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Original research article

The Convergence of AA-Iterative Algorithm for Generalized AK-α-Nonexpansive Mappings in Banach Spaces

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ABSTRACT

In this paper, we are inspired by the collective ideas of the authors mentioned, making us interested in studying about the strong and weak convergence results by using the AA-iterative algorithm (1.14) for the generalized AK-α-nonexpansive mappings (1.5) in uniformly convex Banach spaces . In particular, we provide only sufficient convergence criteria for a suitable procedural method. Our work is more general and unifies the comparable results in the existing literature, for instance, the results given in [1].

Keywords: AK-α-generalized nonexpansive mapping; Banach space; Convergence theorem; Fixed point; Hilbert space

1. Introduction

Let E be a closed convex subset of a Banach space B, and the set of all fixed points of map ξ defined by symbol $Fix(\xi)$. A mapping $\xi : E \to E$ is said to be

(n_i) A contraction if, for all $a, b \in E$, there exists a $\in [0, 1)$ such that

$$\|\xi a - \xi b\| \leq \alpha\|a - b\|. \quad (1.1)$$

(n_{ii}) A nonexpansive mapping if , for all $a, b \in E$ such that

$$\|\xi a - \xi b\| \leq \|a - b\|. \quad (1.2)$$

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(n_{iii}) Quasi-non-expansive if, for all $a, b \in E$ and $a^* \in Fix(\xi)$, we have

$$\|\xi a - a^*\| \leq \|a - a^*\|. \qquad (1.3)$$

In 2008, Suzuki [2] introduced a new type of mapping satisfying Condition (C). A self mapping ξ on E satisfies Condition (C) if for $a, b \in E$ with

$$\frac{1}{2}\|a - \xi a\| \leq \|a - b\| \Longrightarrow \|\xi a - \xi b\| \leq \|a - b\|. \qquad (1.4)$$

The mappings satisfying Condition (C) do not need to be continuous; hence, Condition (C) is weaker than the one depicting non-expansive mappings. However, mappings satisfying Condition (C) are stronger than the one defining quasi-non-expansive mappings. Suzuki [2] studied the existence and convergence results for such mappings.

In 2011, Aoyama and Kohsaka [3] introduced the class of α-nonexpansive mappings in Banach spaces as follows: Let E be a Banach space and let E be a nonempty subset of E. A mapping $\xi : E \to E$ is said to be α-nonexpansive for some real number $0 \leq \alpha < 1$ if

$$\|\xi a - \xi b\| \leq \alpha||\xi a - b|| + \alpha||\xi b - a|| + (1 - 2\alpha)||a - b||,$$

for all $a, b \in E$. Clearly, 0-nonexpansive maps are exactly nonexpansive maps. This mapping was generalized and extended by many authors in several directions; see for instance [2, 4] and references therein.

In 2021, Suanoom and Khuangsatung [5] , we introduced a new class of nonexpansive type of mapping namely, *AK-generalized nonexpansive mapping*, which is more general than an α-nonexpansive mapping in Hilbert spaces as follow.

Definition 1.1 ([5]). Let E be a nonempty closed convex subset of a Hilbert space B. A mapping $\xi : E \to E$ is said to satisfy condition (AK) (or generalized AK-α-nonexpansive) for some real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$ if

$$\begin{aligned}\|\xi a - \xi b\| \leq\ & \alpha_1\|\xi a - a\| + \alpha_2\|\xi b - b\| \\ & + \alpha_3\|\xi a - b\| + \alpha_4\|\xi b - a\| \\ & + (1 - 4\max\alpha)\|a - b\|,\end{aligned} \qquad (1.5)$$

where $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \max\alpha$ for all $a, b \in E$.

Notice that the class of AK-generalized nonexpansive mappings covers several well-known mappings. For example, every α-nonexpansive mappings is an AK-generalized nonexpansive mapping and also 0-nonexpansive maps are exactly nonexpansive maps. Hence we have the following diagram and related research used as a study guide in this work [6, 7].

