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Coincidence Point Theorems on Odered b-Metric Spaces via wt-Distance with Application to Matrix Equations and Numerical Experiments

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ABSTRACT

This article aims to create a new type of generalized contraction mapping to modify the concept of an e_K-simulation function which is defined by Yamaod and Sintunavarat [2019, J. Nonlinear Convex Anal.], we investigate the existence and uniqueness of a point of coincidence in the mapping with respect to a wt- distance in a partially ordered b-metric space which extends the results of Roldán López de Hierro et al. [2015, J. Comput. Appl. Math.]. Furthermore, we prove the existence of Hermitian positive definite solutions of nonlinear matrix equations with some examples and numerical experiments.

Keywords: b-metric spaces ; Coincidence point ; Extended simulation function; Matrix equations; wt-distance

1. Introduction

Recently, there are many results of metric spaces generalized Banach Contraction theorem. In 1996 Kada, Suzuki and Takahashi [12] defined the concept of w-distance which is a generalized metric space and also proved the generalized Caristi's fixed point theorem. Based on this concept, several researchers have focu-sed on w-distance and the extension of the well known classical fixed point result (for more details see [3, 10, 11, 18]). On the other hand, Czerwik [4] formally defined the idea of b-metric spaces covering metric spaces and gave the Banach contraction principle in complete b-metric spaces. In 2014, Hussain, Saadati and Agrawal [8] introduced the concept of wt-distance on b-metric spaces. Later, many authors used

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this concept for proving fixed point result and related topics (for more details see [2, 6, 7, 17]). Motivated by their results, we modify the concept of an e_K-simulation function for extending the results of Roldán López de Hierro et al.[15] to the results in the partially ordered b-metric spaces via wt-distance. Also, we provide an example to illustrate our results. Finally, we apply our results to prove the existence of Hermitian positive definite solutions for nonlinear matrix equations with some examples and numerical experiments.

2. Preliminaries

Now, the definitions of a b-metric space, w-distance, wt-distance and other basic definitions in such spaces. In this paper, we denote the sets of positive integers, non-negative real numbers and real numbers by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ respectively.

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set. For elements $x, y \in X$, we say x, y are comparable with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$.

The subset of $X \times X$ defined by

$$X_{\preceq} = \{(x, y) \in X \times X \,|\, x \preceq y \text{ or } y \preceq x\},$$

is denoted by $X_{\preceq}$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $T, g : X \to X$. We say that T is g non-decreasing with respect to "$\preceq$" if for $x, y \in X$, $gx \preceq gy$ implies $Tx \preceq Ty$.

Definition 2.3. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is said to be a w-distance on X if the following hold:

($w2$) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous (i.e, if $x \in X$ and $y_n \to y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$;

($w3$) for any $\varepsilon > 0$, there is $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Let X be a metric space with metric d. A w-distance p on X is said to be symmetric, if $p(x, y) = p(y, x)$ for all $x, y \in X$. It is obvious that every metric is a w-distance but not conversely. Next, we recall some examples in [19] for show that w-distance is generalized of metric.

Example 2.4. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is a w-distance on X if $p(x, y) = c$ for every $x, y \in X$, where c is a positive real number. But p is not a metric since $p(x, x) = c \neq 0$ for any $x \in X$.

Example 2.5. Let $(X, \|\cdot\|)$ be a normed linear space. A function $p : X \times X \to [0, \infty)$ is a w-distance on X if $p(x, y) = \|x\| + \|y\|$ for every $x, y \in X$.

Definition 2.6. Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $D : X \times X \to \mathbb{R}_+$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions hold:

1. $D(x, y) = 0$ if and only if $x = y$;
2. $D(x, y) = d(y, x)$;
3. $D(x, z) \leq s[D(x, y) + D(y, z)]$.

A pair (X, d) is called a b-metric space with coefficient s.

In Definition 2.6, any metric space is a b-metric space with $s = 1$ and hence the class of b-metric spaces is larger than the class of metric spaces. Some examples of b-metric spaces are given by Berinde [1], Czerwik [**?**], Heinonen [9]. The following well known examples of b-metric showing that b-metric space is real generalization of metric space.

