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On Line Graph Associated with a Non-Commuting Graph for Finite Rings

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ABSTRACT

Let R be a non-commutative ring. A non-commuting graph of R, denoted by Γ_R, is a simple graph with a vertex set consisting of elements in R, except for its center. Any two distinct vertices x and y are adjacent if $xy \neq yx$. A line graph associated with Γ_R, denoted by $L(\Gamma_R)$, is a simple graph in which each vertex of $L(\Gamma_R)$ represents an edge of Γ_R, and two distinct vertices of $L(\Gamma_R)$ are adjacent if their corresponding edges share a common endpoint in Γ_R. This paper provides bounds for seven graph parameters of $L(\Gamma_R)$: minimum degree, maximum degree, order, size, diameter, vertex-connectivity and edge-connectivity. Additionally, we show that the girth of $L(\Gamma_R)$ is exactly 3.

Keywords: Connectivity; Line graph; Non-commuting graph

1. Introduction

Let R be a non-commutative ring and $Z(R)$ be its center. We introduce the non-commuting graph of R, denoted by Γ_R, with vertices $R \setminus Z(R)$. For any distinct $x, y \in R \setminus Z(R)$, x and y are adjacent if and only if $xy \neq yx$. The idea of non-commuting graph of a non-commutative ring was first introduced by Erfanian et al. [5], who were interested in its properties such as diameter, girth, domination number, chromatic number and clique number. The non-commuting graph was also studied by Dutta and Basnet [4], while Wanida et al. [10] investigated the connectivity of non-commuting graphs.

A line graph associated with a simple graph G, denoted by $L(G)$, is a simple graph whose set of vertices constitute the edges of G such that any two distinct vertices are adjacent if the corresponding edges share a common endpoint in G. The essential of a line graph is derived from the fact that it transforms the adjacency relations on edges to adjacency relations on vertices, which is very useful in graph theory. For example, the chromatic index of a graph leads to the chromatic number of its line graph and a matching in a graph leads to an independent set in its line graph. Although Whitney [11]

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and Krausz [7] had previously studied a line graph, Harary and Norman [6] formally defined it. Many mathematicians have since worked on the line graph, including Aleksandra and Zoran [1] and Suthar and Prakash [9].

Given the properties of the non-commuting graph Γ_R, our goal is to investigate the properties of its line graph. In this paper, we focus on a finite non-commutative ring R. We show that the line graph associated with the non-commuting graph is connected. We also consider a lower bound of minimum degree and an upper bound of maximum degree for all vertices of $L(\Gamma_R)$. Additionally, we not only determine a lower bound and an upper bound of order and size of $L(\Gamma_R)$, but also give lower bounds for vertex-connectivity and edge-connectivity. We show that $L(\Gamma_R)$ has a diameter of at most 3 and girth of exactly 3.

2. Preliminaries

Let R be a non-commutative ring. The set of all elements of R that commutes with x is called the **centralizer** of x, denoted by $C_R(x)$. The set of all elements of R that commute with every element of R is called the **center** of R, denoted by $Z(R)$. For any $x \in R$, $C_R(x)$ and $Z(R)$ are additive subgroups of R. An element of R is said to be **non-central** if it does not belong to the center of R. An element x of a ring is a **nilpotent element** if $x^n = 0$ for some positive integer n, where the smallest such n is called the **nilpotency index** of x, denoted by nil(x).

Let G be a simple graph with a vertex set $V(G)$ and an edge set $E(G)$. The number of vertices in G is often called the **order** of G, denoted by $|V(G)|$, while the number of edges is its **size**, denoted by $|E(G)|$. The **degree** of a vertex v in a graph G is the number of edges incident with v and is denoted by deg(v). The **minimum degree** of G is the minimum degree among the vertices of G, denoted by $\delta(G)$. The **maximum degree** of G is the maximum degree among the vertices of G, denoted by Δ(G).

A graph G is said to be **connected** if it contains a u−v path for every pair u, v of distinct vertices. The **distance** between u and v is the smallest length (the number of edges that are encountered) of any u−v path in G, denoted by $d(u,v)$. The greatest distance between any two vertices of a connected graph G is called the **diameter** of G, denoted by diam(G). The **girth** of the graph G, denoted by gr(G), is the length of the shortest cycle in G. A **complete graph** is a graph in which all pairs of distinct vertices are adjacent. A complete graph of order n is denoted by K_n.

The **edge-connectivity** of G, denoted by $\lambda(G)$, is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Similarly, the **vertex- connectivity** of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal from G results in a disconnected or trivial graph.