On the other hand, we will review the knowledge of reproduction processes for use in research. Throughout this research, we will use the symbol $\mathbb{N}_0$ to represent non-negative integers.

Banach [8] proved that fixed points of contraction mappings can be approximated with the Picard iterative algorithm [9]. The Picard sequence fang is defined as follows:

$$\begin{cases} a_1 \in E, \\ a_{n+1} = \xi(a_n)\ n \in \mathbb{N}_0. \end{cases} \qquad (1.6)$$

In 1953, Mann [10] introduced a new iterative algorithm to approximate a fixed point for nonexpansive mappings.The sequence obtained by this algorithm is defined as follows:

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1 - \beta_n)a_n + \beta_n\xi(b_n)\ n \in \mathbb{N}_0, \end{cases} \qquad (1.7)$$

where $\{\beta_n\}$ is in $(0,1)$.

To overcome this problem, Ishikawa [11] introduced a two-step iterative algorithm to approximate the fixed point of pseudocontractive mapping as follows:

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1-\beta_n)a_n + \beta_n \xi(b_n), \\ b_n = (1-\gamma_n)a_n + \gamma_n \xi(a_n),\ n \in \mathbb{N}_0, \end{cases} \tag{1.8}$$

where $\{\beta_n\}$, and $\{\gamma_n\}$ are in $(0,1)$;

The proposed different iterative algorithms. Let start be an initial guess to different iterative algorithms. An iterative sequence is called

(i) Noor iteration [12] in 2000 if

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1-\beta_n)a_n + \beta_n \xi(b_n), \\ b_n = (1-\gamma_n)a_n + \gamma_n \xi(c_n), \\ c_n = (1-\delta_n)a_n + \delta_n \xi(a_n)\ n \in \mathbb{N}_0, \end{cases} \tag{1.9}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are in $(0,1)$;

(ii) Agarwal et al. iteration [13] in 2007 if

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1-\beta_n)\xi(a_n) + \beta_n \xi(b_n), \\ b_n = (1-\gamma_n)a_n + \gamma_n \xi(c_n),\ n \in \mathbb{N}_0, \end{cases} \tag{1.10}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$;

(iii) Abbas et al. iteration [14] in 2014 if

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1-\beta_n)\xi(b_n) + \beta_n \xi(c_n), \\ b_n = (1-\gamma_n)a_n + \gamma_n \xi(c_n), \\ c_n = (1-\delta_n)a_n + \delta_n \xi(a_n), \\ n \in \mathbb{N}_0, \end{cases} \tag{1.11}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are in $(0,1)$;

(iv) Thakur et al. iteration [15] in 2017 if

$$\begin{cases} a_1 \in E, \\ a_{n+1} = (1-\beta_n)\xi(c_n) + \beta_n \xi(b_n), \\ b_n = (1-\gamma_n)c_n + \gamma_n \xi(c_n), \\ c_n = (1-\delta_n)a_n + \delta_n \xi(a_n), \\ n \in \mathbb{N}_0, \end{cases} \tag{1.12}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are in $(0,1)$;

(v) Ullah et al. iteration [16] in 2018 if

$$\begin{cases} a_1 \in E, \\ a_{n+1} = \xi(b_n), \\ b_n = \xi(c_n), \\ c_n = (1-\delta_n)a_n + \delta_n \xi(a_n), \\ n \in \mathbb{N}_0, \end{cases} \tag{1.13}$$

where $\{\delta_n\}$ is in $(0,1)$.

Now, Abbas et al. [17] introduced a new iterative algorithm known as the AA-iterative algorithm, which converges faster than the iterative algorithms mentioned above for the class of enriched contraction and contraction mapping as follows:

$$\begin{cases} a_1 \in E, \\ a_{n+1} = \xi(b_n) \\ b_n = (1-\beta_n)\xi(d_n) + \beta_n \xi(c_n), \\ c_n = (1-\gamma_n)d_n + \gamma_n \xi(d_n), \\ d_n = (1-\delta_n)a_n + \delta_n \xi(a_n), \\ n \in \mathbb{N}_0, \end{cases} \tag{1.14}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are in $(0,1)$.

