \|w_k - z_k\|^2 . \tag{3.6}$$

Since $r_{K(w_k)}(w_k) = \left\|w_k - P_{K(w_k)}(w_k - F(w_k))\right\|$ and $r_{K(w_k)}(w_k) \neq 0$ then

$$w_k \neq P_{K(w_k)}(w_k - F(w_k)) = z_k .$$

That is, $\|w_k - z_k\| \neq 0$. Since $\mu \in (0,1)$, by (3.6), thus $\langle d_k, w_k - x^* \rangle > 0$. Hence, $d_k \neq 0$. □

To obtain the main theorems, it is important to demonstrate the boundedness of a sequence $\{x_k\}$ generated by Algorithm 1 because later it is needed to explain the existence of a subsequence. Moreover, when $\{x_k\}$ is bounded, some sequences related with $\{x_k\}$ can be bounded as well.

Lemma 3.2. *Suppose that the assumptions (A1)-(A3) hold. The sequence $\{x_k\}$ generated by Algorithm 1 is bounded.*

Proof. Let $x^* \in S^*$. Observe that

$$\rho(2-\rho)(1-\mu)^2 \frac{\|w_k - z_k\|^4}{\|d_k\|^2} \geq 0. \tag{3.7}$$

Then, by Proposition 2.1, Lemma 3.1 and (3.7), we have

$$
\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
=\ & \left\|P_{K(w_k)}(w_k - \alpha_k d_k) - x^*\right\|^2 \\
=\ & \left\|P_{K(w_k)}(w_k - \alpha_k d_k) - P_{K(w_k)}(x^*)\right\|^2 \\
\leq\ & \|(w_k - \alpha_k d_k) - x^*\|^2 \\
=\ & \|w_k - x^*\|^2 - 2\alpha_k \langle d_k, w_k - x^* \rangle \\
&+\alpha_k^2 \|d_k\|^2 \\
\leq\ & \|w_k - x^*\|^2 - 2\alpha_k (1-\mu) \|w_k - z_k\|^2 \\
&+\alpha_k^2 \|d_k\|^2 \\
=\ & \|x_k + \gamma_k (x_k - x_{k-1}) - x^*\|^2 \\
&-\rho(2-\rho)(1-\mu)^2 \frac{\|w_k - z_k\|^4}{\|d_k\|^2} \qquad (3.8) \\
\leq\ & \|x_k + \gamma_k (x_k - x_{k-1}) - x^*\|^2 \\
\leq\ & \|x_k - x^*\|^2 \\
&+\gamma_k \left(\|x_k - x^*\|^2 - \|x^* - x_{k-1}\|^2 \right) \\
&+2\gamma_k \|x_k - x_{k-1}\|^2 .
\end{aligned}
$$

It can be concluded that

$$
\begin{aligned}
&\|x_{k+1} - x^*\|^2 - \|x_k - x^*\|^2 \\
&\quad \leq \gamma_k \left(\|x_k - x^*\|^2 - \|x^* - x_{k-1}\|^2 \right) \\
&\qquad +2\gamma_k \|x_k - x_{k-1}\|^2 .
\end{aligned}
$$

Recall that $\gamma_k \leq \overline{\gamma}_k \leq \frac{\xi_k}{\|x_k - x_{k-1}\|^2}$ for all $k \in \mathbb{N}$ such that $x_k \neq x_{k-1}$. Hence,

$$\gamma_k \|x_k - x_{k-1}\|^2 \leq \xi_k$$

for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} \xi_k < +\infty$, then $\sum_{k=1}^{\infty} \gamma_k \|x_k - x_{k-1}\|^2 < +\infty$. From Lemma 2.4, we then get that $\left\{\|x_k - x^*\|^2\right\}$ is a convergent sequence. It implies the boundedness of $\{\|x_k - x^*\|\}$, that is, $\{x_k\}$ is bounded. □

