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Fixed Point of Suzuki-Geraghty Type θ-Contractions in Partial Metric Spaces with Some Applications

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ABSTRACT

The aim of this paper is to introduce the notions of Suzuki-Geraghty type θ- contractions and to establish some fixed point theorems in the setting of complete partial metric spaces and give an example to illustrate these main results. Moreover, we utilize our results to study the existence problem of solutions of nonlinear Hammerstein integral equations.

Keywords: θ-contraction; Fixed point; Geraghty contraction; Hammerstein integral equation; Suzuki contraction

1. Introduction

Let X be a non empty set and $T : X \to X$ be a self mapping. A point $x^* \in X$ is said to be a fixed point of T if $Tx^* = x^*$ and if d is a metric on X, then T is called contraction if there is $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \;\; \forall x, y \in X. \quad (1.1)$$

The newness of fixed point theory in distance spaces discovered in 1922 by Banach [1] which is well known as Banach's contraction principle or Banach's fixed point theorem is as follows:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq kd(x, y), \quad (1.2)$$

for all $x, y \in X$ where $0 \leq k < 1$. Then, T has a unique fixed point in X.

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Due to its importance and simplicity, several authors have obtained many interesting extensions of Banach's contraction principle [6–12].

In 1973, Geraghty[3] generalized Banach's contraction principle as follow:

Theorem 1.2. *Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying the following condition: there exists $\beta \in \mathfrak{F}$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

where $\mathfrak{F}$ denotes the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the following condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0.$$

Then T has a unique fixed point $z \in X$ and $\{T^n x\}$ converges to the point z for each $x \in X$.

In 2009, Suzuki[4] introduced a generalized Banach's contractions in compact metric spaces.

Theorem 1.3. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying*

$$\frac{1}{2}d(x, Tx) < d(x, y) \longrightarrow d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point in X.

In 2014, Jleli and Samet[5] introduced the following notion of a θ-contraction as follows:

- (Θ_1) θ is nondecreasing;
- (Θ_2) for any sequence $\{t_n\}$ in $(0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
- (Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;
- (Θ_4) θ is continuous.

In the sequel, we denote by $\tilde{\Theta}$ the set of all the functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

$(\tilde{\Theta}_1)$ θ is non-decreasing and continuous;

$(\tilde{\Theta}_2)$ $\inf_{t\in(0,\infty)} \theta(t) = 1$.

Example 1.4. It is obvious that the following functions belong to the set $\tilde{\Theta}$:

(1) $\theta_1(t) := e^{e^{-\frac{1}{t^p}}}$ for all $p > 0$;
(2) $\theta_2(t) := 1 + t$ for all $t > 0$;
(3) $\theta_3(t) := e^{\sqrt{t}}$ for all $t > 0$;
(4) $\theta_4(t) := 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha})$ for all $0 < \alpha < 1$ and $t > 0$.

Also, they generalized Banach's fixed point theorem in a generalized metric space, which sometimes is called a Branciari metric space as follows:

Theorem 1.5. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$d(Tx, Ty) \neq 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k$$

for all $x, y \in X$. Then T has a unique fixed point in X.

The aim of this paper is to establish the existence and uniqueness of fixed point in complete partial metric space by using the concept of Suzuki contraction, Geraghty contraction and θ-contraction. The results presented in the paper extend and improve the results in Banach, Suzuki, Geraghty, Jleli and Samet. An example to support the main results is illustrated. In addition, we shall utilize our results to study the existence problem of solutions of nonlinear Hammerstein integral equations.

2. Preliminaries

In this section, we give some definitions, examples and fundamental results.

Definition 2.1. [2] Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ satisfy following properties:

(PM1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$;

(PM2) $p(x,x) \leq p(x,y)$;

(PM3) $p(x,y) = p(y,x)$;

(PM4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called a partial metric on X and the pair (X, p) is known as partial metric space.

In 1995, Matthews [2] proved that every partial metric p on X induces a metric $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

for all $x, y \in X$.

Notice that a metric on a set X is a partial metric d such that $d(x,x) = 0$ for all $x \in X$.

