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Fixed Point of Total Asymptotically Nonexpansive Mappings in Banach Spaces

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ABSTRACT

In this paper, we present a strong convergence theorem for total asymptotically nonexpansive mappings in a real uniformly convex Banach space.

Keywords: Condition (A); Fixed point; Strong convergence; Total asymptotically nonexpansive; Uniformly convex

1. Introduction

Recently, we studied a convergence theorem for pseudocontractive mappings in Hilbert spaces and constructed a modified hybrid algorithm, see [1]. In this work, we provide a strong convergence theorem for total asymptotically nonexpansive mappings in a real uniformly convex Banach space. On the other hand, researchers may decide to focus on enriched contraction mappings, see [2]. The results provide a flexible and effective tool for researching fixed point theorems, nonlinear problems, optimization tasks, control systems, numerical analysis, and machine learning which have a wide range of applications across different mathematical disciplines. However, for effective fixed point approximation, total asymptotically nonexpansive mappings are often used in signal processing, image processing, and optimization applications. In fixed point research, various types of mappings are studied, each with its own properties and characteristics, see [3, 4]. Our continuous motivation when it comes to studying this subject arises from its application as we said earlier.

Let us assume that C is a nonempty closed convex subset of a real Banach space X and we define a self-mapping of C as T. Consequently, we denote $F(T)$ as a set of all fixed point of T.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}, k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$, for each $x, y \in C$ and $n \geq 1$. T is said to be uniformly L - Lipschitzian

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if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for each $x, y \in C$, $n \geq 1$. A mapping $T : C \to C$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for each $x, y \in C$. Note that, if T is asymptotically nonexpansive (AN), then it is uniformly L - Lipschitzian. Let $\mathbb{R}^+ = [0, \infty)$ and $\phi \in \Gamma(\mathbb{R}^+)$ if and only if ϕ is strictly increasing, continuous on $\mathbb{R}^+$ and $\phi(0) = 0$. A mapping $T : C \to C$ is said to be total asymptotically nonexpansive (TAN) [6] if there exist two non-negative real sequences $\{c_n\}, \{d_n\}$ with $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 1$, $\phi \in \Gamma(\mathbb{R}^+)$ such that $\|T^n x - T^n y\| \leq \|x - y\| + c_n\phi(\|x - y\|) + d_n$, for each $x, y \in C$, $n \geq 1$. If $\phi(t) = t$ for each $t \geq 0$ and $d_n = 0$, $n \geq 1$, then TAN reduced to AN.

Arithmetic analysts are still curious about fixing point estimation for different sorts of nonlinear mappings, with modern techniques being created for their application (see citeOK, PS, KLS, SAF, MAWT, GR). Approximating fixed points of the modified Ishikawa iterative conspire under total asymptotically nonexpansive mappings has been examined by a few creators; see, for illustration, Chidume and Ofoedu [13, 14], Kim [15, 16], Kim and Kim [17] and others. For a mapping $T : C \to C$, Suantai and Nammanee [18] consider improving the three-step preparation process in C defined by

$$\langle 1 \rangle \begin{cases} \text{given } x_1 \in C, \\ z_n = (1 - \gamma_n)x_n + \gamma_n T^n x_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$. If $\gamma_n = 0$ for all $n \geq 1$, then iteration $\langle 1 \rangle$ becomes the following modified Ishikawa iteration process (Ishikawa [19]) in C defined by

$$\langle 2 \rangle \begin{cases} \text{given } x_1 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$. If $\beta_n = 0$ for all $n \geq 1$, then iteration process $\langle 2 \rangle$ becomes the following modified Mann iteration process (Mann [20]):

$$\langle 3 \rangle \begin{cases} \text{given } x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

Theorems 1.5 and 2.3 of Schu [22] were extended to uniformly convex Banach spaces in the following findings by Rhoades [21].

Theorem 1.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1, \sum_{n=1}^{\infty}(k_n^r - 1) < \infty, r = \max\{1, p\}$. Then, for any $x \in C$, the sequence $\{x_n\}$ defined by $\langle 3 \rangle$, where $\{\alpha_n\}$ satisfies $a \leq \alpha_n \leq 1 - a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T.*

Theorem 1.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1, \sum_{n=1}^{\infty}(k_n^r - 1) < \infty, r = \max\{1, p\}$. Then, for any $x \in C$, the sequence $\{x_n\}$ defined by $\langle 2 \rangle$, where $\{\alpha_n\}, \{\beta_n\}$ satisfies $a \leq (1-\alpha_n), (1-\beta_n) \leq 1 - a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T.*

The following result, which generalized Theorem 1 of Senter and Dotson [24], was established by Kim in 2012.