In this paper, we are inspired by the collective ideas of the authors mentioned, making us interested in studying about the strong and weak convergence results by using the AA-iterative algorithm (1.14) for the generalized AK-α-nonexpansive mappings (1.5) in uniformly convex Banach spaces

. In particular, we provide only sufficient convergence criteria for a suitable procedural method. Our work is more general and unifies the comparable results in the existing literature, for instance, the results given in [1].

2. Preliminaries

Definition 2.1 ([18])**.** Let E be a nonempty closed convex subset of a Banach space B. A mapping $\xi : E \to E$ is called demiclosed with respect to $b \in B$ if, for each sequence $\{a_n\}$ in E and $a \in E$, $\{a_n\}$ converges weakly to a, and $\{\xi a_n\}$ converges strongly to b, implying that $\xi a = b$.

Definition 2.2 ([19])**.** A Banach space B satisfies Opial's condition if, for each sequence $\{a_n\}$ converging weakly to $a \in B$, the following holds:

$$\liminf_{n\to\infty} \|a_n - a\| \leq \liminf_{n\to\infty} \|a_n - b\|, \tag{2.1}$$

for all $b \in B$, with $a \neq b$.

Definition 2.3 ([20])**.** A mapping $\xi : E \to E$ satisfies Condition (I) if there exists an increasing function $r : [0, \infty) \to [0, \infty)$ with $r(0) = 0$ and $r(t) > 0$, for all $t > 0$, such that $d(a, \xi(a)) \geq \lim_{n\to\infty} r(d(a, Fix(\xi))$ for all $a \in E$ where $d(a_n, A) := \inf\{\|a_n - a^*\| : a^* \in A\}$.

Lemma 2.4 ((Lemma 3.1, [5]))**.** *Let E be a nonempty closed convex subset of a Banach space B and $\xi : E \to E$ be an generalized AK-α-nonexpansive nonexpansive mapping with $Fix(\xi) \neq \emptyset$. Then $Fix(\xi)$ is closed convex and $\|\xi a - p\| \leq \|a - p\|$ for all $a \in E$ and $p \in F(\xi)$.*

3. Main results

In this section, we prove some strong and weak convergence results using AA-iterative scheme (1.14) for generalized AK α-nonexpansive mappings (1.5) in a uniformly convex Banach space B.

Lemma 3.1. Let E be a nonempty subset of a Banach space B and $\xi : E \to E$ be a generalized AK-α-nonexpansive nonexpansive mapping. Then, for all $a, b \in E$:

$$\|a - \xi b\| \leq \|a - b\| + \frac{(1 + \alpha_1 + \alpha_3)}{(1 - \alpha_2 - \alpha_4)}\|a - \xi a\|. \tag{3.1}$$

Proof. Since $\xi : E \to E$ is a generalized AK-α-nonexpansive mapping,

$$\begin{aligned}\|\xi a - \xi b\| \leq{}& \alpha_1\|\xi a - a\| + \alpha_2\|\xi b - b\| \\ &+ \alpha_3\|\xi a - b\| + \alpha_4\|\xi b - a\| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})\|a - b\|,\end{aligned}$$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$. We consider

$$\begin{aligned}\|a - \xi b\| &\leq \|a - \xi a\| + \|\xi a - \xi b\| \\ &\leq \|a - \xi a\| + \alpha_1\|\xi a - a\| + \alpha_2\|\xi b - b\| \\ &+ \alpha_3\|\xi a - b\| + \alpha_4\|\xi b - a\| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) \\ &\leq \|a - \xi a\| + \alpha_1\|\xi a - a\| + \alpha_2\|\xi b - a\| \\ &+ \alpha_2\|a - b\| + \alpha_3\|\xi a - a\| \\ &+ \alpha_3\|a - b\| + \alpha_4\|\xi b - a\| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})\|a - b\|.\end{aligned}$$