Definition 2.2. [2] Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 2.3. *[2]*

(i) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete.

(ii) A sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(x, x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) If $\lim_{n\to\infty} x_n = v$ such that $p(v,v) = 0$ then $\lim_{n\to\infty} p(x_n, y) = p(v, y)$ for every $y \in X$.

3. Fixed points of Suzuki-Geraghty type θ-contractions

In this section, we prove some fixed point theorems for Suzuki-Geraghty type θ-contractions in complete partial metric spaces.

First, we begin with the following definition:

Definition 3.1. Let (X, d) be a partial metric space. A mapping $T : X \to X$ is said to be Suzuki-Geraghty type θ-contraction, if there exist $k \in (0,1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow$$
$$\theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

The following theorem is our main result in this paper:

Theorem 3.2. *Let (X, d) be a complete partial metric space and $T : X \to X$ be a Suzuki-Geraghty type θ-contraction, if* *there exist $k \in (0,1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow$$
$$\theta(d(Tx,Ty)) \leq [\theta(\beta(d(x,y))d(x,y))]^k.$$

Then T has a unique fixed point $z \in X$.

Proof. Let x be an arbitrary point in X. If for some positive integer p such that $T^{p-1}x = T^p x$, then $T^{p-1}x$ will be a fixed point of T. So, without loss of generality, we can assume that $d(T^{n-1}x, T^n x) > 0$ for all $n \geq 1$.
Therefore,

$$\frac{1}{2}d(T^{n-1}x,T^n x) < d(T^{n-1}x,T^n x), \quad \forall n \geq 1. \tag{3.1}$$

Hence from (3.1), for all $n \geq 1$, we have

$$\begin{aligned}
&\theta(d(TT^{n-1}x, TT^n x)) \\
&= \theta(d(T^n x, T^{n+1}x)) \\
&\leq [\theta(\beta(d(T^{n-1}x,T^n x))d(T^{n-1}x,T^n x))]^k \\
&< [\theta(d(T^{n-1}x,T^n x))]^k \\
&< \theta(d(T^{n-1}x,T^n x)).
\end{aligned}$$

So,

$$\theta(d(T^n x,T^{n+1}x)) < \theta(d(T^{n-1}x,T^n x)). \tag{3.2}$$

This implies that

$$\theta(d(T^{n+1}x,T^{n+2}x)) < \theta(d(T^n x,T^{n+1}x)), \tag{3.3}$$

for each $n \geq 0$, Taking $n \to \infty$ in (3.3), we have

$$\theta(d(T^n x,T^{n+1}x)) \to 1.$$

Therefore, from $(\tilde{\Theta}_2)$, it follows that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{3.4}$$

Now, we show that $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Arguing by contradiction, we assume that $\{T^n x\}_{n=1}^{\infty}$ is not a Cauchy sequence in X, that is, there exists $\varepsilon > 0$, we can find the sequence $\{p^n\}_{n=1}^{\infty}$ and $\{q^n\}_{n=1}^{\infty}$ of natural numbers such that

$$\begin{aligned}
&p^n > q^n > n, \quad d(T^{p_n}, T^{q_n}) \geq \varepsilon, \\
&d(T^{p_n - 1}, T^{q_n}) < \varepsilon, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\varepsilon &\leq d(T^{p_n}x, T^{q_n}x) \\
&\leq d(T^{p_n}x, T^{p_n-1}x) + d(T^{p_n-1}x, T^{q_n}x) \\
&\quad - d(T^{p_n-1}x, T^{p_n-1}x) \\
&\leq d(T^{p_n}x, T^{p_n-1}x) + d(T^{p_n-1}x, T^{q_n}x) \\
&< d(T^{p_n}x, T^{p_n-1}x) + \varepsilon.
\end{aligned}$$

Thus, from (3.4) and the above inequality, it follows that

$$\lim_{n\to\infty} d(T^{p_n}x, T^{q_n}x) = \varepsilon. \tag{3.5}$$

From (3.4) and (3.5), we can choose a positive integer $n_0 > 1$ such that

$$\begin{aligned}
&\frac{1}{2}d(T^{p_n}x, TT^{p_n}x) < \frac{1}{2}\varepsilon \\
&< d(T^{p_n}x, T^{q_n}x), \quad \forall n > n_0. \qquad (3.6)
\end{aligned}$$

