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On Ordered Semigroups Containing Covered Bi-Ideals

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ABSTRACT

In this paper, we characterize ordered semigroups containing covered bi-ideals and study some results based on covered bi-ideals. Moreover, in a regular ordered semigroup, we show that, under some conditions, a proper bi-ideal of an ordered semigroup is also a covered bi-ideal.

Keywords: Bi-ideal; Covered bi-ideal; Maximal bi-ideal; Ordered semigroup

1. Introduction and Preliminaries

Ideal theory is the main research in many algebraic structures, for example, rings, semirings, semigroups and ordered semigroups. Given a semigroup S, a proper ideal A of S is called a covered ideal of S if it satisfies $A \subseteq S(S-A)S$ where $S-A$ denote the set of all elements x in S such that $x \notin A$. This notion was introduced and studied by Fabrici in [1, 2]. An ordered semigroup is one of generalizations of semigroups. Later, Changphas and Summaprab discussed the structure of ordered semigroups containing covered ideals in [3] and the structure of ordered semigroups containing covered one-sided ideals in [4]. A bi-ideal of semigroups is one of generalizations of ideals. These are motivated to research in this paper. In this paper, we introduce the concepts of covered bi-ideals of ordered semigroups. We investigate some results based on covered bi-ideals of ordered semigroups. Moreover, in a regular ordered semigroup, we show that a proper bi-ideal of an ordered semigroup, under some conditions, is also a covered bi-ideal.

Now, we include here some basic definitions of ordered semigroups that are necessary for the subsequent results and for more details on ordered semigroups we refer to [2, 5–9].

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By an ordered semigroup we mean a partially ordered set $(S, \leqslant)$ and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$,

$$a \leqslant b \text{ implies } xa \leqslant xb \text{ and } ax \leqslant bx.$$

It is denoted by $(S, \cdot, \leqslant)$. Every semigroup $(S, \cdot)$ can be considered an ordered semigroup $(S, \cdot, \leqslant)$ where $\leqslant := id_S = \{(x, x) \mid x \in S\}$. Throughout this paper, we will denote the ordered semigroup $(S, \cdot, \leqslant)$ by S unless otherwise stated.

Let A, B be non-empty subsets of an ordered semigroup S. The set product AB is defined as follows:

$$AB = \{ab \mid a \in A, b \in B\}$$

and we define $(A]$ by:

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, if $A = \{a\}$, we write aB for $\{a\}B$, similarly for $B = \{b\}$, and we write $(a]$ for $(\{a\}]$. It was observed in [9] that the following conditions hold:

(1) $A \subseteq (A]$;
(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;
(3) $(A](B] \subseteq (AB]$;
(4) $(A] \cup (B] = (A \cup B]$;
(5) $((A]] = (A]$.

Then A is called a subsemigroup of S if $AA \subseteq A$. The concept of bi-ideals in an ordered semigroup has been introduced in [7] as follows: a subsemigroup B of an ordered semigroup S is called a bi-ideal of S if it satisfies the following conditions:

(1) $BSB \subseteq B$;
(2) $B = (B]$, that is, for any $x \in B$ and $y \in S$, $y \leqslant x$ implies $y \in B$.

A bi-ideal B of S is called a proper if $B \subset S$. The symbol $\subset$ stands for proper subset of sets. A proper bi-ideal B of S is said to be maximal if for any bi-ideal A of S such that $B \subseteq A \subseteq S$, then $B = A$ or $A = S$. It is well-known that the intersection of all bi-ideals of S, if it is non-empty, is also a bi-ideal of S. The bi-ideal of S generated by a non-empty set A of S is of the form

$$(A)_B = (A \cup AA \cup ASA].$$

In particular, we write $(\{a\})_B$ as $(a)_B$, and $(a)_B = (a \cup aa \cup aSa]$ which is called the principal bi-ideal [6] of S generated by a.

Finally, in [7, 8], an ordered semigroup S is regular if $a \in (aSa]$ for every $a \in S$, i.e., if for any $a \in S$, $a \leqslant axa$ for some $x \in S$. An element a of an ordered semigroup S is called an idempotent [10] if $a \leqslant a^2$. An ordered semigroup S is called bi-simple [6] if S has no proper bi-ideal.

2. Main Results

In this section, the structure of ordered semigroups containing covered bi-ideals will be discussed.