Theorem 1.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a nonexpansive mapping with satisfying condition (A) and $F(T) \neq \emptyset$. Suppose that for $x_1 \in C$, the sequence $\{x_n\}$ is defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n x_n + (1-\beta_n)Tx_n]$ for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.*

In 2013, Kim [16] generalized the results due to Rhoades [21] by proving that if $T : C \to C$ is an TAN mapping satisfying condition (A); then the squence $\{x_n\}$ defined by $\langle 2 \rangle$ converges strongly to some fixed point of T. According to results above, we were inspired to extend Kim' s results [16] by using the iteration $\langle 1 \rangle$ to define the sequence $\{x_n\}$ under T to be a TAN mapping satisfying condition (A) and prove that the sequence converges strongly to some fixed point of T. Our result generalizes the results due to Rhoades [21] and Kim [16].

2. Preliminaries

We'll collect some helpful results in this section, which will be used in the next phase. If $T : C \to C$ is a mapping and C is a nonempty subset of a real Banach space X, then X is said to be uniformly convex if the modulus of convexity $\delta_X = \delta_X(\epsilon), 0 < \epsilon \leq 2$, of X defined by $\delta_X(\epsilon) = \inf\{1 - \frac{\|x-y\|}{2} : x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$ satisfies the inequality $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Written $x_n \to x$ will denote the sequence $\{x_n\}$ converges strongly to x.

Definition 2.1 ([24])**.** Let $T : C \to C$ be a mapping and $F(T) \neq \emptyset$. T is said to satisfy condition (A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ and $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

Lemma 2.2 ([25])**.** *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative numbers such that $a_{n+1} \leq (1+\delta_n)a_n + b_n, \forall n \in \mathbb{N}$. If $\Sigma_{n=1}^{\infty}\delta_n < \infty$ and $\Sigma_{n=1}^{\infty} b_n < \infty$, then $lim_{n\to\infty} a_n$ exists.*

Lemma 2.3 ([26])**.** *If X is a uniformly convex Banach space, for $x, y \in X$ and $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon)$ for $0 \leq \lambda \leq 1$.*

3. Main Results

First, we prove that $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T)$, where the sequence $\{x_n\}$ is defined by $\langle 1 \rangle$, as the following lemma.

Lemma 3.1. *If C is a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ is a TAN mapping and $F(T) \neq \emptyset$, $\{c_n\}, \{d_n\}$ and ϕ satisfy the following two condition:*

(i) $\exists \alpha, \beta > 0$ such that $\phi(t) \leq \alpha t$ for all $t \geq \beta$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} d_n < \infty$, then $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T)$.

Proof. For any $z \in F(T)$, we let $M := \max\{1, \phi(\beta)\} < \infty$. From (i) and ϕ is a strictly increasing, so

$$\phi(t) \leq \phi(\beta) + \alpha t, \ t \geq 0.$$

It follows that

$$\|T^n x_n - z\| \leq (1 + \alpha c_n)\|x_n - z\| + k_n M, \quad (3.1)$$

where $k_n = c_n + d_n$. From $\langle 1\rangle$ and Eq. (3.1) we have

$$\begin{aligned}&\|z_n - z\|\\ &\leq \gamma_n\|T^n x_n - z\| + (1-\gamma_n)\|x_n - z\|\\ &\leq \gamma_n[(1+\alpha c_n)\|x_n - z\| + k_n M]\\ &\quad + (1-\gamma_n)\|x_n - z\|\\ &= (1+\alpha\gamma_n c_n)\|x_n - z\| + \gamma_n k_n M. \qquad (3.2)\end{aligned}$$