Lemma 3.3. *Suppose that the assumptions (A1)-(A3) hold, then*

$$\lim_{k\to\infty} \frac{\|w_k - z_k\|^2}{\|d_k\|} = 0.$$

Proof. By Proposition 2.1 and (3.8), we have that

$$\rho(2-\rho)(1-\mu)^2 \frac{\|w_k - z_k\|^4}{\|d_k\|^2}$$

$$
\begin{aligned}
&\leq \|x_k + \gamma_k (x_k - x_{k-1}) - x^*\|^2 \\
&\quad - \|x_{k+1} - x^*\|^2 \\
&= \|x_k - x^*\|^2 + 2\gamma_k \langle x_k - x^*, x_k - x_{k-1}\rangle \\
&\quad +\gamma_k^2 \|x_k - x_{k-1}\|^2 - \|x_{k+1} - x^*\|^2 \\
&\leq \|x_k - x^*\|^2 + \gamma_k \Big(\|x_k - x^*\|^2 \\
&\quad + \|x_k - x_{k-1}\|^2 - \|x_{k-1} - x^*\|^2\Big) \\
&\quad +\gamma_k \|x_k - x_{k-1}\|^2 - \|x_{k+1} - x^*\|^2 \\
&\leq \|x_k - x^*\|^2 + \gamma_k \|x_k - x^*\|^2 \\
&\quad -\gamma_k \|x_{k-1} - x^*\|^2 + 2\xi_k \\
&\quad - \|x_{k+1} - x^*\|^2
\end{aligned}
$$

By Lemma 3.2 and $\sum_{k=1}^{\infty} \xi_k < +\infty$, then $\sum_{k=1}^{\infty} \frac{\|w_k - z_k\|^2}{\|d_k\|} < +\infty$ which implies that

$$\lim_{k\to\infty} \frac{\|w_k - z_k\|^2}{\|d_k\|} = 0.$$

□

Theorem 3.4. *Suppose that the assumptions (A1)-(A4) hold. Any accumulation point of a sequence $\{x_k\}$ generated by Algorithm 1 is a solution of the quasi-variational inequality problem (*).*

Proof. Let $x^* \in S^*$. We can obtain that

$$
\begin{aligned}
\|w_k\| &= \|x_k + \gamma_k (x_k - x_{k-1})\| \\
&\leq \|x_k\| + \gamma_k (\|x_k\| + \|x_{k-1}\|),
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_k - x^*\| &= \left\|P_{K(w_k)} (w_k - F(w_k)) - x^*\right\| \\
&\leq \|w_k\| + \|F(w_k)\| + \|x^*\|.
\end{aligned}
$$

Since F is continuous and $\{x_k\}$ is bounded, then $\{w_k\}$ and $\{z_k\}$ are also bounded. In the same way, the boundedness of $\{y_k\}$ and $\{F(y_k)\}$ are held. Now consider $\|x_k - w_k\|$, we have

$$
\begin{aligned}
\|x_k - w_k\|^2 &= \|x_k - x_k - \gamma_k (x_k - x_{k-1})\|^2 \\
&\leq \gamma_k \|x_k - x_{k-1}\|^2 \\
&\leq \xi_k.
\end{aligned}
$$

Since $\sum_{k=1}^{\infty} \xi_k$ converges then $\lim_{k\to\infty} \|x_k - w_k\|^2 = 0$. From $\|x_k - w_k\| \geq 0$ for any $k \in \mathbb{N}$, thus

$$\lim_{k\to\infty} \|x_k - w_k\| = 0. \tag{3.9}$$

By Lemma 3.2, a sequence $\{x_k\}$ contains at least a subsequence weakly convergent to a cluster point, called $\widetilde{x}$. Then there exists a strictly increasing sequence $\{k_i\} \subseteq \mathbb{N}$ such that

$$x_{k_i} \rightharpoonup \widetilde{x} \quad \text{as} \quad i \to \infty. \tag{3.10}$$

Without loss of generality, we may assume, by (3.9), that the sequence $\{k_i\}$ considered is also satisfied

$$w_{k_i} \rightharpoonup \widetilde{x} \quad \text{as} \quad i \to \infty. \tag{3.11}$$

Next we claim that, $\lim_{i\to\infty} \left\|w_{k_i} - z_{k_i}\right\| = 0$.