So, from assumption of the theorem, we have $\forall n \geq n_0$

$$\begin{aligned}
&\theta(d(T^{p_n+1}x, T^{q_n+1}x)) && (3.7) \\
&\leq [\theta(\beta(d(T^{p_n}x, T^{q_n}x))d(T^{p_n}x, T^{q_n}x)]^k. && (3.8)
\end{aligned}$$

Substituting (3.7) into (3.6), then letting $n \to \infty$ and by using the condition $(\Theta_2)'$, (3.4) and (3.5), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \theta(d(T^{p_n+1}x, T^{q_n+1}x)) \\
&\leq \lim_{n\to\infty} [\theta(\beta(d(T^{p_n}x, T^{q_n}x))d(T^{p_n}x, T^{q_n}x))]^k \\
&< \lim_{n\to\infty} [\theta(d(T^{p_n}x, T^{q_n}x))]^k.
\end{aligned}$$

By the triangle inequality, we have

$$\lim_{n\to\infty} \theta(d(T^{p_n+1}x, T^{q_n+1}x)) < \lim_{n\to\infty} [\theta(d(T^{p_n+1}x, T^{q_n+1}x))]^k.$$

This is a contradiction. Therefore $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. By the completeness of (X, d), without loss of generality, wr can assume that $\{T^n x\}_{n=1}^{\infty}$ converges to some point $z \in X$, that is

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{3.9}$$

Now, we claim that

$$\frac{1}{2} d(T^{n+1}x, T^{n+2}x) < d(T^{n+1}x, z), \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Suppose to the contrary that (3.10) is not true. Therefore the following inequality is also not true:

$$\frac{1}{2} d(T^n x, T^{n+1}x) < d(T^n x, z), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

It follows from (3.10) and (3.11) that, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2} d(T^m x, T^{m+1}x) \geq d(T^m x, z)$$
$$\text{and } \frac{1}{2} d(T^{m=1}x, T^{m+2}x) \geq d(T^{m+1}x, z). \tag{3.12}$$

Therefore,

$$2d(T^m x, z) \leq d(T^m x, T^{m+1}x) \leq d(T^m x, z) + d(z, T^{m+1}x).$$

This implies that

$$d(T^m x, z) \leq d(z, T^{m+1}x). \tag{3.13}$$

This together with (3.12) shows that

$$d(T^m x, z) \leq d(z, T^{m+1}x) \leq \frac{1}{2} d(T^{m+1}x, T^{m+2}x). \tag{3.14}$$

Since $\frac{1}{2} d(T^m x, T^{m+1}x) < d(T^m x, T^{m+1}x)$, by the assumption of theorem, we get

$$\theta(d(T^{m+1}x, T^{m+2}x)) \leq [\theta(\beta(d(T^m x, T^{m+1}x))d(T^m x, T^{m+1}x))]^k < [\theta(d(T^m x, T^{m+1}x))]^k. \tag{3.15}$$

Since $k \in (0, 1)$, we obtain

$$[\theta(d(T^m x, T^{m+1}x))]^k < \theta(d(T^m x, T^{m+1}x)).$$

Hence, from condition $(\Theta_1)'$ and (3.15), we have

$$d(T^{m+1}x, T^{m+2}x) < d(T^m x, T^{m+1}x). \tag{3.16}$$

This together with (3.14) shows that

$$\begin{aligned} &d(T^{m+1}x, T^{m+2}x) \\ &< d(T^m x, T^{m+1}x) \\ &\leq d(T^m x, z) + d(z, T^{m+1}x) \\ &\leq \frac{1}{2} d(T^{m+1}x, T^{m+2}x) + \frac{1}{2} d(T^{m+1}x, T^{m+2}x) \\ &= d(T^{m+1}x, T^{m+2}x), \end{aligned}$$

which is a contradiction. Therefore the inequality (3.10) is proved.
By assumption of Theorem 3.2 and (3.10) we have, for every $n \in \mathbb{N}$,

$$\theta(d(TT^{n+1}x, Tz)) \tag{3.17}$$
$$\leq [\theta(\beta(d(T^{n+1}x, z))d(T^{n+1}x, z))]^k. \tag{3.18}$$