From Eq. (3.2) it follows that

$$\begin{aligned}&\|z_n - z\| + c_n\phi(\|z_n - z\|)\\ &\leq \|z_n - z\| + c_n[\phi(\beta) + \alpha\|z_n - z\|]\\ &\leq (1+\alpha\gamma_n c_n)\|x_n - z\| + \gamma_n k_n M\\ &\quad + c_n M + \alpha c_n[(1+\alpha\gamma_n c_n)\|x_n - z\|\\ &\quad + \gamma_n k_n M]\\ &\leq (1+\alpha\gamma_n c_n + \alpha c_n + \alpha^2\gamma_n {c_n}^2)\|x_n - z\|\\ &\quad + \gamma_n k_n M + c_n M + \alpha c_n\gamma_n k_n M\\ &\leq (1+\alpha c_n + \alpha c_n + \alpha^2 {c_n}^2)\|x_n - z\|\\ &\quad + k_n M + c_n M + \alpha c_n k_n M\\ &= (1+\sigma_n)\|x_n - z\| + \delta_n M, \qquad (3.3)\end{aligned}$$

where $\sigma_n = 2\alpha c_n + \alpha^2 {c_n}^2$ and $\delta_n = k_n + c_n + \alpha c_n k_n$. And so

$$\begin{aligned}&\|T^n z_n - z\|\\ &\leq \|z_n - z\| + c_n\phi(\|z_n - z\|) + d_n\\ &\leq (1+\sigma_n)\|x_n - z\| + \delta_n M + d_n M\\ &\leq (1+\sigma_n)\|x_n - z\| + \eta_n M, \qquad (3.4)\end{aligned}$$

where $\eta_n = \delta_n + d_n$. Then we get that

$$\begin{aligned}&\|y_n - z\|\\ &\leq \beta_n\|T^n z_n - z\| + (1-\beta_n)\|x_n - z\|\\ &\leq \beta_n[(1+\sigma_n)\|x_n - z\| + \eta_n M]\\ &\quad + (1-\beta_n)\|x_n - z\|\\ &= (1+\beta_n\sigma_n)\|x_n - z\| + \beta_n\eta_n M. \qquad (3.5)\end{aligned}$$

and

$$\begin{aligned}&\|T^n y_n - z\|\\ &\leq \|y_n - z\| + c_n\phi(\|y_n - z\|) + d_n\\ &\leq \|y_n - z\| + c_n[\phi(\beta) + \alpha\|y_n - z\|] + d_n\\ &\leq (1+\beta_n\sigma_n)\|x_n - z\| + \beta_n\eta_n M\\ &\quad + c_n M + \alpha c_n[(1+\beta_n\sigma_n)\|x_n - z\|\\ &\quad + \beta_n\eta_n M] + d_n M\\ &= (1+\beta_n\sigma_n + \alpha c_n + \alpha\beta_n c_n\sigma_n)\|x_n - z\|\\ &\quad + (\beta_n\eta_n + c_n + \alpha\beta_n c_n\sigma_n + d_n)M\\ &= (1+\varphi_n)\|x_n - z\| + \nu_n M, \qquad (3.6)\end{aligned}$$

where $\varphi_n = \beta_n\sigma_n + \alpha c_n + \alpha\beta_n c_n\sigma_n$ and $\nu_n = \beta_n\eta_n + c_n + \alpha\beta_n c_n\sigma_n + d_n$. Hence

$$\begin{aligned}&\|x_{n+1} - z\|\\ &\leq \alpha_n\|T^n y_n - z\| + (1-\alpha_n)\|x_n - z\|\\ &\leq \alpha_n[(1+\varphi_n)\|x_n - z\| + \nu_n M]\\ &\quad + (1-\alpha_n)\|x_n - z\|\\ &= (1+\alpha_n\varphi_n)\|x_n - z\| + \alpha_n\nu_n M\\ &= (1+\epsilon_n)\|x_n - z\| + \lambda_n M, \qquad (3.7)\end{aligned}$$

where $\epsilon_n = \alpha_n\varphi_n$ and $\lambda_n = \alpha_n\nu_n$.
By Lemma 2.2, we get that $\lim_{n\to\infty}\|x_n - z\|$ exists. □

Another, we will show that $\{x_n\}$ is defined by $\langle 1\rangle$ converges strongly to some $p \in F(T)$.