If $\beta_k > 0$. Since $\{w_k\}$, $\{z_k\}$ and $\{F(y_k)\}$ are bounded, then $\{d_k\}$ is bounded. Hence, by Lemma 3.3, we have

$$\lim_{k\to\infty} \|w_k - z_k\| = 0.$$

If $\beta_k = 0$. Thus, there exists a strictly increasing sequence $\{k_i\} \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \beta_{k_i} = 0. \tag{3.12}$$

By (3.1), for any sufficient β_{k_i}, we get that

$$
\begin{aligned}
&\left\langle F(w_{k_i}) - F\left(\left(1 - \frac{\beta_{k_i}}{\vartheta}\right) w_{k_i} + \frac{\beta_{k_i}}{\vartheta} z_{k_i}\right), w_{k_i} - z_{k_i}\right\rangle \\
&\quad > \mu \left\|w_{k_i} - z_{k_i}\right\|^2.
\end{aligned}
$$

By Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left\langle F\left(w_{k_i}\right) - F\left(\left(1 - \frac{\beta_{k_i}}{\vartheta}\right) w_{k_i} + \frac{\beta_{k_i}}{\vartheta} z_{k_i}\right), w_{k_i} - z_{k_i}\right\rangle \\
&\leq \left\|F\left(w_{k_i}\right) - F\left(\left(1 - \frac{\beta_{k_i}}{\vartheta}\right) w_{k_i} + \frac{\beta_{k_i}}{\vartheta} z_{k_i}\right)\right\| \left\|w_{k_i} - z_{k_i}\right\|
\end{aligned}
$$

which refer to

$$\left\|F\left(w_{k_i}\right)-F\left(\left(1-\frac{\beta_{k_i}}{\vartheta}\right)w_{k_i}+\frac{\beta_{k_i}}{\vartheta}z_{k_i}\right)\right\|>\mu\left\|w_{k_i}-z_{k_i}\right\|.$$

Since F is continuous, by (3.12), it turns out that

$$\lim_{i\to\infty}\left\|w_{k_i}-z_{k_i}\right\|=0. \qquad (3.13)$$

We note that

$$\begin{aligned}\left\|x_{k_i}-z_{k_i}\right\| &= \left\|x_{k_i}-w_{k_i}+w_{k_i}-z_{k_i}\right\| \\ &\leq \left\|x_{k_i}-w_{k_i}\right\|+\left\|w_{k_i}-z_{k_i}\right\|\end{aligned}$$

By (3.9) and (3.13), then

$$\lim_{i\to\infty}\left\|x_{k_i}-z_{k_i}\right\|=0.$$

That is,

$$z_{k_i}\rightharpoonup\widetilde{x} \quad \text{as} \quad i\to\infty \qquad (3.14)$$

because of (3.10). By upper semicontinuous of $K(\cdot)$, (3.11) and (3.14), also since $z_{k_i}\in K(w_{k_i})$, thus, $\widetilde{x}\in K(\widetilde{x})$. Last, we claim that $\langle F(\widetilde{x}),u-\widetilde{x}\rangle\geq 0$ for all $u\in K(\widetilde{x})$. By lower semicontinuous of $K(\cdot)$, and (3.11), for any $u\in K(\widetilde{x})$, there exists a sequence $u_{k_i}\in K(w_{k_i})$ such that $u_{k_i}\rightharpoonup u$ as $i\to\infty$. Due to the fact that $z_{k_i}=P_{K(w_{k_i})}\left(w_{k_i}-F\left(w_{k_i}\right)\right)$, by Lemma 2.2, then we have

$$\left\langle z_{k_i}-w_{k_i}+F\left(w_{k_i}\right),u_{k_i}-z_{k_i}\right\rangle\geq 0 \qquad (3.15)$$

which is

$$\left\langle z_{k_i}-w_{k_i},u_{k_i}-z_{k_i}\right\rangle+\left\langle F\left(w_{k_i}\right),u_{k_i}-z_{k_i}\right\rangle\geq 0.$$