Let R be a non-commutative ring with $Z(R)$ as its center. The **non-commuting graph** of R, denoted by Γ_R, is a simple graph whose vertex set is $R \backslash Z(R)$ such that any two distinct vertices x and y are adjacent if $xy \neq yx$. Thus, deg(x) = $|R|-|C_R(x)|$ for all vertices x of Γ_R.

The **line graph** associated with a non-commuting graph Γ_R, denoted by $L(\Gamma_R)$, is a simple graph in which each vertex represents an edge of Γ_R and two distinct vertices of $L(\Gamma_R)$ are adjacent if and only if their corresponding edges share a common endpoint in Γ_R. If a and b are distinct vertices that are adjacent in Γ_R, then we write $[a, b]$ as a vertex of $L(\Gamma_R)$. It is easy to see that deg($[a, b]$) = deg(a) + deg(b) − 2 for every vertex $[a, b]$ of $L(\Gamma_R)$.

Lemma 2.1 ([8]). *If R is a ring of prime order, then R is commutative.*

Lemma 2.2 ([2]). *The line graph $L(G)$ of a connected graph G is complete if and only if G is isomorphic to a star or K_3.*
Lemma 2.3 ([3]). *If G is a connected graph, then*
1. $\kappa(L(G)) \geq \lambda(G)$ whenever $\lambda(G) \geq 2$,
2. $\lambda(L(G)) \geq 2\lambda(G) - 2$.

Proposition 2.4 ([4]). *Let R be a finite ring. Then Γ_R is connected.*

Theorem 2.5 ([5]). *Let R be a non-commutative ring. Then* $\mathrm{diam}(\Gamma_R) \leq 2$ *and* $\mathrm{gr}(\Gamma_R) = 3$.

Theorem 2.6 ([5]). *Let R be a non-commutative ring. Then Γ_R is complete if and only if $|R| = 4$.*

Lemma 2.7 ([10]). *Let R be a finite non-commutative ring. Then $\delta(\Gamma_R) \geq \frac{|R|}{2}$.*

Theorem 2.8 ([10]). *Let R be a finite non-commutative ring. Then*

$$\frac{|R|}{2} \leq \delta(\Gamma_R) = \lambda(\Gamma_R) \leq |R| - 2.$$

Lemma 2.9. *Let R be a finite ring. If $|R| \leq 3$, then R is commutative.*
Proof. If $|R| = 1$, then R is the zero ring, so R is commutative. In addition, if $|R| = 2$ or 3, then, by Lemma 2.1, R is commutative. □

Lemma 2.10. *Let R be a finite non-commutative ring. Then $|C_R(x)| \leq \frac{|R|}{2}$ for any $x \in R \backslash Z(R)$.*
Proof. Let $x \in R\backslash Z(R)$. Then $|C_R(x)|$ divides $|R|$ by Lagrange theorem. Because x is non-central, $|R| \neq |C_R(x)|$, so $|R| \geq 2|C_R(x)|$ Hence, $|C_R(x)| \leq \frac{|R|}{2}$. □

3. Main Results

In this section, we discuss some properties of $L(\Gamma_R)$. We begin with connectedness and completeness. In Section 3.2, we give a lower bound of $\delta(L(\Gamma_R))$ and an upper bound of $\Delta(L(\Gamma_R))$. We will later, in Section 3.3, find a bound of order and size of $L(\Gamma_R)$. In Section 3.4, we determine the diameter and girth of $L(\Gamma_R)$. Finally, we investigate the lower bounds of both edge and vertex connectivity of $L(\Gamma_R)$.

3.1 Connectedness and completeness

In this part, we show that $L(\Gamma_R)$ is connected and also give a necessary and sufficient condition for the completeness of $L(\Gamma_R)$.

Theorem 3.1. *Let R be a finite non-commutative ring. Then $L(\Gamma_R)$ is connected.*

Proof. Let $[x_1, y_1]$ and $[x_2, y_2]$ be two distinct vertices in $L(\Gamma_R)$. Then x_1, y_1, x_2, y_2 are the vertices in Γ_R. If $y_1 = x_2$, then P_L: $[x_1, y_1], [x_2, y_2]$ is a path in $L(\Gamma_R)$. On the other hand, suppose that $y_1 \neq x_2$. Then Γ_R is connected by Proposition 2.4. Thus, there exists a path P: $x_1, y_1, z_1, z_2,\ldots, z_n, x_2, y_2$ for some vertices $z_1, z_2,\ldots, z_n$ in Γ_R. This implies that P_L: $[x_1, y_1], [y_1, z_1], [z_1, z_2],\ldots, [z_n, x_2], [x_2, y_2]$ is a path in $L(\Gamma_R)$. Hence, $L(\Gamma_R)$ contains a $[x_1, y_1] - [x_2, y_2]$ path. Therefore, $L(\Gamma_R)$ is connected. □