Theorem 3.2. *If C is a nonempty closed convex subset of a uniformly convex space X, $T : C \to C$ is a TAN mapping with satisfying condition (A) and $F(T) \neq \emptyset$, $\{c_n\}, \{d_n\}$ and ϕ satisfy the following two condition :*

(i) $\exists\alpha, \beta > 0$ such that $\phi(t) \leq \alpha t$ for all $t \geq \beta$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} d_n < \infty$, and suppose that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ and $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n = 0$, then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. Let $z \in F(T)$, then by Lemma 3.1, it follows that $\{x_n\}$ is a bounded sequence and WLOG, we assume that $\lim_{n\to\infty}\|x_n - z\| = r > 0$. And let

$$U := \max\{1, \phi(\beta), \sup\|x_n - z\|\} < \infty.$$

From Eq. (3.6) we get that

$$\begin{aligned}\|T^n y_n - z\| &\leq (1+\varphi_n)\|x_n - z\| + \nu_n U\\ &\leq \|x_n - z\| + \varrho_n U, \qquad (3.8)\end{aligned}$$

where $\varrho_n = \varphi_n + \nu_n$. By using Eq. (3.8) and Lemma 3.1, we get that

$$\|x_{n+1}-z\|$$
$$= \|(1-\alpha_n)(x_n-z)+\alpha_n(T^n y_n - z)\|$$
$$\leq (\|x_n - z\| + \varrho_n U)[1- 2\alpha_n(1-\alpha_n)\delta_X(\frac{\|T^n y_n - x_n\|}{\|x_n - z\| + \varrho_n U})].$$

And so
$2\alpha_n(1-\alpha_n)(\|x_n - z\| + \varrho_n U)\delta_X(\frac{\|T^n y_n - x_n\|}{\|x_n - z\| + \varrho_n U})$
$\leq \|x_n - z\| - \|x_{n+1} - z\| + \varrho_n U.$
Hence
$2\alpha_n(1-\alpha_n)(\|x_n - z\| + \varrho_n U)\delta_X(\frac{\|T^n y_n - x_n\|}{\|x_n - z\| + \varrho_n U}) < \infty.$
Since δ_X is a strictly increasing and continuous function and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, we get that $\liminf_{n\to\infty}\|T^n y_n - x_n\| = 0$.
Consider,

$$\|T^{n-1}x_{n-1} - z\|$$
$$\leq \|x_{n-1} - z\| + c_{n-1})\phi(\|x_{n-1} - z\|) + d_{n-1}$$
$$\leq \|x_{n-1} - z\| + c_{n-1})\phi(\beta) + \alpha\|x_{n-1} - z\| + d_{n-1}$$
$$\leq (1+\alpha c_{n-1})\|x_{n-1} - z\| + \varepsilon_{n-1}U,$$

where $\varepsilon_{n-1} = c_{n-1} + d_{n-1}$. Thus

$$\|y_{n-1} - z\|$$
$$\leq \beta_{n-1}\|T^{n-1}x_{n-1} - z\| + (1-\beta_{n-1})\|x_{n-1} - z\|$$
$$\leq \beta_{n-1}\{(1+\alpha c_{n-1})\|x_{n-1} - z\| + \varepsilon_{n-1}U\} + (1-\beta_{n-1})\|x_{n-1} - z\|$$
$$\leq (1+\alpha c_{n-1})\|x_{n-1} - z\| + \varepsilon_{n-1}U.$$

Similarity to Eq. (3.6), it follows that

$$\|T^{n-1}y_{n-1} - z\|$$
$$\leq (1+\varphi_{n-1})\|x_{n-1} - z\| + \nu_{n-1}U$$
$$\leq (1+\varphi_{n-1})\|x_{n-1} - z\| + \nu_{n-1}U$$
$$\leq \|x_{n-1} - z\| + \omega_{n-1}U, \qquad (3.9)$$

where $\omega_{n-1} = \varphi_{n-1} + \nu_{n-1}$. By using Eq. (3.9) and Lemma 3.1, we get that

$$\|x_n - z\|$$
$$= \|(1-\alpha_{n-1})(x_{n-1} - z) + \alpha_{n-1}(T^{n-1}y_{n-1} - z)\|$$
$$\leq (\|x_{n-1} - z\| + \omega_{n-1}U)[1 - 2\alpha_{n-1}(1- \alpha_{n-1})\delta_X(\frac{\|T^{n-1}y_{n-1} - x_{n-1}\|}{\|x_{n-1} - z\| + \omega_{n-1}U})].$$