This implies that

$$\begin{aligned}(1 - \alpha_2 - \alpha_4)\|a - \xi y\| &\leq (1 + \alpha_1 + \alpha_3)\|a - \xi a\| \\ &+ (1 + \alpha_2 + \alpha_3 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})\|a - b\|.\end{aligned}$$

From $2\alpha_2 + \alpha_3 + \alpha_4 < 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, we get

$$\begin{aligned}\|a - \xi b\| &\leq \frac{(1 + \alpha_1 + \alpha_3)}{(1 - \alpha_2 - \alpha_4)}\|a - \xi a\| \\ &+ \frac{(1 + \alpha_2 + \alpha_3 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})}{(1 - \alpha_2 - \alpha_4)}\|a - b\| \\ &\leq \frac{(1 + \alpha_1 + \alpha_3)}{(1 - \alpha_2 - \alpha_4)}\|a - \xi a\| + \|a - b\|.\end{aligned}$$

The complete proof. ∎

Lemma 3.2. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ a generalized AK-α-nonexpansive mapping with $Fix(\xi) \neq \emptyset$. If $\{a_n\}$ is a sequence defined by AA-iterative algorithm (1.14), then $\lim_{n\to\infty} \|a_n - a^*\|$ exists for all $a^* \in Fix(\xi)$.

Proof. Let $a^* \in Fix(\xi)$. Since ξ satisfies Condition (AK-α), with Lemma 2.4, ξ is quasi-nonexpansive mapping, that is, $\|\xi a - p\| \leq \|a - p\|$, for $p \in Fix(\xi)$. By Iterative algorithm (1.14), we have

$$\begin{aligned}
&\|d_n - a^*\| \\
&= \|(1-\sigma_n)a_n + \sigma_n\xi(a_n) - a^*\| \\
&\leq (1-\sigma_n)\|a_n - a^*\| + \sigma_n\|\xi(a_n) - a^*\| \\
&\leq (1-\sigma_n)\|a_n - a^*\| + \sigma_n\|a_n - a^*\| \\
&= \|a_n - a^*\|. \qquad (3.2)
\end{aligned}$$

If $t_n = (1-\rho_n)d_n + \rho_n\xi(d_n)$, then

$$\|c_n - a^*\| = \|\xi(t_n) - a^*\|. \qquad (3.3)$$

Now,

$$\begin{aligned}
&\|\xi(t_n) - a^*\| \\
&\leq \|t_n - a^*\| \\
&\leq \|(1-\rho_n)d_n + \rho_n\xi(d_n) - a^*\| \\
&\leq (1-\rho_n)\|d_n - a^*\| + \rho_n\|\xi(d_n) - a^*\| \\
&\leq (1-\rho_n)\|d_n - a^*\| + \rho_n\|d_n - a^*\| \\
&\leq \|d_n - a^*\| \\
&\leq \|a_n - a^*\|. \qquad (3.4)
\end{aligned}$$

So,

$$\|c_n - a^*\| \leq \|a_n - a^*\|. \qquad (3.5)$$

Now, take $u_n = (1-\eta_n)\xi(d_n) + \eta_n\xi(c_n)$,

$$\begin{aligned}
&\|b_n - a^*\| \\
&\leq \|\xi(u_n) - a^*\| \\
&\leq \|u_n - a^*\| \\
&\leq \|(1-\eta_n)\xi(d_n) + \eta_n\xi(c_n) - a^*\| \\
&\leq (1-\eta_n)\|\xi(d_n) - a^*\| + \eta_n\|\xi(c_n) - a^*\| \\
&\leq (1-\eta_n)\|d_n - a^*\| + \eta_n\|c_n - a^*\| \\
&\leq (1-\eta_n)\|a_n - a^*\| + \eta_n\|a_n - a^*\| \\
&\leq \|a_n - a^*\|. \qquad (3.6)
\end{aligned}$$

Now,

$$\begin{aligned}
\|a_{n+1} - a^*\| &\leq \|\xi(b_n) - a^*\| \\
&\leq \|b_n - a^*\|. \qquad (3.7)
\end{aligned}$$

Thus,

$$\|a_{n+1} - a^*\| \leq \|a_n - a^*\|. \qquad (3.8)$$

This shows that $\{\|a_n - a^*\|\}$ is decreasing and bounded from the below sequence for each $a^* \in Fix(\xi)$. Hence, $\lim_{n\to\infty} \|a_n - a^*\|$ exists. ∎

Lemma 3.3. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ a generalized AK-α-nonexpansive mapping. If $\{a_n\}$ is a sequence defined by AA-iterative algorithm (1.14), then $Fix(\xi) \neq \emptyset$ if and only if $\{a_n\}$ is bounded and $\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0$.