Theorem 3.2. *Let R be a finite non-commutative ring. Then $L(\Gamma_R)$ is complete if and only if $|R| = 4$.*

Proof. Suppose $|R| = 4$. By Theorem 2.6, Γ_R is complete. Next, we let $x \in V(\Gamma_R)$. As x is non-central, we get $x \neq 0$ and $C_R(x) \neq R$. Since $|R| = 4$, we have $|C_R(x)| = 1$ or 2. Note that $0, x \in C_R(x)$, so $|C_R(x)| = 2$. As a result, $\deg(x) = |R| - |C_R(x)| = 2$. Hence, $\deg(x) = 2$ for every $x \in V(\Gamma_R)$, which implies Γ_R is isomorphic to K_3. By Lemma 2.2, $L(\Gamma_R)$ is complete.

Conversely, suppose $|R| \neq 4$. If $|R| \leq 3$, then R is commutative by Lemma 2.9, which is a contradiction. Thus, $|R| > 4$. By Theorem 2.6, Γ_R is not complete, and not isomorphic to K_3. Since $\deg(v) \geq \delta(\Gamma_R) \geq$

$\frac{|R|}{2} > 2$ for every $v \in V(\Gamma_R)$, Γ_R is not isomorphic to a star graph. Hence, by Lemma 2.2, $L(\Gamma_R)$ is not complete. □

3.2 Minimum and maximum degree

We investigate a lower bound of $\delta(L(\Gamma_R))$ and an upper bound of $\Delta(L(\Gamma_R))$.

Theorem 3.3. *Let R be a finite non-commutative ring. Then* $\delta(L(\Gamma_R)) \geq |R| - 2$.

Proof. Let $[x, y]$ be a vertex of $L(\Gamma_R)$. Then x and y are vertices of Γ_R. By Lemma 2.10, we have $|C_R(x)| \leq \frac{|R|}{2}$ and $|C_R(y)| \leq \frac{|R|}{2}$, so

$$|R|-|C_R(x)| \geq \frac{|R|}{2} \text{ and } |R|-|C_R(y)| \geq \frac{|R|}{2}.$$

Then

$$\begin{aligned} \deg([x, y]) &= \deg(x) + \deg(y) - 2 \\ &= (|R|-|C_R(x)|) + (|R|-|C_R(y)|) - 2 \\ &\geq \frac{|R|}{2} + \frac{|R|}{2} - 2 \\ &= |R| - 2. \end{aligned}$$

Hence, $\delta(L(\Gamma_R)) \geq |R| - 2$. □

Theorem 3.4. *Let R be a finite non-commutative ring. Then* $\Delta(L(\Gamma_R)) \leq 2|R| - 6$.

Proof. Let $[x, y]$ be a vertex of $L(\Gamma_R)$. Then x and y are vertices of Γ_R. This implies that x and y are non-central elements in R. Since $0, x \in C_R(x)$ and $0, y \in C_R(y)$, $|C_R(x)| \geq 2$ and $|C_R(y)| \geq 2$. Hence, $\deg(x) = |R| - |C_R(x)| \leq |R| - 2$ and $\deg(y) = |R| - |C_R(y)| \leq |R| - 2$. Therefore,

$$\begin{aligned} \deg([x, y]) &= \deg(x) + \deg(y) - 2 \\ &\leq (|R| - 2) + (|R| - 2) - 2 \\ &= 2|R| - 6. \end{aligned}$$

Hence $\Delta(L(\Gamma_R)) \leq 2|R| - 6$. □

The following example shows that the bounds given in Theorems 3.3 and 3.4 are sharp.

Example 3.5. Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{Z}_2 \right\}$ and $\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus, R is a non-commutative ring such that $|R| = 4$. Γ_R and its line graph are shown in the following figure.

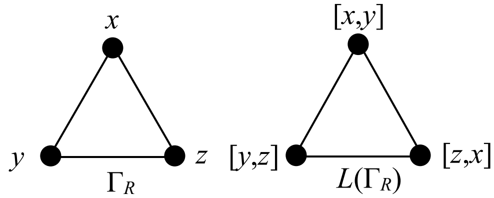


Fig. 1. The non-commuting graph of R and line graph of Γ_R.

Thus, $\delta(L(\Gamma_R)) = 2 = |R| - 2$,
and $\Delta(L(\Gamma_R)) = 2 = 2|R| - 6$. #

In Theorem 3.4, we consider any arbitrary finite non-commutative ring R. If R is a ring with identity, then we can improve the upper bound of $\Delta(L(\Gamma_R))$ by the following corollary.