And so we get that $\liminf_{n\to\infty}\|T^{n-1}y_{n-1} - x_{n-1}\| = 0$.
Since $\{x_n\}$ is a bounded sequence and T is a TAN mapping, we have

$$\|y_n - x_n\|$$
$$= \beta_n\|T^n z_n - x_n\|$$
$$\leq \beta_n[\|T^n z_n - T^n x_n\| + \|T^n x_n - x_n\|]$$
$$\leq \|T^n z_n - T^n x_n\| + \beta_n\|T^n x_n - x_n\|$$
$$\leq \|z_n - x_n\| + c_n\phi(\|z_n - x_n\|) + d_n + \beta_n\|T^n x_n - x_n\|$$
$$\leq \|z_n - x_n\| + c_n\{\phi(\beta) + \alpha\|z_n - x_n\|)\} + d_n + \beta_n\|T^n x_n - x_n\|$$
$$\leq (1+\alpha c_n)\|z_n - x_n\| + d_n + \beta_n\|T^n x_n - x_n\|$$
$$\leq (1+\alpha c_n)\gamma_n\|T^n x_n - x_n\| + d_n U + \beta_n\|T^n x_n - x_n\|$$
$$\leq (1+\alpha c_n)\gamma_n W + d_n U + \beta_n W, \qquad (3.10)$$

where $W = \sup_{n\geq 1}\|T^n x_n - x_n\|$. By Eq. (3.10) and $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n = 0$, so $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and so $\liminf_{n\to\infty}\|T^n y_n - y_n\| = 0$. It implies that

$$\liminf_{n\to\infty}\|T^{n-1}y_{n-1} - y_{n-1}\| = 0.$$

Consider,

$$\|T^{n-1}x_{n-1} - x_{n-1}\|$$
$$\leq \|T^{n-1}x_{n-1} - y_{n-1}\| + \|T^{n-1}y_{n-1} - x_{n-1}\|$$
$$\leq \|x_{n-1} - y_{n-1}\| + c_{n-1}\phi(\|x_{n-1} - y_{n-1}\|) + d_{n-1} + \|T^{n-1}y_{n-1} - x_{n-1}\|.$$

It implies that $\liminf_{n\to\infty}\|T^{n-1}x_{n-1} - x_{n-1}\| = 0$. Since

$$\|x_n - x_{n-1}\|$$
$$\leq \alpha_{n-1}\|T^{n-1}y_{n-1} - x_{n-1}\|$$
$$\leq \|T^{n-1}y_{n-1} - y_{n-1}\| + \|y_{n-1} - x_{n-1}\|,$$

it implies that $\liminf_{n\to\infty}\|x_n - x_{n-1}\| = 0$.
Consider,

$$\|T^{n-1}x_n - x_n\|$$
$$\leq \|T^{n-1}x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|$$
$$\leq 2\|x_n - x_{n-1}\| + c_{n-1}\phi(\|x_n - x_{n-1}\|) + d_{n-1} + \|T^{n-1}x_{n-1} - x_{n-1}\|.$$

It implies that $\liminf_{n\to\infty}\|T^{n-1}x_n - x_n\| = 0$. Since

$$\|x_n - Tx_n\|$$
$$\leq \|x_n - y_n\| + \|y_n - T^n y_n\|$$

$$+ \|T^n y_n - T^n x_n\| + \|T^n x_n - Tx_n\|$$
$$\leq 2\|x_n - y_n\| + c_n\phi(\|x_n - y_n\|) + d_n$$
$$+ \|y_n - T^n y_n\| + \|T^n x_n - Tx_n\|.$$

Since T is a uniformly continuous, it implies that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.
By T satisfies condition (A), so $f(d(x_n, F(T))) \leq \|x_n - Tx_n\|$, for all $n \geq 1$.
From (3.7), it follows that

$$\|x_{n+1} - z\| \leq (1 + \epsilon_n)\|x_n - z\| + \lambda_n U. \quad (3.11)$$

Thus $\liminf_{n\to\infty} \|x_{n+1} - z\| \leq (1+\epsilon_n) \liminf_{n\to\infty} \|x_n - z\| + \lambda_n U$. By Lemma 2.2, we have $\liminf_{n\to\infty} d(x_n, F(T)) = c$, for some $c \in \mathbb{R}$. We claim that $c = 0$. Assume that $c = \liminf_{n\to\infty} d(x_n, F(T)) > 0$. Then we can choose $N_0 \in \mathbb{N}$ such that $0 < c/2 < d(x_n, F(T))$ for all $n \geq N_0$. Since T satisfies condition (A), it implies that

$$0 < f(\frac{c}{2}) \leq f(d(x_{n_i}, F(T)))$$
$$\leq \|x_{n_i} - Tx_{n_i}\| \to 0 \text{ as } i \to \infty,$$

which is a contradiction. So $c = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Since $\sum_{n=1}^{\infty} \epsilon_n < \infty$, so $\prod_{n=1}^{\infty}(1 + \epsilon_n) := L < \infty$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, we get

$$d(x_n, F(T)) < \frac{\epsilon}{4L + 4} \text{ and } \sum_{i=n_0}^{\infty} \lambda_i < \frac{\epsilon}{4U}. \quad (3.12)$$