Proof. With Lemma 3.2 above, $\lim_{n\to\infty} \|a_n - a^*\|$ exists, and $\{a_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|a_n - a^*\| = k. \qquad (3.9)$$

From Eqs. (3.2), (3.5), (3.6) and (3.9), we have

$$\limsup_{n\to\infty} \|b_n - a^*\| \leq \limsup_{n\to\infty} \|a_n - a^*\| = k, \qquad (3.10)$$

$$\limsup_{n\to\infty} \|c_n - a^*\| \leq \limsup_{n\to\infty} \|a_n - a^*\| = k, \qquad (3.11)$$

$$\limsup_{n\to\infty} \|d_n - a^*\| \leq \limsup_{n\to\infty} \|a_n - a^*\| = k. \qquad (3.12)$$

It follows from Lemma 2.4 that

$$\limsup_{n\to\infty} \|\xi(a_n) - a^*\| \leq \limsup_{n\to\infty} \|a_n - a^*\| = k, \quad (3.13)$$

$$\|a_{n+1} - a^*\| = \|\xi(b_n) - \xi(a^*)\| \leq \|b_n - a^*\|. \quad (3.14)$$

By taking lim inf as $n \to \infty$, we obtain

$$\begin{aligned} k &\leq \liminf_{n\to\infty} \|b_n - a^*\| \\ &\leq \limsup_{n\to\infty} \|b_n - a^*\| \\ &\leq k, \end{aligned} \quad (3.15)$$

we have

$$\lim_{n\to\infty} \|b_n - a^*\| = k. \quad (3.16)$$

Now, from

$$\begin{aligned} \|a_{n+1} - a^*\| &\leq \|b_n - a^*\| \\ &\leq \|\xi(c_n) - \xi(a^*)\| \\ &\leq \|c_n - a^*\|, \end{aligned} \quad (3.17)$$

we obtain that

$$\begin{aligned} k &\leq \liminf_{n\to\infty} \|c_n - a^*\| \\ &\leq \limsup_{n\to\infty} \|c_n - a^*\| \\ &\leq k, \end{aligned} \quad (3.18)$$

we have

$$\lim_{n\to\infty} \|c_n - a^*\| = k. \quad (3.19)$$

From

$$\|a_{n+1} - a^*\| \leq \|c_n - a^*\| \leq \|d_n - a^*\|, \quad (3.20)$$

we obtain that

$$\begin{aligned} k &\leq \liminf_{n\to\infty} \|d_n - a^*\| \\ &\leq \limsup_{n\to\infty} \|d_n - a^*\| \\ &\leq k, \end{aligned} \quad (3.21)$$

we have

$$\lim_{n\to\infty} \|d_n - a^*\| = k. \quad (3.22)$$

And,

$$\begin{aligned} k &\leq \lim_{n\to\infty} \|d_n - a^*\| \\ &= \lim_{n\to\infty} \|(1-\sigma_n)a_n + \sigma_n\xi(a_n) - a^*\| \\ &\leq \lim_{n\to\infty} ((1-\sigma_n)\|a_n - a^*\| + \sigma_n\|\xi(a_n) - a^*\|) \\ &\leq \lim_{n\to\infty} ((1-\sigma_n)\|a_n - a^*\| + \sigma_n\|a_n - a^*\|) \\ &\leq \lim_{n\to\infty} \|a_n - a^*\| \\ &\leq k. \end{aligned} \quad (3.23)$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} \|(1-\sigma_n)(a_n - a^*) \\ +\sigma_n(\xi(a_n) - a^*)\| = k. \end{aligned} \quad (3.24)$$