Definition 2.1. Let S be an ordered semigroup. A proper bi-ideal B of S is called a covered bi-ideal (CB-ideal) of S if

$$B \subseteq ((S - B)S(S - B)].$$

Example 2.2. Let $S = \{a, b, c, d, e\}$ and the multiplication and the partial order on S are defined by

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\leqslant = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (c, c), (d, d), (e, e)\}.$$

In [8], we have that S is an ordered semigroup. We obtain that the proper bi-ideals of S are $\{a\}, \{a, b\}$, $\{a, c\}, \{a, d\}, \{a, e\}, \{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$ and $\{a, c, e\}$. Moreover, we can deduce that the CB-ideals of S are $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}$ and $\{a, e\}$.

First, we characterize when a proper bi-ideal of an ordered semigroup is not a CB-ideal.

Theorem 2.3. *Let S be an ordered semigroup. If S contains two different proper bi-ideals B_1 and B_2 such that $B_1 \cup B_2 = S$, then B_1 and B_2 are not CB-ideals of S.*

Proof. Assume that S contains two different proper bi-ideals B_1 and B_2 such that $B_1 \cup B_2 = S$. Since $B_1 \cup B_2 = S$, it implies that $S - B_1 \subseteq B_2$ and $S - B_2 \subseteq B_1$. Suppose that B_1 is a CB-ideal of S. Then

$$\begin{aligned} B_1 &\subseteq ((S - B_1)S(S - B_1)] \\ &\subseteq (B_2SB_2] \\ &\subseteq (B_2] = B_2. \end{aligned}$$

Since $B_1 \cup B_2 = S$, it follows that $S = B_2$, which is a contradiction. Similarly, if B_2 is a CB-ideal of S, then

$$\begin{aligned} B_2 &\subseteq ((S - B_2)S(S - B_2)] \\ &\subseteq (B_1SB_1] \\ &\subseteq (B_1] = B_1. \end{aligned}$$

Thus, $S = B_1$, which is a contradiction. Hence, the assertion holds. □

Corollary 2.4. *If an ordered semigroup S contains two different maximal proper bi-ideals such that union of two different maximal bi-ideals is a bi-ideal, then maximal bi-ideals are not CB-ideals.*

Proof. Assume that S contains two different maximal proper bi-ideals B_1 and B_2 such that $B_1 \cup B_2$ is a bi-ideal of S. Then $B_1 \subset B_1 \cup B_2$. Since B_1 is a maximal proper bi-ideal of S, we obtain $B_1 \cup B_2 = S$. Hence, by Theorem 2.3, neither B_1 nor B_2 is a CB-ideal of S. □

Theorem 2.5. *Let B_1 be a CB-ideal of an ordered semigroup S and B_2 be a bi-ideal of S. If $B_1 \cap B_2$ is a non-empty, then $B_1 \cap B_2$ is a CB-ideal of S.*

Proof. Let B_1 be a CB-ideal of an ordered semigroup S and B_2 be a bi-ideal of S. Suppose that $B_1 \cap B_2 \neq \varnothing$. Clearly, $B_1 \cap B_2$ is a bi-ideal of S. Since B_1 is a CB-ideal of S, we have $B_1 \subseteq ((S - B_1)S(S - B_1)]$ and $B_1 \cap B_2$ is a proper bi-ideal of S. Hence,

$$\begin{aligned} B_1 \cap B_2 \subseteq B_1 &\subseteq ((S - B_1)S(S - B_1)] \\ &\subseteq ((S - (B_1 \cap B_2))S(S - (B_1 \cap B_2))]. \end{aligned}$$

This implies that $B_1 \cap B_2$ is a CB-ideal of S. □

The following corollary follows directly from Theorem 2.5.

Corollary 2.6. *If B_1 and B_2 are CB-ideals of an ordered semigroup S such that $B_1 \cap B_2$ is a non-empty, then $B_1 \cap B_2$ is a CB-ideal of S.*

Theorem 2.7. *Let S be an ordered semigroup. If S is not bi-simple such that there are not any two proper bi-ideals in which their intersection is empty, then S contains a CB-ideal.*

Proof. Assume that S is not bi-simple such that there are not any two proper bi-ideals in which their intersection is empty. Then S contains a proper bi-ideal B. Now, we show that $((S-B)S(S-B)]$ is a bi-ideal of S. Let $B_1 = ((S - B)S(S - B)]$. Consider

$$\begin{aligned} &B_1B_1 \\ &= ((S - B)S(S - B)]((S - B)S(S - B)] \\ &\subseteq ((S - B)(S(S - B))((S - B)S)(S - B)] \\ &\subseteq ((S - B)SS(S - B)] \\ &\subseteq ((S - B)S(S - B)] = B_1. \end{aligned}$$

