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A Note on Regularity of Certain Semigroups of Transformations Preserving a Reflexive and Transitive Relation

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ABSTRACT

In this paper, we introduce semigroups of transformations that preserve a reflexive and transitive relation. These semigroups are generalizations of well-known classes of transformation semigroups, namely semigroups of transformations preserving an equivalence relation and semigroups of order-preserving transformations. Additionally, we investigate the necessary and sufficient conditions for elements within such a semigroup, in certain cases, to be regular. These results extend the existing knowledge in the aforementioned semigroups.

Keywords: Reflexive relation; Regular element; Transformation semigroup; Transitive relation

1. Introduction

Let X be a nonempty set and $T(X)$ the set of all functions from X into X. Then $T(X)$ is a semigroup under the operation of composition. It is well-known that each $\alpha \in T(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X)$, that is, $T(X)$ is a regular semigroup (see [1], for details). Since every semigroup can be embedded in $T(X)$, for some appropriate set X, the structural properties on such type of semigroups have been researched extensively. For extending results, generalized transformation semigroups have been continually constructed.

For an equivalence relation E on a set X, let

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

Then $T_E(X)$ is a subsemigroup of $T(X)$. In

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particular, if $E = \{(x, x) : x \in X\}$ or $E = X \times X$, then $T_E(X) = T(X)$. However, in general, $T_E(X)$ is not regular. All regular elements within such semigroups have been completely described in [2]. Several properties of $T_E(X)$ have been identified, including those in [3–8].

Let $(X, \leq)$ be a partially ordered set. A map $\alpha : X \to X$ is said to be order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X$. We denoted by $OT(X)$ the semigroup of all order-preserving transformations on a partially ordered set X. Such a semigroup is a subsemigroup of $T(X)$ and plays an important role in the study of algebraic systems. Necessary and sufficient conditions for any elements of $OT(X)$ to be regular were discovered in [9]. Many others properties of $OT(X)$ have also been explored, see, for example [10–14].

In this paper, we introduce a subsemigroup of $T(X)$, which is a generalization of $T_E(X)$ and $OT(X)$, by letting σ_{RT} be a reflexive and transitive relation on X and

$$T(X, \sigma_{RT}) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in \sigma_{RT} \text{ implies } (x\alpha, y\alpha) \in \sigma_{RT}\}.$$

$T(X, \sigma_{RT})$ is called the semigroup of full transformations on X which preserve σ_{RT}. Particularly, $T(X, \sigma_{RT}) = T_{\sigma_{RT}}(X)$ whenever σ_{RT} is symmetric. Moreover, if (X, σ_{RT}) is a partially ordered set, then $T(X, \sigma_{RT}) = OT(X)$.

2. Main Results

Here, we present necessary and sufficient conditions for elements in $T(X, \sigma_{RT})$ to be regular, where σ_{RT} satisfies the condition, $\forall x, y \in X, (x, y) \in \sigma_{RT}$ or $(y, x) \in \sigma_{RT}$. Moreover, we prove that if $T(X, \sigma_{RT})$ is a regular semigroup, then $\sigma_{RT} = X \times X$ or σ_{RT} is a totally order relation on X. Let X be a nonempty set and σ_{RT} an arbitrary reflexive and transitive relation on X. For each $x \in X$, let

$$\overline{x} = \{y \in X : (x, y), (y, x) \in \sigma_{RT}\} \text{ and}$$
$$\overline{X} = \{\overline{x} : x \in X\}.$$

Obviously, $\bigcup \overline{X} = X$. Especially, for the case σ_{RT} is an equivalence relation, $\overline{x}$ is an equivalence class of σ_{RT} containing x and $\overline{X}$ is readily a partition of X. Although σ_{RT} is not an equivalence relation, $\overline{X}$ is a partition of X, as shown in the following lemma.