Let $n, m \geq n_0$ and $z \in F(T)$. Then by (3.11), we have

$$\|x_n - x_m\| \leq \|x_n - z\| + \|x_m - p\|$$
$$\leq \prod_{i=n_0}^{n-1}(1 + \epsilon_i)\|x_{n_0} - z\| + U\sum_{i=n_0}^{n-1}\lambda_i$$
$$+ \prod_{i=n_0}^{m-1}(1 + \epsilon_i)\|x_{n_0} - z\| + U\sum_{i=n_0}^{m-1}\lambda_i$$
$$\leq 2[\prod_{i=n_0}^{\infty}(1 + \epsilon_i)\|x_{n_0} - z\| + U\sum_{i=n_0}^{\infty}\lambda_i]. \quad (3.13)$$

Taking the infimum over $z \in F(T)$ on (3.13) and by (3.12), we get that

$$\|x_n - x_m\| \leq 2[\prod_{i=n_0}^{\infty}(1 + \epsilon_i)d(x_{n_0}, F(T))$$
$$+ U\sum_{i=n_0}^{\infty}\lambda_i]$$
$$< 2[(1 + L)\frac{\epsilon}{4L + 4} + U\frac{\epsilon}{4U}] = \epsilon,$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Thus $\lim_{\infty} x_n = p$. Then $d(p, F(T)) = 0$. Since $F(T)$ is a closed set, we have $p \in F(T)$. Hence $\{x_n\}$ converges strongly to some fixed point of T □

The following remark was introduced by Kim [15].

Remark 3.3.

1. *Senter and Dotson [24] as saying that if $T : CtoC$ is fully continuous, it fulfills demicompact, and if T is continuous and demicompact, it meets condition (A).*
2. *If $a \leq \alpha_n \leq b$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\lim_{n\to\infty} = 0$. However, the converse is not true.*

Finally, we shall demonstrate a mapping $T : C \to C$ that satisfies every requirement of T in Theorem 3.2, but not Lipschitzian and hence not asymptotically nonexpansive.

Example 3.4. Let $X := \mathbb{R}$ and $C := [0, \frac{3}{4}]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{4}, & \text{if } x \in [0, \frac{1}{4}]; \\ (x - \frac{3}{4})^2, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

Since $T^n x = \frac{1}{4}$ for each $x \in C$, $n \geq 2$ and $F(T) = \{\frac{1}{4}\}$, T is both uniformly continuous and TAN on C. We will show that T satisfies condition (A) as follows :
if $x \in [0, \frac{1}{4}]$, then $|x - \frac{1}{4}| = |x - Tx|$.

Similarly, if $x \in [\frac{1}{4}, \frac{3}{4}]$, then $|x - \frac{1}{4}| = x - \frac{1}{4}$ and $(x - \frac{3}{4})^2 \le \frac{1}{4}$.
Thus $x - \frac{1}{4} \le x - (x - \frac{3}{4})^2 = |x - Tx|$, that is, $|x - \frac{1}{4}| \le |x - Tx|$.
Hence, $d(x, F(T)) = |x - \frac{1}{4}| \le |x - Tx|$ for all $x \in C$. Next, we will show that T is not a Lipschitzian mapping which proved by contradiction. Suppose that there exists $L > 0$ such that

$$|Tx - Ty| \le L|x - y|, \text{ for all } x, y \in C.$$

Consider, if we take $x = \frac{3}{4} - \frac{L}{4(L+1)} > \frac{1}{4}$ and $y = \frac{3}{4}$, then

$$(x - \frac{3}{4})^2 = |Tx - Ty| \le L|x - y| = L(\frac{3}{4} - x).$$

And so

$$\frac{3}{4} - x \ge L \iff \frac{L}{4(L+1)} \ge L,$$

it is a contradiction.

4. Conclusion

This work has established a new theorem of fixed point approximation for total asymptotically nonexpansive mappings, which is a more generalized than asymptotically nonexpansive mappings and nonexpansive mappings. Our results have been created as an alternative for applying to other research.

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