Letting $i\to\infty$, by (3.11) and (3.14), hence $\langle F(\widetilde{x}),u-\widetilde{x}\rangle\geq 0$ for all $u\in K(\widetilde{x})$. Therefore $\widetilde{x}$ is a solution of the quasi-variational inequality problem (*). □

Theorem 3.5. *Suppose that the assumptions (A1)-(A4) hold. Let $\{x_k\}$ be a sequence generated by Algorithm 1. If F is strictly monotone at an accumulation point $\widetilde{x}$ of $\{x_k\}$, then the sequence converges weakly to $\widetilde{x}$.*

Proof. Since $\widetilde{x}$ is an accumulation point of $\{x_k\}$, by Theorem 3.4, we have that $\widetilde{x}$ is a solution of the quasi-variational inequality problem (*). Let $x^*\in S^*$. Then, it follows that $\langle F(x^*),z_k-x^*\rangle\geq 0$. Letting $k\to\infty$, thus $\langle F(x^*),\widetilde{x}-x^*\rangle\geq 0$. By pseudo monotonicity of F, we can see that

$$\langle F(\widetilde{x}),\widetilde{x}-x^*\rangle\geq 0.$$

Since $x^*\in K(x_k)$, using upper continuity of $K(\cdot)$, we obtain that $x^*\in K(\widetilde{x})$. Since $\widetilde{x}$ is a solution of the quasi-variational inequality problem (*) then $\langle F(\widetilde{x}),x^*-\widetilde{x}\rangle\geq 0$. That is, $\langle F(\widetilde{x}),x^*-\widetilde{x}\rangle=0$. In the similar way, we get that $\langle F(x^*),x^*-\widetilde{x}\rangle=0$. It can be seen that $\langle F(\widetilde{x}),x^*-\widetilde{x}\rangle=\langle F(x^*),x^*-\widetilde{x}\rangle=0$, i.e., $\langle F(x^*)-F(\widetilde{x}),x^*-\widetilde{x}\rangle=0$. Since F is strictly monotone at $\widetilde{x}$, therefore $\widetilde{x}=x^*\in S^*$. Since every accumulation point of $\{x_k\}$ is x^*. Therefore, the sequence converges weakly to $\widetilde{x}$. □

Since the previous results which is a weakly convergence of the sequences generated by Algorithm 1 remain in the Hilbert space, so, when the space is restricted to be a finite dimensional space, it turned out the ensuing corollary.

Corollary 3.6. *Suppose that the assumptions (A1)-(A4) hold. Let $\{x_k\}$ be a sequence in a finite dimensional real vector space generated by Algorithm 1. If F is strictly monotone at an accumulation point $\widetilde{x}$ of $\{x_k\}$, then the sequence converges strongly to $\widetilde{x}$.*

4. Numerical results

In this section, to illustrate how Algorithm 1 behaves, some examples are included and demonstrated using MATLAB. For the stopping criterion

$$r_{K(x_k)}(x_k):=\left\|x_k-P_{K(x_k)}(x-F(x_k))\right\|,$$

we now terminate the numerical methods by selecting a tolerance ϵ.

From now on, we assign the parameters in Algorithm 1 as $\epsilon = 10^{-6}$, $\mu = 0.3$, $\vartheta = 0.5$, $\rho = 1.99$, $c = 0.95$ and $\gamma = 0.6\overline{7}$ with $\xi_k = \frac{1}{k^2}$ such that $\sum_{k=1}^{\infty} \xi_k < +\infty$ for the computational experiments. The following experiments are reported for the information of number of iterations, CPU time in a second unit, and the approximate solution which referred to a last iterative point.