Example 2.7. The set of real numbers together with the functional $D : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$, defined by

$$D(x, y) := |x - y|^2,$$

for all $x, y \in \mathbb{R}$, is a b-metric space with coefficient $s = 2$. However, we obtain that d is not a metric on X since the ordinary triangle inequality is not satisfied. Indeed,

$$D(2, 4) > D(2, 3) + D(3, 4).$$

In 2014, Hussain, Saadati and Agrawal [8] introduced the concept of wt-distance as follow :

Definition 2.8. Let (X, D) be a $b-$metric space. A function $\mathcal{P} : X \times X \to [0, \infty)$ is said to be a wt-distance on X if the following hold:

$(wt1)$ for all $x, y, z \in X$, $\mathcal{P}(x, z) \le K(\mathcal{P}(x, y) + \mathcal{P}(y, z))$;

$(wt2)$ for any $x \in X, \mathcal{P}(x, \cdot) : X \to [0, \infty)$ is $K-$lower semi-continuous (i.e, if $x \in X$ and $y_n \to y \in X$, then $\mathcal{P}(x, y) \le \liminf_{n\to\infty} K\mathcal{P}(x, y_n)$;

$(wt3)$ for $\varepsilon > 0$, there is $\delta > 0$ such that $\mathcal{P}(z, x) \le \delta$ and $\mathcal{P}(z, y) \le \delta$ imply $D(x, y) \le \varepsilon$.

Example 2.9. [8] Let (X, D) be a $b-$metric space. Then the metric D is a wt-distance on X.

Example 2.10. ([8]) Let $D_1 = (x - y)^2$. A function $\mathcal{P} : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $\mathcal{P}(x, y) = \|x\|^2 + \|y\|^2$ for every $x, y \in \mathbb{R}$ is a wt-distance on $\mathbb{R}$.

Example 2.11. ([8]) Let $D_1 = (x - y)^2$. A function $\mathcal{P} : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $\mathcal{P}(x, y) = \|y\|^2$ for every $x, y \in \mathbb{R}$ is a wt-distance on $\mathbb{R}$.

The following two lemmas are crucial for our consideration.

Lemma 2.12. *([8]) Let (X, D) be a $b-$metric space with constant $K \ge 1$ and $\mathcal{P}$ be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero. Then the following conditions hold (for $x, y, z \in X$):*

1. *for any $n \in N$, if $\mathcal{P}(x_n, y) \le \alpha_n$ and $\mathcal{P}(x_n, z) \le \beta_n$ then $y = z$. Particularly, if $\mathcal{P}(x, y) = 0$ and $\mathcal{P}(x, z) = 0$, then $y = z$;*
2. *for any $n \in N$, if $\mathcal{P}(x_n, y_n) \le \alpha_n$ and $\mathcal{P}(x_n, z) \le \beta_n$ then $\{y_n\}$ converges to z;*
3. *for any $n, m \in N$ with $m > n$, if $\mathcal{P}(x_n, x_m) \le \alpha_n$ then $\{x_n\}$ is Cauchy sequence;*
4. *for any $n \in N$, $\mathcal{P}(y, x_n) \le \alpha_n$ then $\{x_n\}$ is Cauchy sequence.*

3. A Class of Simulation Functions

In 2015, Khojasteh *et al.* [13] introduced the concept of a simulation function and a $\mathcal{Z}$-contraction mapping. They proved many results for $\mathcal{Z}$-contraction mappings and established the existence result of a unique fixed point for $\mathcal{Z}$-contraction mappings in metric spaces. Here, we review some basic knowledge related to our investigation from [13].

Definition 3.1 ([13])**.** A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

$(\zeta1)$ $\zeta(0, 0) = 0$;

$(\zeta2)$ $\zeta(t, s) < s - t$ for all $t, s > 0$;

($\zeta 3$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the class of all simulation functions by $\mathcal{Z}$. 3

Example 3.2 ([13]). Let $\zeta_1, \zeta_2, \zeta_3 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

(a) $\zeta_1(t,s) = \psi(s) - \phi(t)$ for all $t, s \in [0,\infty)$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$;

(b) $\zeta_2(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;

(c) $\zeta_3(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \to [0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.

Then ζ_1, ζ_2 and ζ_3 are simulation functions.