Thus, $B_1B_1 \subseteq B_1$, and so B_1 is a subsemigroup of S. And, we consider

$$\begin{aligned}
&B_1SB_1\\
&= ((S-B)S(S-B)]S((S-B)S(S-B)]\\
&\subseteq ((S-B)S](S](S(S-B)]\\
&\subseteq ((S-B)SS](S(S-B)]\\
&\subseteq ((S-B)SS(S-B)]\\
&\subseteq ((S-B)S(S-B)] = B_1.
\end{aligned}$$

So, we obtain that $B_1SB_1 \subseteq B_1$. Since $B_1 = ((S-B)S(S-B)]$, it implies that

$$\begin{aligned}
(B_1] &= (((S-B)S(S-B)]]\\
&= ((S-B)S(S-B)] = B_1.
\end{aligned}$$

Hence, B_1 is a bi-ideal of S. By assumption, $B \cap B_1 \neq \varnothing$. Let $B' = B \cap B_1$. Then B' is a proper bi-ideal of S. Since $B' \subseteq B$, we have $S - B \subseteq S - B'$. Since $B' \subseteq B_1$, it implies that

$$\begin{aligned}
B' \subseteq B_1 &= ((S-B)S(S-B)]\\
&\subseteq ((S-B')S(S-B')].
\end{aligned}$$

This shows that B' is a CB-ideal of S. □

The following theorem gives necessary and sufficient conditions for every proper bi-ideal of a regular ordered semigroup is a CB-ideal.

Theorem 2.8. *Let S be a regular ordered semigroup. If for any proper bi-ideal B of S such that for any $a \in B$, $(a)_B \subseteq (b)_B$ for some $b \in S - B$, then B is a CB-ideal of S.*

Proof. Assume that B is a proper bi-ideal of S such that $a \in B$, $(a)_B \subseteq (b)_B$ for some $b \in S - B$. Since S is regular, there exists $x \in S$ such that $b \leqslant bxb$. Since $b \in S - B$, we obtain $b \leqslant bxb \in (S-B)S(S-B)$. It implies that $b \in ((S-B)S(S-B)]$. By the proof of Theorem 2.7, $((S-B)S(S-B)]$ is a bi-ideal of S. So, we have

$$\begin{aligned}
bb &\in ((S-B)S(S-B)]((S-B)S(S-B)]\\
&\subseteq ((S-B)S(S-B)]
\end{aligned}$$

and

$$\begin{aligned}
&bSb\\
&\subseteq ((S-B)S(S-B)]S((S-B)S(S-B)]\\
&\subseteq ((S-B)S(S-B)].
\end{aligned}$$

Thus,

$$\begin{aligned}
(b)_B &= (b \cup bb \cup bSb]\\
&= (b] \cup (bb] \cup (bSb]\\
&\subseteq (((S-B)S(S-B)]]\\
&= ((S-B)S(S-B)].
\end{aligned}$$

Hence,

$$a \in (a)_B \subseteq (b)_B \subseteq ((S-B)S(S-B)].$$

This shows that $B \subseteq ((S-B)S(S-B)]$. Therefore, B is a CB-ideal of S. □

Example 2.9. Let $S = \{a, b, c, d, f, 1\}$ and the multiplication and the partial order of S are defined by

$\cdot$	a	b	c	d	f	1
a	a	a	a	a	a	a
b	a	b	a	d	a	b
c	a	f	c	c	f	c
d	a	b	d	d	b	d
f	a	f	a	c	a	f
1	a	b	c	d	f	1

$$\begin{aligned}
\leqslant = \{&(a,a),(a,b),(a,c),(a,d),(a,f),\\
&(b,b),(c,c),(d,d),(f,f),(1,1)\}.
\end{aligned}$$

In [11], we have that S is an ordered semigroup. Clearly, $x \leqslant xxx$ where $x \in \{a, b, c, d, 1\}$, and we have $f \leqslant fdf = f$. Thus, S is a regular ordered semigroup. We can obtain that the proper bi-ideals of S are $B_1 = \{a\}, B_2 = \{a, b\}, B_3 = \{a, c\}, B_4 = \{a, d\}, B_5 = \{a, f\}, B_6 = \{a, b, d\}, B_7 = \{a, b, f\}, B_8 = \{a, c, d\}, B_9 = \{a, c, f\}$ and $B_{10} = \{a, b, c, d, f\}$. One may easily verify that for every principal bi-ideal $(x)_B \subseteq B_i$ for all $x \in B_i$ and $i =$

$1, 2, \ldots, 10$, and we have $1 \in S - B_i$ where $i = 1, 2, \ldots, 10$ such that $(x)_B \subseteq (1)_B$. By Theorem 2.8, B_i is a CB-ideal of S for all $i = 1, 2, \ldots, 10$.