Lemma 2.1. *$\overline{X}$ is a partition of X.*

Proof. Let $\overline{x}, \overline{y} \in \overline{X}$ be such that $\overline{x} \cap \overline{y} \neq \emptyset$. Then there exists $z \in \overline{x} \cap \overline{y}$. Hence, $(x, z), (z, x), (y, z), (z, y) \in \sigma_{RT}$. By the transitivity of σ_{RT}, we get $(x, y), (y, x) \in \sigma_{RT}$. Let $a \in \overline{x}$. Then $(x, a), (a, x) \in \sigma_{RT}$, which implies $(a, y), (y, a) \in \sigma_{RT}$. Thus, $a \in \overline{y}$ and so $\overline{x} \subseteq \overline{y}$. Similarly, we can show that $\overline{y} \subseteq \overline{x}$. Therefore, $\overline{X}$ is a partition of X. □

Note, for $\alpha \in T(X, \sigma_{RT})$, that $\overline{x}\alpha \subseteq \overline{x\alpha}$, since $(x, z), (z, x) \in \sigma_{RT}$ implies $(x\alpha, z\alpha), (z\alpha, x\alpha) \in \sigma_{RT}$ for all $z \in \sigma_{RT}$. Hereafter, we denote by $\mathrm{Reg}(T(X, \sigma_{RT}))$ the set of all regular elements of $T(X, \sigma_{RT})$.

Lemma 2.2. *Let $\alpha \in T(X, \sigma_{RT})$ and $\overline{x} \in \overline{X}$, such that $\overline{x} \cap X\alpha \neq \emptyset$. If $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$, then there exists $\overline{y} \in \overline{X}$ such that $\overline{x} \cap X\alpha = \overline{y}\alpha$.*

Proof. Assume $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, \sigma_{RT})$. Choose $\overline{y} = \overline{x\beta} \in \overline{X}$. By the above note, we have $\overline{x}\beta \subseteq \overline{x\beta} = \overline{y}$. Let $z \in \overline{x} \cap X\alpha$. Then there exists $x' \in X$, such that $z = x'\alpha$. $z = x'\alpha = x'\alpha\beta\alpha = z\beta\alpha \in \overline{x}\beta\alpha \subseteq \overline{x\beta}\alpha = \overline{y}\alpha$. This implies $\overline{x} \cap X\alpha \subseteq \overline{y}\alpha$. Since $\overline{X}$ is a partition of X and $\overline{x} \cap \overline{y}\alpha \neq \emptyset$, we have $\overline{y}\alpha \subseteq \overline{x}$. Therefore, $\overline{x} \cap X\alpha = \overline{y}\alpha$. □

For each $\emptyset \neq A \subseteq X$ and $x \in X$, let

$$U(A, x) = \{y \in A : (x, y) \in \sigma_{RT}\} \text{ and}$$

$$L(A, x) = \{y \in A : (y, x) \in \sigma_{RT}\}.$$

Evidently, $U(A, x), L(A, x) \subseteq A$. In particular, if $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$, we have the following properties.

Lemma 2.3. *Let $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$ and let $x \in X$.*

1. *If $L(X\alpha, x) = X\alpha$, then there exists $m \in X\alpha$ such that $L(X\alpha, m) = X\alpha$.*
2. *If $U(X\alpha, x) = X\alpha$, then there exists $n \in X\alpha$ such that $U(X\alpha, n) = X\alpha$.*

Proof. (1) Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, \sigma_{RT})$ and $L(X\alpha, x) = X\alpha$. Choose $m = x\beta\alpha \in X\alpha$. To show that $L(X\alpha, m) = X\alpha$, we let $y \in X\alpha = L(X\alpha, x)$. Then $y = x'\alpha$ for some $x' \in X$ and $(y, x) \in \sigma_{RT}$. Hence, $y = x'\alpha = x'\alpha\beta\alpha = y\beta\alpha \subseteq X\alpha = L(X\alpha, x)$. Thus, $(y, m) = (y\beta\alpha, x\beta\alpha) \in \sigma_{RT}$, and so $y \in L(X\alpha, m)$. Therefore, $L(X\alpha, m) = X\alpha$, as required.

(2) It can be proved similar to (1). □

We now characterize all regular elements in $T(X, \sigma_{RT})$, where σ_{RT} is an arbitrary reflexive and transitive relation satisfies the following condition:

$$\forall x, y \in X, (x, y) \in \sigma_{RT} \text{ or } (y, x) \in \sigma_{RT}. \qquad (*)$$

Especially, Lemmas 2.3 and 2.4 give the necessary conditions for this case. The following lemma shows all remaining such conditions.