Corollary 3.6. *Let R be a finite non-commutative ring with identity* $1 \neq 0$. *Then* $\Delta(L(\Gamma_R)) \leq 2|R| - 8$.

Proof. For any vertex $[x, y]$ in $L(\Gamma_R)$, we have $0, 1, x \in C_R(x)$ and $0, 1, y \in C_R(y)$. This implies $|C_R(x)| \geq 3$ and $|C_R(y)| \geq 3$. Thus, $\deg([x, y]) \leq (|R|-3) + (|R|-3) - 2 = 2|R| - 8$. Therefore, $\Delta(L(\Gamma_R)) \leq 2|R| - 8$. □

Furthermore, if R contains a non-central nilpotent element, we get the upper bound of $\delta(L(\Gamma_R))$ of Corollary 3.7.

Corollary 3.7. *Let R be a finite non-commutative ring containing a non-central nilpotent element of nilpotency index n. Then* $\delta(L(\Gamma_R)) \leq 2|R| - n - 4$.

Proof. Let x be a non-central nilpotent element of nilpotency index n. By Theorem 3.1, there exists $y \in R\backslash Z(R)$ such that x is adjacent to y. Then $0, x, x^2, \ldots, x^{n-1} \in C_R(x)$, which implies $|C_R(x)| \geq n$. Since both $0, y \in C_R(y)$, $|C_R(y)| \geq 2$. Thus,

$$\deg([x, y]) \leq (|R| - n) + (|R| - 2) - 2 = 2|R| - n - 4.$$

Therefore, $\delta(L(\Gamma_R)) \leq 2|R| - n - 4$. ☐

Corollary 3.8. *Let R be a finite non-commutative ring such that* $\min\{\text{nil}(x) \mid x \in R\backslash Z(R)\} = k$. *If* $xy \neq yx$ *for some non-central nilpotent elements x and y, then* $\delta(L(\Gamma_R)) \leq 2(|R| - k - 1)$.

Proof. Let z be a non-central nilpotent element of nilpotency index m. Then $m \geq k$. This implies that $|C_R(z)| \geq m \geq k$. Suppose that $xy \neq yx$ for some non-central nilpotent elements x and y. Then $[x, y] \in V(L(\Gamma_R))$. Since $|C_R(z)| \geq k$ for all non-central nilpotent element z, $|C_R(x)| \geq k$ and $|C_R(y)| \geq k$, so

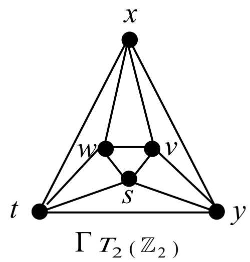
$$\deg([x, y]) \leq (|R| - k) + (|R| - k) - 2 = 2(|R| - k - 1)$$

and $\delta(L(\Gamma_R)) \leq 2(|R| - k - 1)$. ☐

In the following examples, we give a ring that satisfies Corollaries 3.6, 3.7 and 3.8.

Example 3.9. Let $T_2(\mathbb{Z}_2)$ be the ring of upper triangular 2×2 matrices over $\mathbb{Z}_2$. We let $\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $s = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $t = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and $w = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Thus, $T_2(\mathbb{Z}_2)$ is a non-commutative ring of order 8 with identity u. The non- commuting graph of $T_2(\mathbb{Z}_2)$ and the line graph of $\Gamma_{T_2(\mathbb{Z}_2)}$ are shown in the following figures.



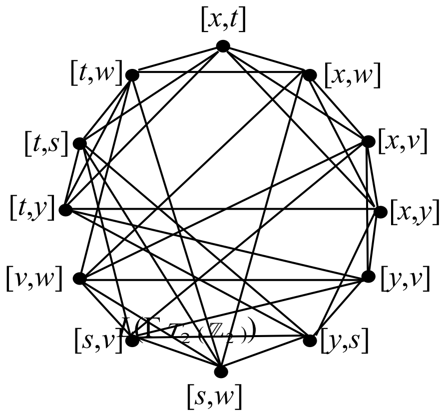


Fig. 2. The non-commuting graph of $T_2(\mathbb{Z}_2)$ and line graph of $\Gamma_{T_2(\mathbb{Z}_2)}$.

As $\Delta(L(\Gamma_{T_2(\mathbb{Z}_2)})) = 6 < 8 = 2|R| - 8$, Corollary 3.6 holds. #

Example 3.5. Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{Z}_4 \right\}$ and $\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $v_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $v_6 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $v_7 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, $v_8 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $v_9 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, $v_{10} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $v_{11} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$, $v_{12} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $v_{13} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$, $v_{14} = \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$, $v_{15} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$.

The non-commuting graph Γ_S is shown as Fig. 3.