By Lemma 3.3 , we obtain

$$\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0. \quad (3.25)$$

Conversely, suppose $\{a_n\}$ is bounded and

$$\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0. \quad (3.26)$$

Let $a^* \in A(E, \{a_n\})$. Through Lemma 3.1, we have

$$\begin{aligned} r(\xi, \{a_n\}) &= \limsup_{n\to\infty} \|a_n - \xi(a^*)\| \\ &\leq \limsup_{n\to\infty} (\|a_n - a^*\| \\ &\quad + \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|a_n - \xi(a_n)\|) \\ &\leq \limsup_{n\to\infty} \|a_n - a^*\| \\ &\leq r(a^*, \{a_n\}) \\ &\leq r(E, \{a_n\}), \end{aligned} \quad (3.27)$$

which implies that $\xi(a^*) \in A(E, \{a_n\})$. Since B is uniformly convex, $A(E, \{a_n\})$ is a singleton. Hence, we have $\xi(a^*) = a^*$. ∎

Theorem 3.4. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ a generalized AK-α-nonexpansive mapping and $I - \xi$ is demiclosed at zero. If $\{a_n\}$ is a sequence defined by the AA-iterative algorithm (1.14), then $\{a_n\}$ converges weakly to a point of $Fix(\xi)$, provided that B satisfies Opial's condition.

Proof. Let $a^* \in Fix(\xi)$. Through Lemma 3.2, $\lim_{n\to\infty} \|a_n - a^*\|$ exists. Now, we show that $\{a_n\}$ has a unique weak subsequential limit in $Fix(\xi)$. Suppose a, b are weak limits of subsequences $\{a_{n_i}\}$ and $\{a_{n_j}\}$ of $\{a_n\}$, respectively. From Lemma 3.3, we have $\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0$. Moreover, from $I - \xi$ is demiclosed at zero. This implies that $(I - \xi)a = 0$, that is $\xi(a) = a$. Now, we show the uniqueness. Suppose that $\xi(b) = b$. If $a \neq b$ then by using Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|a_n - a\| &= \lim_{n\to\infty} \|a_{n_i} - a\| \\
&< \lim_{n\to\infty} \|a_{n_i} - b\| \\
&= \lim_{n\to\infty} \|a_n - b\| \\
&= \lim_{n\to\infty} \|a_{n_j} - b\| \\
&< \lim_{n\to\infty} \|a_{n_j} - a\| \\
&= \lim_{n\to\infty} \|a_n - a\|,
\end{aligned}$$

a contradiction; so, $a = b$. Consequently, $\{a_n\}$ converges weakly to a point of $Fix(\xi)$. ∎

Theorem 3.5. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ a generalized AK-α-nonexpansive mapping. If $\{a_n\}$ is a sequence defined by the AA-iterative algorithm (1.14), then $\{a_n\}$ converges to a point of $Fix(\xi)$, if and only if $\liminf_{n\to\infty} d(a_n, Fix(\xi)) = 0$ or $\limsup_{n\to\infty} d(a_n, Fix(\xi)) = 0$ where $d(a_n, A) := \inf\{\|a_n - a^*\| : a^* \in A\}$.

Proof. $(\Rightarrow)$ If $\{a_n\}$ converges to a point $a^* \in Fix(\xi)$, then $\liminf_{n\to\infty} d(a_n, Fix(\xi)) = 0$ and $\limsup_{n\to\infty} d(a_n, Fix(\xi)) = 0$.
$(\Leftarrow)$ Suppose that $\liminf_{n\to\infty} d(a_n, Fix(\xi)) = 0$. From Lemma 3.2, $\lim_{n\to\infty} \|a_n - a^*\|$ exists for all $a^* \in Fix(\xi)$, thus, by assumption, $\lim_{n\to\infty} d(a_n, Fix(\xi)) = 0$. We now show that $\{a_n\}$ is a Cauchy sequence in E. For given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$, such that, for all $n \geq m_0$, $d(a_n, Fix(\xi)) < \frac{\varepsilon}{2}$ that is $\inf\{\|a_n - a^*\| : a^* \in Fix(\xi)\} < \frac{\varepsilon}{2}$. In particular, $\inf\{\|a_n - a^*\| : a^* \in Fix(\xi)\} < \frac{\varepsilon}{2}$. Therefore, there exists $a^* \in Fix(\xi)$ such that $\|a_n - a^*\| < \frac{\varepsilon}{2}$. Now, for $m, n \geq m_0$,