Lemma 2.4. *Let $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$ and $x \in X$. If both $L(X\alpha, x)$ and $U(X\alpha, x)$ are nonempty, then one of the following holds:*

1. *there exists $u \in U(X\alpha, x)$ such that $U(X\alpha, x) \subseteq U(X\alpha, u)$;*
2. *there exists $l \in L(X\alpha, x)$ such that $L(X\alpha, x) \subseteq L(X\alpha, l)$.*

Proof. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, \sigma_{RT})$ and $L(X\alpha, x) \neq \emptyset \neq U(X\alpha, x)$. Since $x\beta\alpha \in X\alpha \subseteq X$, by the property of σ_{RT}, we get $(x, x\beta\alpha) \in \sigma_{RT}$ or $(x\beta\alpha, x) \in \sigma_{RT}$.

Case 1: $(x, x\beta\alpha) \in \sigma_{RT}$. Then we set $u = x\beta\alpha \in U(X\alpha, x)$. Let $y \in U(X\alpha, x)$. Thus $y \in X\alpha$ and $(x, y) \in \sigma_{RT}$. Hence, $y = x'\alpha$ for some $x' \in X$ and $(u, y) = (x\beta\alpha, x'\alpha) = (x\beta\alpha, x'\alpha\beta\alpha) = (x\beta\alpha, y\beta\alpha) \in \sigma_{RT}$. This implies $y \in U(X\alpha, u)$ and so $U(X\alpha, x) \subseteq U(X\alpha, u)$.

Case 2: $(x\beta\alpha, x) \in \sigma_{RT}$. Then we set $l = x\beta\alpha \in L(X\alpha, x)$. Let $y \in L(X\alpha, x)$. Thus $y \in X\alpha$ and $(y, x) \in \sigma_{RT}$. Hence, $y = x'\alpha$ for some $x' \in X$ and $(y, l) = (x'\alpha, x\beta\alpha) = (x'\alpha\beta\alpha, x\beta\alpha) = (y\beta\alpha, x\beta\alpha) \in \sigma_{RT}$. This implies $y \in L(X\alpha, l)$ and so $L(X\alpha, x) \subseteq L(X\alpha, l)$. □

Theorem 2.5. *Let $\alpha \in T(X, \sigma_{RT})$. Then $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$ if and only if, for each $x \in X$, the following four conditions hold.*

1. *If $\overline{x} \cap X\alpha \neq \emptyset$, then there exists $\overline{y} \in \overline{X}$ such that $\overline{x} \cap X\alpha = \overline{y}\alpha$.*
2. *If $L(X\alpha, x) = X\alpha$, then there exists $m \in X\alpha$ such that $L(X\alpha, m) = X\alpha$.*
3. *If $U(X\alpha, x) = X\alpha$, then there exists $n \in X\alpha$ such that $U(X\alpha, n) = X\alpha$.*
4. *If $L(X\alpha, x) \neq \emptyset$ and $U(X\alpha, x) \neq \emptyset$, then there exists $u \in U(X\alpha, x)$ such that $U(X\alpha, u) = U(X\alpha, x)$ or there exists $l \in L(X\alpha, x)$ such that $L(X\alpha, l) = L(X\alpha, x)$.*

Proof. If $\alpha \in \mathrm{Reg}(T(X,\sigma_{RT}))$, then (1) – (3) hold by Lemmas 2.2 – 2.3. From Lemma 2.4, there exists $u \in U(X\alpha,x)$ such that $U(X\alpha,x) \subseteq U(X\alpha,u)$. Let $y \in U(X\alpha,u)$. Then $y \in X\alpha$ and $(u,y) \in \sigma_{RT}$. Since $(x,u) \in \sigma_{RT}$ and by transitivity of σ_{RT}, we have $y \in U(X\alpha,x)$. Hence $U(X\alpha,u) = U(X\alpha,x)$. Similarly, we get $L(X\alpha,l) = L(X\alpha,x)$.