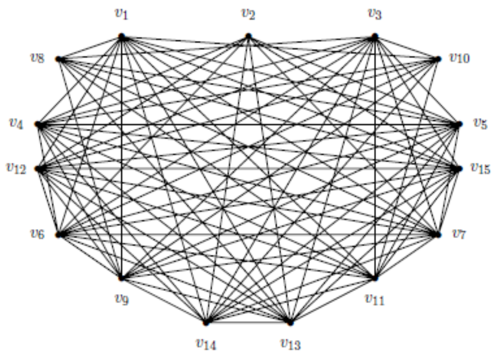


Fig. 3. The non-commuting graph of ring S

Therefore, we have $|S| = 16$, $Z(S) = \{\bar{0}\}$ and $\min\{\text{nil}(x) \mid x \in S\backslash Z(S)\} = 2$. In addition, v_1 and v_9 are the non-central nilpotent element such that $v_1 v_9 \neq v_9 v_1$. Then $\delta(L(\Gamma_S)) = 18 < 30 = 2(|S| - 2 - 1)$, which satisfies the inequality in Corollary 3.8. #

The following results are related to Theorems 3.3 and 3.4.

Proposition 3.12. *There is no finite non-commutative ring R such that* $\delta(L(\Gamma_R)) = 1$.

Proof. Let R be a finite non-commutative ring such that $\delta(L(\Gamma_R)) = 1$. By Theorems 3.3 and 3.4, $|R| - 2 \leq 1 \leq \Delta(L(\Gamma_R)) \leq 2|R| - 6$. From the left inequality, we get $|R| \leq 3$. By the right inequality, we have $|R| \geq 3.5$, which is a contradiction. □

Proposition 3.13. *There is no finite non-commutative ring R such that* $\Delta(L(\Gamma_R)) = 1$.

Proof. This is similar to the proof of Proposition 3.12. □

3.3 Order and size

We give some results for a bound of the order of $L(\Gamma_R)$ and determine the order when $|R| = 4$.

Theorem 3.14. *Let R be a finite non-commutative ring. Then*

$$|V(L(\Gamma_R))| \leq \frac{1}{2}(|R| - |Z(R)|)(|R| - |Z(R)| - 1).$$

Proof. Recall that $2|E(\Gamma_R)| = \sum_{v \in V(\Gamma_R)} \deg(v)$ and $|V(\Gamma_R)| = |R| - |Z(R)|$. Because $\deg(v) \leq |R| - |Z(R)| - 1$ for all $v \in V(\Gamma_R)$,

$$\begin{aligned} |V(L(\Gamma_R))| &= |E(\Gamma_R)| \\ &= \frac{1}{2} \sum_{v \in V(\Gamma_R)} \deg(v) \\ &\leq \frac{1}{2}(|R| - |Z(R)|)(|R| - |Z(R)| - 1). \end{aligned}$$ □

Corollary 3.15. *Let R be a finite non-commutative ring. Then*

$$|V(L(\Gamma_R))| = \frac{1}{2}(|R| - |Z(R)|)(|R| - |Z(R)| - 1)$$

if and only if $|R| = 4$.

Proof. Notice that $|V(L(\Gamma_R))| = |E(\Gamma_R)| = \frac{1}{2}(|R| - |Z(R)|)(|R| - |Z(R)| - 1)$ if and only if Γ_R is complete. By Theorem 2.6, the non-commuting graph of ring R is complete if and only if $|R| = 4$. □

Theorem 3.16. *Let R be a finite non-commutative ring. Then*

$$|V(L(\Gamma_R))| \geq \frac{|R|}{4}(|R| - |Z(R)|).$$

Proof. By Theorem 2.7, we have $\deg(v) \geq \delta(\Gamma_R) \geq \frac{|R|}{2}$ for all $v \in V(\Gamma_R)$. Thus,

$$\begin{aligned} |V(L(\Gamma_R))| &= |E(\Gamma_R)| \\ &= \frac{1}{2} \sum_{i=1}^{|R|-|Z(R)|} \deg(v_i) \\ &\geq \frac{1}{2} \sum_{i=1}^{|R|-|Z(R)|} \frac{|R|}{2} \\ &= \frac{|R|}{4}(|R| - |Z(R)|). \end{aligned}$$ □

In next theorem, we give a bound of the size of $L(\Gamma_R)$.