Definition 3.3 ([13]). Let (X,d) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $T : X \to X$ is said to be a $\mathcal{Z}$-contraction mapping with respect to ζ if it satisfies

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0,$$

for all $x, y \in X$.

In 2015, Roldán López de Hierro et al. [15] modified the class of simulation functions as follows.

Definition 3.4 ([15]). A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(Ξ_1) $\zeta(0,0) = 0$;

(Ξ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(Ξ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the class of all modify simulation functions by ζ.

In 2017, Roldán López de Hierro and Samet [16] introduced the class of extended simulation functions, which covers the class of simulation functions.

Definition 3.5 ([16]). A function $\theta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called an extended simulation function (for short, an e-simulation function) if it satisfies the following conditions:

(θ_1) for every $a, b > 0$, $\theta(a,b) < b - a$;

(θ_2) for any sequences $\{a_n\}$, $\{b_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l \in (0,\infty),$$

$$b_n > l \ \Rightarrow \ \limsup_{n\to\infty} \theta(a_n, b_n) < 0;$$

(θ_3) for any sequence $\{a_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} a_n = l \in [0,\infty),$$

$$\theta(a_n, l) \geq 0 \ \Rightarrow \ l = 0.$$

The class of all e-simulation functions is denoted by Θ.

Example 3.6. Let $\theta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\theta(\alpha,\beta) = \begin{cases} 1 - \alpha & \text{if } \alpha = 0; \\ \frac{\beta}{2} - \alpha & \text{if } \alpha \neq 0. \end{cases}$$

Then $\theta \in \Theta$, but $\theta \notin \mathcal{Z}$ see [16] for more details.

Recently, Yamaod and Sintunavarat [20] introduced the class of simulation functions in the sense of b-metric space.

Definition 3.7 ([20])**.** Let s be a real number such that $s \geq 1$. A function $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is said to be an s-simulation function if it satisfies the following conditions hold

(ζ_{s1}) $\zeta(\alpha,\beta) < \beta - \alpha$ for all $\alpha,\beta > 0$;

(ζ_{s2}) if $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,\infty)$ such that

$$\begin{aligned} 0 < \liminf_{n\to\infty} \alpha_n &\leq \left(\limsup_{n\to\infty} \beta_n\right) \\ &\leq s^2\left(\liminf_{n\to\infty} \alpha_n\right), \end{aligned}$$

and

$$\begin{aligned} 0 < \liminf_{n\to\infty} \beta_n &\leq s\left(\limsup_{n\to\infty} \alpha_n\right) \\ &\leq s^2\left(\liminf_{n\to\infty} \beta_n\right), \end{aligned}$$

then

$$\limsup_{n\to\infty} \zeta(\alpha_n,\beta_n) < 0.$$

Most recently, Yamaod and Sintunavarat [21] introduced the class of e_K-simulation functions in the sense of b-metric space.

Definition 3.8. Let K be a be real number such that $K \geq 1$. A function $\theta_K : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is said to be an e_K-simulation function if the following conditions hold

(θ_{K1}) for every $a,b > 0$, $\theta_K(a,b) < b-a$;

(θ_{K2}) for any sequences $\{a_n\}$, $\{b_n\} \subset (0,\infty)$, if there exists $L \in (0,\infty)$ such that

$$\begin{aligned} 0 < L \leq \liminf_{n\to\infty} a_n &\leq K\left(\limsup_{n\to\infty} b_n\right) \\ &\leq K^2\left(\liminf_{n\to\infty} a_n\right) \\ &\leq K^3 L, \end{aligned}$$

$$\begin{aligned} 0 < L \leq \liminf_{n\to\infty} b_n &\leq K\left(\limsup_{n\to\infty} a_n\right) \\ &\leq K^2\left(\liminf_{n\to\infty} b_n\right) \\ &\leq K^3 L \end{aligned}$$

and $b_n > L$, then

$$\limsup_{n\to\infty} \theta_K(a_n,b_n) < 0.$$

Now, we note that Θ_K is the class of all e_K-simulation functions.

4. Coincidence Point Theorems for Simulation Function

This section provides a $(\theta_K, g)_{\mathcal{P}}$ contraction by using the concept of Simulation function, and obtain the existence of coincidence point theorems for such mapping in complete $b-$metric space via wt-distance function. First, we modified the concept of an e_K-simulation function which is defined by [21].