On the other hand, suppose that all four aforementioned conditions hold. Let $\overline{x} \in \overline{X}$. Then, either $\overline{x} \cap X\alpha \neq \emptyset$ or $\overline{x} \cap X\alpha = \emptyset$. If $\overline{x} \cap X\alpha \neq \emptyset$, from (1), we choose $\overline{y} \in \overline{X}$ such that $\overline{x} \cap X\alpha = \overline{y}\alpha$. For each $a \in \overline{x} \cap X\alpha$, we choose $a' \in \overline{y}$ such that $a = a'\alpha$. Define $\beta_{\overline{x}} : \overline{x} \to \overline{y}$ by

$$a\beta_{\overline{x}} = \begin{cases} a' & \text{if } x \in X\alpha, \\ y & \text{otherwise.} \end{cases}$$

Clearly, $\overline{x}\beta_{\overline{x}} \subseteq \overline{y}$. Suppose that $\overline{x} \cap X\alpha = \emptyset$.

- If for all $y \in X\alpha$, $(y,x) \in \sigma_{RT}$, then $L(X\alpha,x) = X\alpha$, from (2), there exists $n \in X\alpha$ such that $L(X\alpha,n) = X\alpha$. Define $\beta_{\overline{x}} : \overline{x} \to X$ by $a\beta_{\overline{x}} = n'$ for all $a \in \overline{x}$.

- If for all $y \in X\alpha$, $(x,y) \in \sigma_{RT}$, then $U(X\alpha,x) = X\alpha$, from (3), there exists $m \in X\alpha$ such that $U(X\alpha,m) = X\alpha$. Define $\beta_{\overline{x}} : \overline{x} \to X$ by $a\beta_{\overline{x}} = m'$ for all $a \in \overline{x}$.

- If there exist $m,n \in X\alpha$ such that $(n,x),(x,m) \in \sigma_{RT}$, then $U(X\alpha,x) \neq \emptyset$ and $L(X\alpha,x) \neq \emptyset$, we first consider the case of $u \in U(X\alpha,x)$, such that $U(X\alpha,u) = U(X\alpha,x)$ exists. In this case, we define $\beta_{\overline{x}} : \overline{x} \to X$ by $a\beta_{\overline{x}} = u'$ for all $a \in \overline{x}$. For the case $U(X\alpha,u) \neq U(X\alpha,x)$ for all $u \in U(X\alpha,x)$, from (4), there exists $l \in L(X\alpha,x)$ such that $L(X\alpha,l) = L(X\alpha,x)$. Define $\beta_{\overline{x}} : \overline{x} \to X$ by $a\beta_{\overline{x}} = l'$ for all $a \in \overline{x}$.

We notice from three cases that $\beta_{\overline{x}}$ is a constant mapping. Now, we define $\beta : X \to X$ by $x\beta = x\beta_{\overline{x}}$ for all $x \in X$. Since $\overline{X}$ is a partition of X, we have β is well-defined. Let $(x,y) \in \sigma_{RT}$. If $(y,x) \in \sigma_{RT}$, then $\overline{x} = \overline{y}$. It follows from the definition of $\beta_{\overline{x}}$ that $(x\beta, y\beta) = (x\beta_{\overline{x}}, y\beta_{\overline{x}}) \in \sigma_{RT}$. Suppose that $(y,x) \notin \sigma_{RT}$.

Case 1: $\overline{x} \cap X\alpha \neq \emptyset$ and $\overline{y} \cap X\alpha \neq \emptyset$. Let $x_1 \in \overline{x} \cap X\alpha$ and $y_1 \in \overline{y} \cap X\alpha$. By the definition of β, we obtain $x_1\beta = x_1'$ where $\overline{x}\beta \subseteq \overline{x_1'}$ and $y_1\beta = y_1'$ where $\overline{y}\beta \subseteq \overline{y_1'}$. Note that $(x_1', y_1') \in \sigma_{RT}$ or $(y_1', x_1') \in \sigma_{RT}$. If $(x_1', y_1') \notin \sigma_{RT}$, then $(y_1, x_1) = (y_1'\alpha, x_1'\alpha) \in \sigma_{RT}$. By the transitivity of σ_{RT}, we have $(y,x) \in \sigma_{RT}$ which is a contradiction. Therefore $(x_1\beta_{\overline{x}}, y_1\beta_{\overline{x}}) = (x_1', y_1') \in \sigma_{RT}$. Since $x\beta \in \overline{x}\beta_{\overline{x}} \subseteq \overline{x_1'}$ and $y\beta \in \overline{y}\beta_{\overline{y}} \subseteq \overline{y_1'}$, we have $(x\beta, x_1'), (y_1', y\beta) \in \sigma_{RT}$. By the transitivity of σ_{RT}, we obtain $(x\beta, y\beta) \in \sigma_{RT}$.