Theorem 3.17. *Let R be a finite non-commutative ring. Then*

$$\frac{|R|}{8}(|R|-2)(|R|-|Z(R)|) \le |E(L(\Gamma_R))|$$
$$\le \frac{1}{2}(|R|-3)(|R|-|Z(R)|)(|R|-|Z(R)|-1).$$

Proof. From Theorem 3.3, $\deg(v) \ge \delta(L(\Gamma_R)) \ge |R| - 2$ for all $v \in V(L(\Gamma_R))$. Then by Theorem 3.16,

$$|E(L(\Gamma_R))| = \frac{1}{2}\sum_{v\in V(L(\Gamma_R))}\deg(v)$$
$$\ge \frac{1}{2}|V(L(\Gamma_R))|\delta(L(\Gamma_R))$$
$$\ge \frac{1}{2}\frac{|R|}{4}(|R|-|Z(R)|)(|R|-2)$$
$$= \frac{|R|}{8}(|R|-2)(|R|-|Z(R)|).$$

By Theorem 3.4, $\deg(v) \le \Delta(L(\Gamma_R)) \le 2|R|-6$ for all $v \in V(L(\Gamma_R))$. By Theorem 3.14,

$$|E(L(\Gamma_R))| = \frac{1}{2}\sum_{v\in V(L(\Gamma_R))}\deg(v)$$
$$\le \frac{1}{2}|V(L(\Gamma_R))|\,\Delta(L(\Gamma_R))$$
$$\le \frac{1}{2}\left(\frac{1}{2}\right)(|R|-|Z(R)|)(|R|-|Z(R)|-1)(2|R|-6)$$
$$= \frac{1}{2}(|R|-3)(|R|-|Z(R)|)(|R|-|Z(R)|-1).$$ □

The next example shows that the bounds given in Theorems 3.14, 3.16 and 3.17 are sharp.

Example 3.18. From Example 3.5, $|R| = 4$, $Z(R) = \{\bar{0}\}$. Thus,

$$|V(L(\Gamma_R))| = \frac{1}{2}(|R|-|Z(R)|)(|R|-|Z(R)|-1)$$
$$= \frac{|R|}{4}(|R|-|Z(R)|)$$
$$= 3$$

and

$$|E(L(\Gamma_R))| = \frac{|R|}{8}(|R|-2)(|R|-|Z(R)|)$$
$$= \frac{1}{2}(|R|-3)(|R|-|Z(R)|)(|R|-|Z(R)|-1)$$
$$= 3.$$ #

3.4 Diameter and girth

We next show that the diameter of $L(\Gamma_R)$ is at most 3 and that every non-commutative ring R has girth of exactly 3.

Theorem 3.19. *Let R be a finite non-commutative ring. Then* $\mathrm{diam}(L(\Gamma_R)) \le 3$.

Proof. Since R is a finite non-commutative ring, $|R| \ge 4$ by Lemma 2.1. If $|R| = 4$, then Γ_R is complete by Theorem 2.6. Thus, Γ_R is actually K_3, so $V(\Gamma_R) = 3$. On the other hand, we assume that $|R| > 4$. Then $|V(\Gamma_R)| = |R| - |Z(R)| \ge \frac{|R|}{4} > 2$, so $V(\Gamma_R) \ge 3$.

In both cases, we can let $[x_1, y_1]$ and $[x_2, y_2]$ be distinct vertices in $L(\Gamma_R)$. By Lemma 2.10, $|C_R(y_1)| \le \frac{|R|}{2}$ and $|C_R(x_2)| \le \frac{|R|}{2}$, so $|C_R(y_1) \cup C_R(x_2)| = |C_R(y_1)| + |C_R(x_2)| - |C_R(y_1) \cap C_R(x_2)| \le |R| - |C_R(y_1) \cap C_R(x_2)|$. Note that $C_R(y_1) \cap C_R(x_2)$ is nonempty as it contains 0, so $|C_R(y_1) \cup C_R(x_2)| < |R|$, which implies $R \ne C_R(y_1) \cup C_R(x_2)$. Since $Z(R) \subseteq C_R(y_1) \cup C_R(x_2)$, there exists $z \in R \setminus Z(R)$ such that $z \notin C_R(y_1) \cup C_R(x_2)$ and a path P: y_1, z, x_2 in Γ_R. Then we consider the following cases:

1. If $z = x_1$ and $z = y_2$, then P_{L1}: $[x_1, y_1]$, $[x_2, y_2]$ is a path in $L(\Gamma_R)$.
2. If $z = x_1$ and $z \ne y_2$, then P_{L2}: $[x_1, y_1]$, $[x_1, x_2]$, $[x_2, y_2]$ is a path in $L(\Gamma_R)$.
3. If $z \ne x_1$ and $z = y_2$, then P_{L3}: $[x_1, y_1]$, $[y_1, y_2]$, $[x_2, y_2]$ is a path in $L(\Gamma_R)$.
4. If $z \ne x_1$ and $z \ne y_2$, then P_{L4}: $[x_1, y_1]$, $[y_1, z]$, $[z, x_2]$, $[x_2, y_2]$ is a path in $L(\Gamma_R)$.