We first sample with the instance from Harker [3].

Example 4.1. Consider a two-person game, each player selects one number in the interval $[0, 10]$ where the sum of their numbers must not be greater than 15. For player $i = 1, 2$, the cost functions f_i and the strategy set K_i are given by

$$\begin{aligned} f_1(a,b) &= a^2 + \frac{8}{3}ab - 34a, \\ f_2(a,b) &= b^2 + \frac{5}{4}ab - \frac{97}{4}b, \\ K_1(b) &= \{0 \le a \le 10 : a \le 15 - b\}, \\ K_2(a) &= \{0 \le b \le 10 : b \le 15 - a\}. \end{aligned}$$

For the quasi-variational inequality formulation, we have $F(a,b) = (\nabla_a f(a,b), \nabla_b f(a,b))^T$, that is,

$$F(a,b) = (2a + \frac{8}{3}b - 34, 2b + \frac{5}{4}a - \frac{97}{4})^T.$$

The set of the solution of the problem is a point $(5,9)^T$ with the line segment $[(9,6)^T, (10,5)^T]$. Therefore, all assumptions (A1)-(A4) are satisfied. The computational results of this example are in Table 1.

Next we consider another example which is improved from Outrata [23] in 1995. This example related with the Stackelberg-Cournot-Nash equilibrium problem.

Table 1. The result of the Example 4.1

	CPU(s)	Number of iterations	Aprroximate solution	
			a	b
$x_0 = x_1 = (0,0)^T$				
Algorithm [14]	0.021100	236	5	9
Algorithm 1	0.016472	131	5	9
$x_0 = x_1 = (10,0)^T$				
Algorithm [14]	208.172546	11,509,127	10	5
Algorithm 1	206.785195	4,944,104	10	5
$x_0 = x_1 = (10,10)^T$				
Algorithm [14]	0.020714	257	5	9
Algorithm 1	0.018858	121	5	9
$x_0 = x_1 = (0,10)^T$				
Algorithm [14]	0.020462	166	5	9
Algorithm 1	0.017471	69	5	9
$x_0 = x_1 = (5,5)^T$				
Algorithm [14]	0.023259	256	5	9
Algorithm 1	0.018474	117	5	9
$x_0 = (15,4)^T$ and $x_1 = (20,40)^T$				
Algorithm [14]	208.086972	11509128	10	5
Algorithm 1	0.013735	79	5	9
$x_0 = (30,40)^T$ and $x_1 = (50,20)^T$				
Algorithm [14]	0.021946	238	5	9
Algorithm 1	0.019332	141	5	9
$x_0 = (20,2)^T$ and $x_1 = (12,3)^T$				
Algorithm [14]	198.409778	11509129	10	5
Algorithm 1	124.078450	4944045	10	5

Example 4.2. Consider oligopoly which is a small market structure, for n vendors selling the same products, they do not cooperate each other. Let q be a quantity of the products on the market which are consumed by purchasers. The demand in the market depends on the quantity. Define the inverse demand curve $p : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$p(q) = 5000^{\frac{1}{\eta}} q^{-\frac{1}{\eta}},$$

where η is a positive parameter termed demand elasticity. Next, let f_i be a function of cost of production given by

$$f_i(x_i) = a_i x_i + \frac{b_i}{b_i + 1} c_i^{-\frac{1}{b_i}} x_i^{\frac{b_i+1}{b_i}},$$

where a_i, b_i and c_i are positive parameters.

For the setting of the generalized Nash equilibrium problem, it can be written as

$$\left.\begin{array}{ll} \text{minimize} & f_i(x_i) - x_i p\left(x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} x_j\right) \\ \text{subject to} & x_i \in X_i := \left\{x_i \in \mathbb{R}_+ : x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} x_j \leq N\right\} \end{array}\right\} \tag{4.1}$$

where N is a joint production bound.