Case 2: $\overline{x} \cap X\alpha \neq \emptyset$ and $\overline{y} \cap X\alpha = \emptyset$. Let $x_1 \in \overline{x} \cap X\alpha$. By the definition of σ_{RT}, we have $(x_1, y) \in \sigma_{RT}$. Then $L(X\alpha, y) \neq \emptyset$.

Subcase 2.1: $L(X\alpha,y) = X\alpha$. From the definition of β, $y\beta = m'$ where $m \in X\alpha$ and $L(X\alpha,m) = X\alpha$. Since $x_1 \in X\alpha$, we have $(x_1, m) \in \sigma_{RT}$. From the transitivity of σ_{RT}, we get $(x,m) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (x\beta, m\beta) \in \sigma_{RT}$.

Subcase 2.2: $L(X\alpha,y) \neq X\alpha$. Then $U(X\alpha,y) \neq \emptyset$. If $y\beta = u'$ where $u \in U(X\alpha,y)$ and $U(X\alpha,u) = U(X\alpha,y)$, then $(y,u) \in \sigma_{RT}$. From the transitivity of σ_{RT} and $(x,y),(y,u) \in \sigma_{RT}$, we get $(x,u) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (x\beta, u\beta) \in \sigma_{RT}$. If $y\beta = l'$ where $l \in L(X\alpha,y)$ and $L(X\alpha,l) = L(X\alpha,y)$, then $(x_1, l) \in \sigma_{RT}$ since $x_1 \in L(X\alpha,y)$. From the transitivity of σ_{RT}, we have $(x,l) \in$

σ_{RT}. It follows from Case 1 that $(x\beta, y\beta) = (x\beta, l\beta) \in \sigma_{RT}$.

Case 3: $\overline{x} \cap X\alpha = \emptyset$ and $\overline{y} \cap X\alpha \neq \emptyset$. Let $y_1 \in \overline{y} \cap X\alpha$. Then $(x, y_1) \in \sigma_{RT}$ and so $U(X\alpha, x) \neq \emptyset$.

Subcase 3.1: $U(X\alpha, x) = X\alpha$. From the definition of β, $x\beta = n'$ where $n \in X\alpha$ and $U(X\alpha, n) = X\alpha$. Since $y_1 \in X\alpha$, we have $(n, y_1) \in \sigma_{RT}$. From the transitivity of σ_{RT}, we get $(n, y) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (n\beta, y\beta) \in \sigma_{RT}$.

Subcase 3.2: $U(X\alpha, x) \neq X\alpha$. Then $L(X\alpha, x) \neq \emptyset$. If $x\beta = u'$ where $u \in U(X\alpha, x)$ and $U(X\alpha, u) = U(X\alpha, x)$, then $(u, y_1) \in \sigma_{RT}$ since $y_1 \in U(X\alpha, x)$. From the transitivity of σ_{RT} and $(u, y_1), (y_1, y) \in \sigma_{RT}$, we get $(u, y) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (u\beta, y\beta) \in \sigma_{RT}$. If $x\beta = l'$ where $l \in L(X\alpha, x)$ and $L(X\alpha, l) = L(X\alpha, x)$, then $(l, x) \in \sigma_{RT}$. From the transitivity of σ_{RT}, we get $(l, y) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (l\beta, y\beta) \in \sigma_{RT}$.

Case 4: $\overline{x} \cap X\alpha = \emptyset$ and $\overline{y} \cap X\alpha = \emptyset$.