On the other hand, from (3.9) we know that $T^n x \to z$. So, we have

$$d(T^{n+1}x, z) \to d(z, Tz). \tag{3.19}$$

Now, we claim that $d(z, Tz) = 0$. In fact, if $d(z, Tz) > 0$, letting $n \to \infty$ in (3.17) and by using (3.9), (3.19) and the condition $(\Theta_1)'$, we have

$$\theta(d(z, Tz))$$

$$= \lim_{n\to\infty} \theta(d(TT^{n+1}x, Tz))$$
$$\leq \lim_{n\to\infty} [\theta(\beta(d(T^{n+1}x, z))d(T^{n+1}x, z))]^k$$
$$= [\theta(\beta(d(z, Tz))d(z, Tz))]^k$$
$$< [\theta(d(z, Tz))]^k$$
$$< \theta(d(z, Tz)).$$

This is a contradiction. Therefore, $z = Tz$, i.e., z is a fixed point of T. □

Remark 3.3. *Theorem 3.2 is a generalization and improvement of the main results in Suzuki[4]*

It follows from Definition 3.1 that if $T : X \to X$ is a Geraghty type θ-contraction, then $T : X \to X$ is a Suzuki-Geraghty type θ-contraction. Hence, from Theorem 3.2 we can obtain the following existence theorem of fixed point for Geraghty type θ-contractions.

Definition 3.4. Let (X, d) be a partial metric space. A mapping $T : X \to X$ is said to be Suzuki-Geraghty type θ-contraction, if there exist $k \in (0, 1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$d(Tx, Ty) > 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(\beta(d(x, y))d(x, y))]^k. \quad (3.20)$$

Theorem 3.5. *Let (X, d) be a complete partial metric space and $T : X \to X$ be a Suzuki-Geraghty type θ-contraction, if there exist $k \in (0, 1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$d(x, y) > 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(\beta(d(x, y))d(x, y))]^k. \quad (3.21)$$

Then T has a unique fixed point $z \in X$.

Corollary 3.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a Suzuki-Geraghty type θ-contraction, if there exist $k \in (0, 1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(\beta(d(x, y))d(x, y))]^k.$$

Then T has a unique fixed point $z \in X$.

Corollary 3.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a Suzuki-Geraghty type θ-contraction, if there exist $k \in (0, 1)$, $\theta \in \tilde{\Theta}$ and $\beta \in \mathfrak{F}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$d(x, y) > 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(\beta(d(x, y))d(x, y))]^k.$$

Then T has a unique fixed point $z \in X$.

Now, we give an example to illustrate Theorem 3.2 as follows:

Example 3.8. Let $X = \{\frac{1}{2}, \frac{1}{3}, 1, 2, 3\}$ and $d(x, y) = \max\{x, y\}$ for all $x, y \in X$. We define a mapping $T : X \to X$ by

$$T(x) = \frac{1}{x},$$

a function $\beta : [0, \infty) \to [0, 1)$ by

$$\beta(x) = \frac{x}{x+1},$$

and a function $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(x) = 1 + x.$$

Now, we show that the condition (3.1) of Theorem 3.2 is satisfied. Consider the following cases:

Case I. If $x = 1$ and $y = 2$, then we have

$$\frac{1}{2}d(1, T(1)) < d(1, 2)$$
$$\frac{1}{2}d(1, 1) < d(1, 2)$$

$$\frac{1}{2} \cdot 1 < 2$$
$$\frac{1}{2} < 2$$

implies that

$$\begin{aligned}
\theta(d(T(1), T(2))) &\leq [\theta(\beta(d(1,2))d(1,2))]^k \\
\theta(d(1, \frac{1}{2})) &\leq [\theta(\beta(2) \cdot 2)]^k \\
\theta(1) &\leq [\theta(\frac{2}{3} \cdot 2)]^k \\
2 &\leq [\theta(\frac{4}{3})]^k \\
2 &\leq [\frac{7}{3}]^k, \text{ choose } k = \frac{9}{10} \\
2 &\leq [\frac{7}{3}]^{\frac{9}{10}}
\end{aligned}$$