Theorem 2.10. *Let S be a regular ordered semigroup. If B is a bi-ideal of S such that for any element of B is an idempotent, then any CB-ideal B_1 of B is also a CB-ideal of S.*

Proof. Assume that B is a bi-ideal of S such that for any element of B is an idempotent. Now, we will show that B is a regular subsemigroup of S. Obviously, B is a subsemigroup of S. Let $a \in B \subseteq S$. By assumption, we have $a \leqslant a^2$. Since S is a regular, then there exists $b \in S$ such that

$$\begin{aligned} a \leqslant aba \leqslant a^2ba^2 &= a(aba)a \\ &\in a(BSB)a \\ &\subseteq aBa. \end{aligned}$$

Thus, $a \in (aBa]$. This implies that B is a regular subsemigroup of S. Next, let B_1 be a CB-ideal of B. We claim that B_1 is a bi-ideal of S. Obviously, B_1 is a subsemigroup of S. Let $b_1, b_2 \in B_1 \subseteq B$ and $s \in S$. By assumption, we have $b_1 \leqslant b_1^2$ and $b_2 \leqslant b_2^2$. Also, $b_1sb_2 \in B_1SB_1 \subseteq BSB \subseteq B$. Suppose that $b' = b_1sb_2 \in B_1SB_1 \subseteq B$. Since B is a regular subsemigroup of S, then there exists $b_3 \in B$ such that

$$\begin{aligned} b' \leqslant b'b_3b' &= (b_1sb_2)b_3(b_1sb_2) \\ &\leqslant (b_1^2sb_2^2)b_3(b_1^2sb_2^2) \\ &\in (B_1^2SB_1^2)B(B_1^2SB_1^2) \\ &\subseteq B_1^2SBSB_1^2 \\ &\subseteq B_1^2SB_1^2 \\ &= B_1B_1SB_1B_1 \\ &\subseteq B_1(BSB)B_1 \\ &\subseteq B_1BB_1 \subseteq B_1. \end{aligned}$$

So, we obtain $b' \in (B_1] = B_1$. Thus, $B_1SB_1 \subseteq B_1$. Suppose that $x \in B_1 \subseteq B$ and $y \in S$ such that $y \leqslant x$. Since B is a bi-ideal of S, it implies that $y \in (B] = B$. Since $y \in B$, $y \leqslant x$ and B_1 is a bi-ideal of B, it follows that $y \in B_1$. Hence, B_1 is a bi-ideal of S. Since B_1 is a CB-ideal of B, we have $B_1 \subseteq ((B - B_1)B(B - B_1)]$ and $B_1 \subset B \subseteq S$. Thus, $\varnothing \neq B - B_1 \subseteq S - B_1$, and so

$$\begin{aligned} B_1 &\subseteq ((B - B_1)B(B - B_1)] \\ &\subseteq ((S - B_1)S(S - B_1)]. \end{aligned}$$

This shows that B_1 is a CB-ideal of S. □

Finally, we give the example description for any CB-ideal of bi-ideal is also a CB-ideal of an ordered semigroup.

Example 2.11. Let $S = \{a, b, c, d, e\}$ and the multiplication and the partial order on S are defined by

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leqslant = \{(a, a), (a, b), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, e)\}.$$

In [8], we have that S is an ordered semigroup. One can check that S is a regular ordered semigroup. We have $B = \{a, b, c, d\}$ is a proper bi-ideal of S and for every element of B is an idempotent. Moreover, we have $B_1 = \{a, c, d\}$ is a proper bi-ideal of B. One can also check that B_1 is a CB-ideal of both B and S.

3. Conclusion

From this paper, the results of covered bi-ideals in ordered semigroups are proved. In Theorem 2.8, we give the condition for a proper bi-ideal of an ordered semigroup is a covered bi-ideal. Moreover, we show the remarkable results of covered bi-ideals of an ordered semigroup in Theorems 2.3, 2.5, 2.7 and 2.10. In the future work, we can extend these results to algebraic hyperstructures, for example semihypergroups, ordered semihypergroups, etc.

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