$$\begin{aligned}
\|a_{n+m} - a_n\| &\leq \|a_{n+m} - a^*\| + \|a^* - a_n\| \\
&\leq \|a_{m_0} - a^*\| + \|a^* - a_{m_0}\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon. \qquad (3.28)
\end{aligned}$$

This is to show that $\{a_n\}$ is a Cauchy sequence in E. As E. is a closed subset of a Banach space B, there is a point $a \in E$, such that $\lim_{n\to\infty} a_n = a$. Now, $\lim_{n\to\infty} d(a_n, Fix(\xi)) = 0$. Hence, $a \in Fix(\xi)$. ∎

Theorem 3.6. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ be a generalized AK-α-nonexpansive mapping. If $\{a_n\}$ is a sequence defined with the AA-iterative algorithm (1.14) and $Fix(\xi) \neq \emptyset$, then $\{a_n\}$ converges strongly to a fixed point of ξ.

Proof. From $Fix(\xi) \neq \emptyset$. Thus, by using Lemma 3.3, we have $\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0$. Since E is compact, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to a^*$ for some

$a^* \in E$. Through Lemma 3.1, we have for $k \geq 1$,

$$\|a_{n_k} - \xi(a^*)\| \leq \|a_{n_k} - \xi(a_{n_k})\| + \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|a_{n_k} - a^*\|. \tag{3.29}$$

On taking the limit to be $k \to \infty$, we obtain that $a_{n_k} \to \xi(a^*)$. This implies that $\xi(a^*) = a^*$, that is, a $a^* \in Fix(\xi)$. In addition, $\lim_{n\to\infty} \|a_n - a^*\|$ exists by Lemma 3.2. Thus, $\{a^*\}$ is the limit of a sequence $\{a_n\}$. ∎

Now, we prove a strong convergence result using condition (I) in Definitions 2.3.

Theorem 3.7. Let E be a nonempty closed convex subset of a uniformly convex Banach space B and $\xi : E \to E$ be a generalized AK-α-nonexpansive mapping satisfying Condition (I). If $\{a_n\}$ is a sequence defined with the AA-iterative algorithm (1.14) , then $\{a_n\}$ converges strongly to a fixed point of ξ.

Proof. By using Lemma 3.3, we have $\lim_{n\to\infty} \|a_n - \xi(a_n)\| = 0$. From Condition (I) and (3.15), we obtain that

$$0 \leq \lim_{n\to\infty} r(d(a_n, Fix(\xi)) \leq \lim_{n\to\infty} \|a_n - \xi(a_n)\|,$$

which implies that $0 = \lim_{n\to\infty} r(d(a_n, Fix(\xi))$. Since r is an increasing function satisfying $r(0) = 0$, $r(t) > 0$ for all $t > 0$. Hence, we have that $0 = \lim_{n\to\infty} d(a_n, Fix(\xi)$, Now, all the conditions of Theorem 3.5 are satisfied; therefore, $\{a_n\}$ converges strongly to a fixed point of ξ. ∎

Example 3.1. Let $E = [0, 2]$ be a nonempty closed convex subset of a Hilbert space $(E = \mathbb{R}, \|\cdot\| = |\cdot|)$. Suppose that $\xi : [0, 2] \to [0, 2]$ be given by $\xi a = \tan a + \cot a$, for all $a \in [0, 2]$. We can easily prove that ξ is an generalized AK-α-generalized nonexpansive.

Competing interests

The authors declare that they have no competing interests.

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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