Case II. If $x = 1$ and $y = 3$, then we have

$$\frac{1}{2}d(1, T(1)) < d(1,3)$$
$$\frac{1}{2}d(1,1) < d(1,3)$$
$$\frac{1}{2} \cdot 1 < 3$$
$$\frac{1}{2} < 3$$

implies that

$$\begin{aligned}
\theta(d(T(1), T(3))) &\leq [\theta(\beta(d(1,3))d(1,3))]^k \\
\theta(d(1, \frac{1}{3})) &\leq [\theta(\beta(3) \cdot 3)]^k \\
\theta(1) &\leq [\theta(\frac{3}{4} \cdot 3)]^k \\
2 &\leq [\theta(\frac{9}{4})]^k \\
2 &\leq [\frac{13}{4}]^k, \text{ choose } k = \frac{9}{10} \\
2 &\leq [\frac{13}{4}]^{\frac{9}{10}}
\end{aligned}$$

Similarly, for other cases, we can check that the condition 3.1 holds, that is T is a Suzuki-Geraghty type θ-contraction. Thus, all assumptions of Theorem 3.2 are satisfied. Therefore, 1 is a unique fixed point of T.

4. Application

Motivated by the fact that a large class of boundary value problem can be converted to a Hammerstein integral equation, in this section, we consider the application of Theorem 3.2 to the Hammerstein type integral equation:

$$x(t) = h(t) + \int_0^t K(t,s) f(s, x(s)) ds, \tag{4.1}$$

where the unknown function $x(t)$ take real values. Let $X = C([0, E])$ be the space of all real continuous functions defined on $[0, E]$. It is well known that $C([0, E])$ endowed with the partial metric

$$p(x,y) = ||x - y|| + ||x|| + ||y||, \tag{4.2}$$

where $||u||_\tau = \max_{t \in [0,E]} |u(t)|e^{-\tau t}$, $\forall u \in X$, $\tau \geq 1$ is chosen arbitrary Note that p is also a partial metric on X and that

$$\begin{aligned}
d_p(x,y) &:= 2p(x,y) - p(x,x) - p(y,y) \\
&= 2||x - y||_\tau.
\end{aligned} \tag{4.3}$$

Hence, (X, p) is a complete as the metric space $(X, ||\cdot||_\tau)$ is complete.

Theorem 4.1. *Assume the conditions as follows:*

(i) $f \in C([0, E]) \times \mathbb{R}$, $h \in X$ *and* $K \in C([0, E]) \times C([0, E])$ *such that* $K(t,s) \geq 0$;

(ii) $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing for all* $t \in [0, E]$;

(iii) there exists $\tau \in [1, +\infty)$ *such that*

$$|f(t,x) - f(t,y)| \leq \frac{1}{2}\tau e^{-\tau}|x - y|,$$
$$\forall x, y \in X, t \in [0, E];$$

(iv) $\max_{t,s \in [0,E]} |K(t,s)| \leq 1$.

Then, the Hammerstein integral equation (4.1) has a unique solution.

Proof. We first show that the mapping $T : X \to X$ defined by, $\forall t \in [0, E]$

$$T(x)(t) = h(t) + \int_0^t K(t,s) f(s, x(s)) ds. \tag{4.4}$$

Now, by condition (iii) and (iv), for each $x, y \in C([0, E]), t \in [0, E]$, we have

$$\begin{aligned}
&|Tx(t) - Ty(t)| \\
&= |\int_0^t K(t,s)(f(s,x(s)) - f(s,y(s)))ds| \\
&\leq \int_0^t |K(t,s)||(f(s,x(s)) - f(s,y(s)))ds| \\
&\leq \int_0^t |(f(s,x(s)) - f(s,y(s)))ds| \\
&\leq \int_0^t \frac{1}{2}\tau e^{-\tau}|x(s) - y(s)|ds \\
&\leq \frac{1}{2}\tau e^{-\tau}\int_0^t e^{s\tau}|x(s) - y(s)|e^{-s\tau}ds \\
&\leq \frac{1}{2}\tau e^{-\tau}||x - y||_\tau \int_0^t e^{-s\tau}ds \\
&= \frac{1}{2}\tau e^{-\tau}||x - y||_\tau \frac{e^{\tau t}}{\tau} \\
&= \frac{1}{2}\tau e^{-\tau(1-t)}||x - y||_\tau.
\end{aligned}$$