We note that (4.1) is a convex minimization problem when $\eta > 1$. Suppose that there are five vendors, consider $n = 5$, in the market with the same lower production bound, 1 unit, and upper production bound, 150 units. It obtain the mapping

$$F_i(x) \equiv \left(a_i + \frac{x_i}{c_i}^{\frac{1}{b_i}} + \left(\frac{5000}{q}\right)^{\frac{1}{\eta}}\left(\frac{x_i}{\eta q} - 1\right)\right)_{i=1}^{5}$$

where $q = \sum_{i=1}^{5} x_i$ with $X_i = \{x_i \in \mathbb{R}_+ : 1 \leq x_i \leq 150\}$ for all $i = 1, 2, \cdots, 5$. Hence, the set-valued mapping, for joint production bound $N = 700$, we have

$$\begin{aligned} K_i(x_{-i}) &= K_i(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_5) \\ &= \left\{1 \leq x_i \leq 150, x_i \leq 700 - \sum_{\substack{j=1 \\ j\neq i}}^{5} x_j\right\}. \end{aligned}$$

Then (A1)-(A4) are satisfied. Using Algorithm 1, set $\eta = 1.1$ and the parameters a_i, b_i and c_i in Table 2:

Table 2. The parameters a_i, b_i and c_i in Example 4.2

	Vandor 1	Vandor 2	Vandor 3	Vandor 4	Vandor 5
a_i	10	8	6	4	2
b_i	1.2	1.1	1.0	0.9	0.8
c_i	5	5	5	5	5

Then we can obtain the numerical results in Table 3.

From the above examples, it can be seen that all joint constraint set are distributed by all players. Also, a generalized Nash equilibrium problem can be solved and the solutions are achieved using Algorithm 1 through the quasi-variational inequality problem.

Table 3. The result of the Example 4.2 in case n = 5

	CPU(s)	Number of iterations	Approximate solution				
			Vendor 1	Vendor 2	Vendor 3	Vendor 4	Vendor 5
$x_0 = x_1 = (50, 50, 50, 50, 50)^T$							
Algorithm [14]	0.023884	148	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.022809	85	36.9325	41.8181	43.7066	42.6592	39.1790
$x_0 = x_1 = (10, 10, 10, 10, 10)^T$							
Algorithm [14]	0.021545	138	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.021218	51	36.9325	41.8181	43.7066	42.6592	39.1790
$x_0 = x_1 = (5, 10, 15, 20, 25)^T$							
Algorithm [14]	0.023746	120	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.022643	62	36.9325	41.8181	43.7066	42.6592	39.1790
$x_0 = (90, 90, 90, 90, 90)^T$ and $x_1 = (60, 60, 60, 60, 60)^T$							
Algorithm [14]	0.024635	133	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.023324	60	36.9325	41.8181	43.7066	42.6592	39.1790
$x_0 = (20, 40, 60, 80, 100)^T$ and $x_1 = (100, 90, 80, 70, 60)^T$							
Algorithm [14]	0.021361	135	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.020906	76	36.9325	41.8181	43.7066	42.6592	39.1790
$x_0 = (200, 400, 200, 400, 200)^T$ and $x_1 = (100, 200, 300, 400, 500)^T$							
Algorithm [14]	0.017610	127	36.9325	41.8181	43.7066	42.6592	39.1790
Algorithm 1	0.015973	72	36.9325	41.8181	43.7066	42.6592	39.1790

5. Conclusion

The sequence generated by the algorithm, known as the inertial projection-like method, is weakly convergent in a Hilbert space. The proposed algorithm gives better results in both the number of iterations and the CPU time scopes with the solution of the quasi-variational inequality. In other words, the generalized Nash equilibrium is reached. Even though all parameters are significantly important of the characteristic of the algorithm, which affects the performance of the method, the algorithm easily works with simple computation.

6. Acknowledgements

This research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2021 under project number 64A306000005. Moreover, the authors are grateful to the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT, for providing a great working environment. The last author is supported by the KMUTT Research Fund. Moreover, the first author would like to thank Science Achievement Scholarship of Thailand (SAST). Last, the authors would like to express our thanks to referees for their valuable comments and suggestions.