Hence, in every case, $d([x_1, y_1], [x_2, y_2]) \le 3$ and $\mathrm{diam}(L(\Gamma_R)) \le 3$. □

Theorem 3.20. *Let R be a finite non-commutative ring. Then* $\mathrm{gr}(L(\Gamma_R)) = 3$.

Proof. Let x and y be non-central elements such that $xy \neq yx$. Thus, $x + y \in R \setminus Z(R)$. Since both $x(x+y) \neq (x+y)x$ and $y(x+y) \neq (x+y)y$, C: x, $x+y$, y, x is a cycle in Γ_R. This implies C_L: $[x, x+y]$, $[x+y, y]$, $[y, x]$, $[x, x+y]$ is also a cycle in $L(\Gamma_R)$. Therefore, $\mathrm{gr}(L(\Gamma_R)) = 3$. □

3.5 Connectivity

We determine the vertex-connectivity ($\kappa(G)$) and edge-connectivity ($\lambda(G)$) of $L(\Gamma_R)$. It is well known [11] that $\lambda(G) \le \kappa(G) \le \delta(G)$ for every graph G.

Theorem 3.21. *Let R be a finite non-commutative ring. Then* $\lambda(L(\Gamma_R)) \ge |R| - 2$.

Proof. By Theorem 2.8, $\lambda(\Gamma_R) \ge \frac{|R|}{2}$. By Lemma 2.3, we have $\lambda(L(\Gamma_R)) \ge 2\lambda(\Gamma_R) - 2 \ge 2\frac{|R|}{2} - 2 = |R| - 2$. □

Theorem 3.22. *Let R be a finite non-commutative ring. Then* $\kappa(L(\Gamma_R)) \ge \frac{|R|}{2}$.

Proof. By Theorem 2.8 and Lemma 2.9, we have $\lambda(\Gamma_R) \ge \frac{|R|}{2} \ge \frac{4}{2} = 2$. By Lemma 2.3, $\kappa(L(\Gamma_R)) \ge \lambda(\Gamma_R) \ge \frac{|R|}{2}$. □

In Example 3.5, the ring R has $\kappa(L(\Gamma_R)) = 2 = \frac{|R|}{2}$ and $\lambda(L(\Gamma_R)) = 2 = |R| - 2$. The bounds given in Theorems 3.21 and Theorem 3.22 are therefore sharp.

In addition, we can show that for every non-commutative ring R, $\lambda(L(\Gamma_R)) \ge 2$.

Proposition 3.23. *There is no finite non-commutative ring R such that* $\lambda(L(\Gamma_R)) = 1$.

Proof. Let R be a finite non-commutative ring such that $\lambda(L(\Gamma_R)) = 1$. By Theorem 3.21, we get $|R| \le 3$. By Lemma 2.9, R is commutative, a contradiction. □

We next give necessary and sufficient conditions for $\lambda(L(\Gamma_R)) = 2$.

Theorem 3.24. *Let R be a finite non-commutative ring. Then* $|R| = 4$ *if and only if* $\lambda(L(\Gamma_R)) = 2$.

Proof. Suppose $|R| = 4$. By Theorem 2.6, $L(\Gamma_R)$ is K_3, so that $\lambda(L(\Gamma_R)) = 2$. Conversely, suppose that $\lambda(L(\Gamma_R)) = 2$. By Theorem 3.21 and 3.4, $|R| - 2 \le 2 \le \delta(L(\Gamma_R)) \le \Delta(L(\Gamma_R)) \le 2|R| - 6$. Thus, $|R| = 4$. □

Corollary 3.25. *Let R be a finite non-commutative ring such that* $|R| \ge 5$. *Then* $\lambda(L(\Gamma_R)) \ge 3$.

If R is a ring with identity, we can improve the lower bound in Corollary 3.25.

Proposition 3.26. *Let R be a finite non-commutative ring with identity. Then* $\lambda(L(\Gamma_R)) \ge 4$.

Proof. Let R be a finite non-commutative ring with identity. By Proposition 3.23, $\lambda(L(\Gamma_R)) \ge 2$. Suppose that $\lambda(L(\Gamma_R)) < 4$. If $\lambda(L(\Gamma_R)) = 2$, then $|R| - 2 \le 2 \le 2|R| - 8$. Thus, $|R| \le 4$ and $|R| \ge 5$, which is a contradiction. If $\lambda(L(\Gamma_R)) = 3$, then $|R| - 2 \le 3 \le 2|R| - 8$. Thus, $|R| \le 5$ and $|R| \ge 5.5$, which is also a contradiction. Therefore, $\lambda(L(\Gamma_R)) \ge 4$. □

The only case in which $\lambda(L(\Gamma_R)) = 4$ arises when the cardinality of R is 6, as in the following theorem.