This implies that $|Tx(t) - Ty(t)|e^{-\tau t} \leq \frac{1}{2}e^{-\tau}||x - y||_\tau$. Hence, we have

$$\begin{aligned}
d_\tau(Tx, Ty) &= \max_{t \in [0,E]}\{|Tx(t) - Ty(t)|e^{-\tau t}\} \\
&\leq \frac{1}{2}e^{-\tau}||x - y||_\tau.
\end{aligned} \tag{4.5}$$

Since $\theta(t) = e^{\sqrt{t}} \in \tilde{\Theta}, t > 0$ and $\beta \in \mathfrak{F}$, we have

$$\begin{aligned}
e^{\sqrt{d_\tau(Tx,Ty)}} &\leq e^{\sqrt{\frac{1}{2}e^{-\tau}||x-y||_\tau}} \\
&= e^{\sqrt{\frac{1}{2}\beta(||x-y||_\tau)||x-y||_\tau}} \\
&= [e^{\sqrt{\beta(||x-y||_\tau)||x-y||_\tau}}]^k
\end{aligned} \tag{4.6}$$

where $k = \sqrt{\frac{1}{2}}$ and $\beta(||x - y||_\tau) = e^{-\tau}$. Therefore T is a Geraghty type θ-contraction. By Theorem 3.2, T has a unique fixed point $x^* \in X$, i.e., x^* is the unique solution of the nonlinear Hammerstein imtegral equation (4.1). □

References

[1] Banach S. Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundamenta Mathematice 1992; 3:133-181.

[2] Mathews SG. Partial metric topology, Proceedings of the 11th Summer Conference on General Topology and Applications 1995;728: 183-197.

[3] Geraghty M. On contractive mappings, Proceedings of The American Mathematical Society 1973; 40:604-608.

[4] Suzuki T. A new type of fixed point theorem in metric spaces, Nonlinear Analysis 2009; 71:5313-5317.

[5] Jleli M. and Samet B. A new generalization of the Banach contraction principle, Journal of Inequalities and Applications 2014; 38.

[6] Jleli M., Karapinar E. and Samet B. Further generalizations of the Banach contraction principle, Journal of Inequalities and Applications 2014; 439.

[7] Liu XD., Chang SS. Xiao Y. and Zhao LC. Existence of fixed points for Θ-type contraction and Θ-type Suzuki contraction in complete metric spaces, Fixed Point Theory and Applications 2016; 8.

[8] Saipara P., Khammahawong K. and Kumam P. Fixed point theorem for a generalized almost Hardy-Rogers-type F contraction on metric-like space, Mathematical Methods in the Applied Sciences 2019; 42:5898-5919.

[9] Onsod W., Kumam P. and Cho YJ. Fixed point of $\alpha - \Theta-$Geraghty type and $\Theta-$Geraghty graphic type contraction, Applied General Topology 2017; 18:153-171.

[10] Onsod W.,Saleewong T. and Kumam P. Fixed and periodic point results for generalized Geraghty contractions in partial metric spaces with application, Thai journal of Mathematics 2020; 18:1247-1260.

[11] Pansuwan A. and Sintunavarat W. The new hybrid iterative algorithm for numerical Reckoning fixed point of Suzuki's generalized nonexpansive mappings with numerical experiments, Thai journal of Mathematics 2021; 19:167-168.

[12] Sintunavarat W. and Kumam P. Coupled coincidence and coupled common fixed point theorems in patially ordered metric spaces, Thai journal of Mathematics 2012; 10:551-563.