Subcase 4.1: $L(X\alpha, x) = X\alpha$. Then $L(X\alpha, y) = X\alpha$. From the definition of β, $x\beta = m_1'$ where $m_1 \in X\alpha$, $L(X\alpha, m_1) = X\alpha$ and $y\beta = m_2'$ where $m_2 \in X\alpha$, $L(X\alpha, m_2) = X\alpha$. Since $m_1 \in X\alpha$, we have $(m_1, m_2) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (m_1\beta, m_2\beta) \in \sigma_{RT}$.

Subcase 4.2: $\emptyset \neq L(X\alpha, x) \neq X\alpha$ and $L(X\alpha, y) = X\alpha$. Then $U(X\alpha, x) \neq \emptyset$. From the definition of β, $y\beta = m'$ where $m \in X\alpha$ and $L(X\alpha, m) = X\alpha$ and $x\beta = k'$ where $k \in X\alpha$. It follows that $(k, m) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (k\beta, m\beta) \in \sigma_{RT}$.

Subcase 4.3: $\emptyset \neq L(X\alpha, y) \neq X\alpha$ and $\emptyset \neq L(X\alpha, x) \neq X\alpha$. Then $U(X\alpha, x) \neq \emptyset$ and $U(X\alpha, y) \neq \emptyset$. Then $U(X\alpha, y) \subseteq U(X\alpha, x)$ and $L(X\alpha, x) \subseteq L(X\alpha, y)$. If $x\beta = u_1'$ where $u_1 \in U(X\alpha, x)$, $U(X\alpha, u_1) = U(X\alpha, x)$ and $y\beta = u_2'$ where $u_2 \in U(X\alpha, y)$, $U(X\alpha, u_2) = U(X\alpha, y)$, then $(u_1, u_2) \in \sigma_{RT}$ since $u_2 \in U(X\alpha, y) \subseteq U(X\alpha, x)$. It follows from Case 1 that $(x\beta, y\beta) = (u_1\beta, u_2\beta) \in \sigma_{RT}$. If $x\beta = l_1'$ where $l_1 \in L(X\alpha, x)$, $L(X\alpha, l_1) = L(X\alpha, x)$ and $y\beta = l_2'$ where $l_2 \in L(X\alpha, y)$, $L(X\alpha, l_2) = L(X\alpha, y)$, then $(l_1, l_2) \in \sigma_{RT}$ since $l_1 \in L(X\alpha, x) \subseteq L(X\alpha, y)$. It follows from Case 1 that $(x\beta, y\beta) = (l_1\beta, l_2\beta) \in \sigma_{RT}$. If $x\beta = u_1'$ where $u_1 \in U(X\alpha, x)$, $U(X\alpha, u_1) = U(X\alpha, x)$ and $y\beta = l_2'$ where $l_2 \in L(X\alpha, y)$, $L(X\alpha, l_2) = L(X\alpha, y)$. By definition of β, we get $U(X\alpha, y) \neq U(X\alpha, u)$ for all $u \in U(X\alpha, y)$. If $U(X\alpha, x) = U(X\alpha, y)$, then $u_1 \in U(X\alpha, x) = U(X\alpha, y)$ and $U(X\alpha, u_1) = U(X\alpha, x) = U(X\alpha, y)$. It is a contradiction. Hence $U(X\alpha, x) \neq U(X\alpha, y)$. There exists $w \in U(X\alpha, x)$ and $w \notin U(X\alpha, y)$. Thus $w \in X\alpha$ such that $(x, w) \in \sigma_{RT}$ and $(w, y) \in \sigma_{RT}$. So $w \in L(X\alpha, y)$. Hence $(w, l_2) \in \sigma_{RT}$. By the transitivity of σ_{RT}, $(x, l_2) \in \sigma_{RT}$. Therefore $l_2 \in U(X\alpha, x)$ and then $(u_1, l_2) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (u_1\beta, l_2\beta) \in \sigma_{RT}$. If $x\beta = l_1'$ where $l_1 \in L(X\alpha, x)$, $L(X\alpha, l_1) = L(X\alpha, x)$ and $y\beta = u_2'$ where $u_2 \in U(X\alpha, y)$, $U(X\alpha, u_2) = U(X\alpha, y)$, then $(l_1, x) \in \sigma_{RT}$ since $l_1 \in L(X\alpha, x)$ and $(y, u_2) \in \sigma_{RT}$ since $u_2 \in U(X\alpha, y)$. By the transitivity of σ_{RT}, we have $(l_1, u_2) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (l_1\beta, u_2\beta) \in \sigma_{RT}$.