Theorem 3.27. *Let R be a finite non-commutative ring with identity. Then* $|R| = 6$ *if and only if* $\lambda(L(\Gamma_R)) = 4$.

Proof. Assume that $|R| = 6$. Then $|R| - 2 = 4$ and $2|R| - 8 = 4$. Since $|R| - 2 \le \lambda(L(\Gamma_R)) \le 2|R| - 8$, we have $4 \le \lambda(L(\Gamma_R)) \le 4$, which implies $\lambda(L(\Gamma_R)) = 4$. Conversely, suppose that $\lambda(L(\Gamma_R)) = 4$. By Theorem 3.21 and 3.6,

$|R|-2 \le 4 \le \delta(L(\Gamma_R)) \le \Delta(L(\Gamma_R)) \le 2|R|-8$. Then $|R|=6$. □

By combining Proposition 3.26 and Theorem 3.27, we obtain the following corollary.

Corollary 3.28. *Let R be a finite non-commutative ring with identity. If $|R| \ne 6$, then $\lambda(L(\Gamma_R)) \ge 5$.*

By Theorem 3.21, we also obtain the following corollary.

Corollary 3.29. *Let R be a finite non-commutative ring such that $|R| \ge 7$. Then $\lambda(L(\Gamma_R)) \ge 5$.*

Proposition 3.30 shows $\kappa(L(\Gamma_R)) \ge 2$ for every non-commutative ring R.

Proposition 3.30. *There is no finite non commutative ring R such that $\kappa(L(\Gamma_R))=1$.*

Proof. Let R be a finite non-commutative ring such that $\kappa(L(\Gamma_R))=1$. By Theorems 3.22 and 3.4, we obtain

$\frac{|R|}{2} \le 1 \le \delta(L(\Gamma_R)) \le \Delta(L(\Gamma_R)) \le 2|R|-6$. Then $|R| \le 2$ and $|R| \ge 3.5$, which is a contradiction. □

Additionally, the case $\kappa(L(\Gamma_R))=2$ only happens when R has cardinality 4.

Theorem 3.31. *Let R be a finite non-commutative ring. Then $|R|=4$ if and only if $\kappa(L(\Gamma_R))=2$.*

Proof. Assume that $|R|=4$. Then $\frac{|R|}{2}=2$ and $2|R|-6=2$. As $\frac{|R|}{2} \le \kappa(L(\Gamma_R)) \le 2|R|-6$, we have $2 \le \kappa(L(\Gamma_R)) \le 2$, which implies $\kappa(L(\Gamma_R))=2$. On the other hand, suppose that $\kappa(L(\Gamma_R))=2$. By Theorem 3.22 and 3.4,

$\frac{|R|}{2} \le 2 \le \delta(L(\Gamma_R)) \le \Delta(L(\Gamma_R)) \le 2|R|-6$. Then $|R|=4$. □

Corollary 3.32. *Let R be a finite non-commutative ring such that $|R| \ge 5$. Then $\kappa(L(\Gamma_R)) \ge 3$.*

We can develop a better lower bound of $\kappa(L(\Gamma_R))$ whenever R is a ring with identity.

Proposition 3.33. *Let R be a finite non-commutative ring with identity. Then $\kappa(L(\Gamma_R)) \ge 3$.*

Proof. By Proposition 3.30, we obtain $\kappa(L(\Gamma_R)) \ge 2$. Suppose $\kappa(L(\Gamma_R))=2$. Then $\frac{|R|}{2} \le 2 \le 2|R|-8$. Thus, $|R| \le 4$ and $|R| \ge 5$, which leads to a contradiction. Therefore, $\kappa(L(\Gamma_R)) \ge 3$. □

4. Conclusion

We study the graph theoretical properties of a line graph associated with a non-commuting graph of a finite non-commutative ring R. We prove that $L(\Gamma_R)$ is connected and present a lower bound of minimum degree and an upper bound of maximum degree. In particular, when R is a finite non-commutative ring with identity, an upper bound of maximum degree is developed. We also develop an upper bound of minimum degree when R contains a non-central nilpotent element. We give bounds of order and size of $L(\Gamma_R)$ and determine order when $|R| = 4$. Furthermore, diameter of $L(\Gamma_R)$ is at most 3 and girth of exactly 3. The lower bounds for vertex-connectivity and edge-connectivity are given and both are at least 2. Especially, these lower bounds are improved when R is a finite non-commutative ring with identity.

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