Definition 4.1. Let K be a real number such that $K \geq 1$. A function $\theta_K : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is said to be a modified e_K-simulation function if it satisfies the following conditions hold

(θ^M_{K1}) $\theta_K(a,b) < b-a$ for every $a,b > 0$;

(θ^M_{K2}) for any sequences $\{a_n\}$, $\{b_n\} \subset (0,\infty)$ and $a_n < b_n$ for all $n \in \mathbb{N}$, if there exists $L \in (0,\infty)$ such that

$$\begin{aligned} 0 < L \leq \liminf_{n\to\infty} a_n &\leq K\left(\limsup_{n\to\infty} b_n\right) \\ &\leq K^2\left(\liminf_{n\to\infty} a_n\right) \\ &\leq K^3 L, \end{aligned}$$

$$0 < L \leq \liminf_{n\to\infty} b_n \leq K\left(\limsup_{n\to\infty} a_n\right) \leq K^2\left(\liminf_{n\to\infty} b_n\right) \leq K^3 L,$$

and $b_n > L$, then

$$\limsup_{n\to\infty} \theta_K(a_n, b_n) < 0.$$

The class of all modified e_K-simulation functions is denoted by Θ_K^M.

Proposition 4.2. *If $\theta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfies (θ_{K2}^M), then it also satisfies (Ξ_3).*

Proof. Follow Proposition 2.5 in [21] with $\{a_n\}, \{b_n\} \subset (0,\infty)$ such that $a_n < b_n$ for all $n \in \mathbb{N}$, we obtain the result. □

Definition 4.3. Let (X, D) be a complete $b-$metric space with constant $K \geq 1$ and $\mathcal{P}$ be a wt-distance on X. Suppose that $T, g : X \to X$ is called a $(\theta_K, g)_{\mathcal{P}}$ contraction, if there exists $\theta_K \in \Theta_K^M$ such that

$$\theta_K(\mathcal{P}(Tx, Ty), \mathcal{P}(gx, gy)) \geq 0 \qquad (4.1)$$

for all $x, y \in X_{\preceq}$ with $gx \neq gy$.

Theorem 4.4. *Let $(X, D, \preceq)$ be a $b-$metric space with constant $K \geq 1$ and $\mathcal{P}$ be a wt-distance on X. Suppose that $T, g : X \to X$ is a $(\theta_K, g)_{\mathcal{P}}$ contraction, and T is g non-decreasing with $T(X) \subseteq g(X)$ satisfying the following conditions:*

(i) $(g(X), D)$ (or $(T(X), D)$) is complete,

(ii) there exists $x_0 \in X$ such that $(gx_0, Tx_0) \in X_{\preceq}$,

(iii) for every $x \in X$ with $(x, Tx) \in X_{\preceq}$,

$$\inf\{\mathcal{P}(gx, gy) + \mathcal{P}(gx, Tx\} > 0,$$

for every $y \in X$ with $gy \neq Ty$.

Thus T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and one the following hold.

(a) if $\{gx_n\}$ contains a coincidence point of T and g.

(b) the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

Thus $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then T and g have a unique coincidence point.

Proof. Let $x_0 \in X$ such that $gx_0 \neq Tx_0$ and $(gx_0, Tx_0) \in X_{\preceq}$. Since $T(X) \subseteq g(X)$ then there exists $x_1 \in X$ such that $Tx_0 = gx_1$ that is $(gx_0, gx_1) \in X_{\preceq}$. Since T is g non-decreasing with respect to $\preceq$, we get $(Tx_0, Tx_1) \in X_{\preceq}$. Again $T(X) \subseteq g(X)$ then there exists $x_2 \in X$ such that $Tx_1 = gx_2$ that is $(gx_1, gx_2) \in X_{\preceq}$, and T is g non-decreasing with respect to $\preceq$, we get $(Tx_1, Tx_2) \in X_{\preceq}$. Continuing of this process, we obtain the sequences $\{x_n\}$ such that $Tx_n = gx_{n+1}$ and

$$(Tx_n, Tx_m) \in X_{\preceq},$$

for any $n, m \in \mathbb{N}$. If $\{x_n\}$ contains a coincidence point of T and g, that is