Subcase 4.4: $L(X\alpha, x) = \emptyset$ and $\emptyset \neq L(X\alpha, y) \neq X\alpha$. Then $U(X\alpha, x) = X\alpha$. From the definition of β, $x\beta = n'$ where $n \in X\alpha$, $U(X\alpha, n) = X\alpha$ and $y\beta = k'$ where $k \in X\alpha$. It follows that $(n, k) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (n\beta, k\beta) \in \sigma_{RT}$.

Subcase 4.5: $L(X\alpha, x) = \emptyset$ and $L(X\alpha, y) = \emptyset$. Then $U(X\alpha, x) =$

$U(X\alpha, y) = X\alpha$. From the definition of β, $x\beta = n_1'$ where $n_1 \in X\alpha$, $U(X\alpha, n_1) = X\alpha$ and $y\beta = n_2'$ where $n_2 \in X\alpha$, $U(X\alpha, n_2) = X\alpha$. Since $n_2 \in X\alpha$, we have $(n_1, n_2) \in \sigma_{RT}$. It follows from Case 1 that $(x\beta, y\beta) = (n_1\beta, n_2\beta) \in \sigma_{RT}$.

From four cases, we obtain $\beta \in T(X, \sigma_{RT})$. Finally, for each $x \in X$, we have $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)\beta_{\overline{x\alpha}}\alpha = (x\alpha)'\alpha = x\alpha$. We conclude that $\alpha = \alpha\beta\alpha$. □

Corollary 2.6. *Let $\alpha \in T(X, \sigma_{RT})$. Then $\alpha \in \mathrm{Reg}(T(X, \sigma_{RT}))$ if and only if, for each $x \in X$, the following two conditions hold.*

1. *If $\overline{x} \cap X\alpha \neq \emptyset$, then there exists $\overline{y} \in \overline{X}$ such that $\overline{x} \cap X\alpha = \overline{y}\alpha$.*
2. *If $L(X\alpha, x) \neq \emptyset$ or $U(X\alpha, x) \neq \emptyset$, then there exists $u \in U(X\alpha, x)$ such that $U(X\alpha, u) = U(X\alpha, x)$ or there exists $l \in L(X\alpha, x)$ such that $L(X\alpha, l) = L(X\alpha, x)$.*

Theorem 2.7. *If $T(X, \sigma_{RT})$ is a regular semigroup, then $\sigma_{RT} = X \times X$ or σ_{RT} is a totally order relation on X.*

Proof. Assume that $\sigma_{RT} \neq X \times X$ and σ_{RT} is not a totally ordered on X. Then there exists $y, z \in X$ such that $(z, y), (y, z) \in \sigma_{RT}$ and $y \neq z$. There exists $a \in \overline{x}$ such that $(a, z) \notin \sigma_{RT}$ or $(z, a) \notin \sigma_{RT}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} y & \text{if } x \in \overline{z}, \\ z & \text{otherwise.} \end{cases}$$

Clearly that $X\alpha = \{y, z\}$. Thus α preserves relation σ_{RT}. Note that $\overline{z} \cap X\alpha = \{y, z\} \neq \overline{x}\alpha$ for all $\overline{x} \in \overline{X}$. Hence α is not regular. □

3. Conclusion

Our research yields crucial finding on the regularity of elements in set $T(X, \sigma_{RT})$, with the satisfaction of condition $(*)$. Specifically, we establish that a regular semigroup $T(X, \sigma_{RT})$ implies $\sigma_{RT} = X \times X$ or σ_{RT} is a totally ordered relation. Nevertheless, there is room for further advancement in this field by either eliminating condition $(*)$ or substituting it with weaker conditions, thus paving the way for future investigations.

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