References

[1] Facchinei F, Kanzow C, Sagratella S. Solving quasi-variational inequalities via their KKT conditions. Mathematical Programming. 2014;144(12,Ser.A):369-412.

[2] Bensoussan A, Lions JL. Propriétés des inéquations quasi variationnelles decroissantes. In: Analyse convexe et ses applications (Comptes Rendus Conf., Saint Pierre de Chartreuse, 1974); 1974. p.66-84. Lecture Notes in Econom. and Math. Systems, Vol.102.

[3] Harker PT. Generalized Nash games and quasi-variational inequalities. European Journal of Operational Re search. 1991;54(1):81-94.

[4] Pang JS, Fukushima M. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput Manag Sci. 2005;2(1):21-56.

[5] Kočvara M, Outrata JV. On a class of quasi-variational inequalities. Optimization Methods and Software 1995;5(4):275-95.

[6] JingYuan W, Smeers Y. Spatial Oligopolistic Electricity Models with Cournot Generators and Regulated Transmission Prices. Operations Research. 1999;47(1):102-12.

[7] Cach J. Solution set in a special case of generalized Nash equilibrium games. Kybernetika (Prague). 2001;37(1):21-37.

[8] Noor MA, Rassias TM. Projection Methods for Monotone Variational In equalities. Journal of Mathematical Analysis and Applications. 1999;237(2):405-12.

[9] Iusem AN. An iterative algorithm for the variational inequality problem. Mat Apl Comput. 1994;13(2):103-14.

[10] Antipin AS, Jaćimović M, Mijajlović N. Extragradient method for solving quasivariational inequalities. Optimization. 2018;67(1):103-12.

[11] Van NTT, Strodiot JJ, Nguyen VH, Vuong PT. An extragradienttype method for solving nonmonotone quasiequilibrium problems. Optimization. 2018;67(5):651-64.

[12] Ye ML. A cutting hyperplane projection method for solving generalized quasi-variational inequalities. J Oper Res Soc China. 2016;4(4):483-501.

[13] Ye M. A halfspace projection method for solving generalized Nash equilibrium problems. Optimization. 2017;66(7):1119-34.

[14] Zhang J, Qu B, Xiu N. Some projection-like methods for the generalized Nash equilibria. Comput Optim Appl. 2010;45(1):89-109.

[15] Poljak BT. Some methods of speeding up the convergence of iterative methods. Ž. Vyčisl. Mat i Mat. Fiz. 1964;4:791-803.

[16] Chen C, Ma S, Yang J. A General Inertial Proximal Point Algorithm for Mixed Variational Inequality Problem. SIAM Journal on Optimization. 2015;25(4):2120-42.

[17] Thong DV, Hieu DV. Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. Journal of Computational and Applied Mathematics. 2018;341:80-98.

[18] Cholamjiak P, Shehu Y. Inertial forward-backward splitting method in Banach spaces with application to compressed sensing. Appl Math. 2019;64(4):409-35.

[19] Shehu Y, Gibali A, Sagratella S. Inertial projection-type methods for solving quasi-variational inequalities in real Hilbert spaces. J Optim Theory Appl. 2020;184(3):877-94.

[20] Bauschke HH, Combettes PL. Convex analysis and monotone operator theory in Hilbert spaces. 2nd ed. CMS Books in Mathematics/Ouvrages de Math□matiques de la SMC. Springer, Cham; 2017.

[21] Zarantonello EH. Projections on convex sets in Hilbert space and spectral theory. I. Projections on convex sets. In: Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971); 1971. p. 237-341.

[22] Maingé PE. Convergence theorems for inertial KM-type algorithms. J Comput Appl Math. 2008;219(1):223-36.

[23] Outrata J, Zowe J. A numerical approach to optimization problems with variational inequality constraintsMath Programming. 1995;68(1,Ser.A):105-30.