$$gx_{n_0} = gx_{n_0+1} = Tx_{n_0} \text{ for some } n_0 \in \mathbb{N},$$

the proof is finished. Suppose that $gx_n = gx_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$. First, we will show that

$$\lim_{n\to\infty} \mathcal{P}(gx_n, gx_{n+1}) = 0. \qquad (4.2)$$

By (4.1) and (θ^M_{K1}), we observe that

$$\begin{aligned} 0 &\leq \theta_K(\mathcal{P}(Tx_n, Tx_{n+1}), \mathcal{P}(gx_n, gx_{n+1})) \\ &= \theta_K(\mathcal{P}(gx_{n+1}, gx_{n+2}), \mathcal{P}(gx_n, gx_{n+1})) \\ &< \mathcal{P}(gx_n, gx_{n+1}) - \mathcal{P}(gx_{n+1}, gx_{n+2}) \end{aligned} \tag{4.3}$$

for all $n \in \mathbb{N}$. It follow that

$$\mathcal{P}(gx_{n+1}, gx_{n+2}) < \mathcal{P}(gx_n, gx_{n+1}) \tag{4.4}$$

for all $n \in \mathbb{N}$. This mean that the sequence $\{\mathcal{P}(gx_n, gx_{n+1})\}$ is a decreasing sequence which converges to some real number $r \geq 0$. Suppose that $r > 0$. Putting $a_n = \mathcal{P}(gx_{n+1}, gx_{n+2})$ and $b_n = \mathcal{P}(gx_n, gx_{n+1})$, then $a_n < b_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} \liminf_{n\to\infty} a_n &= \limsup_{n\to\infty} a_n \\ &= r \\ &= \liminf_{n\to\infty} b_n \\ &= \limsup_{n\to\infty} b_n. \end{aligned}$$

Hence,

$$\begin{aligned} 0 < r \leq \liminf_{n\to\infty} a_n &\leq K\left(\limsup_{n\to\infty} b_n\right) \\ &\leq K^2\left(\liminf_{n\to\infty} a_n\right) \\ &\leq K^3 r, \end{aligned}$$

and

$$\begin{aligned} 0 < r \leq \liminf_{n\to\infty} b_n &\leq K\left(\limsup_{n\to\infty} a_n\right) \\ &\leq K^2\left(\liminf_{n\to\infty} b_n\right) \\ &\leq K^3 r, \end{aligned}$$

By (θ^M_{K2}) with $L = r$, we get

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \theta_K(\mathcal{P}(gx_{n+1}, gx_{n+2}), \mathcal{P}(gx_n, gx_{n+1})) \\ &= \limsup_{n\to\infty} \theta_K(a_n, b_n) \\ &< 0, \end{aligned}$$

which is a contradiction and thus $r = 0$. That is (4.2) hold. Next, we will prove that

$$\lim_{m,n\to\infty} \mathcal{P}(gx_n, gx_m) = 0. \tag{4.5}$$

Suppose to the contrary that there is $\varepsilon > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that for each $k \in \mathbb{N}$, n_k is the smallest number such that

$$\mathcal{P}(gx_{n_k}, gx_{m_k-1}) < \varepsilon \leq \mathcal{P}(gx_{n_k}, gx_{m_k}) \tag{4.6}$$

and $k \leq n_k < m_k$. By $(wt1)$, we have

$$\begin{aligned} \varepsilon &\leq \mathcal{P}(gx_{n_k}, gx_{m_k}) \\ &\leq K[\mathcal{P}(gx_{n_k}, gx_{m_k-1}) + \mathcal{P}(gx_{m_k-1}, gx_{m_k})] \\ &< K\varepsilon + K\mathcal{P}(gx_{m_k-1}, gx_{m_k}). \end{aligned}$$

Using (4.2), we get

$$\varepsilon \leq \limsup_{k\to\infty} \mathcal{P}(gx_{n_k}, gx_{m_k}) \leq K\varepsilon. \tag{4.7}$$

Similarly, we obtain

$$\varepsilon \leq \liminf_{k\to\infty} \mathcal{P}(gx_{n_k}, gx_{m_k}) \leq K\varepsilon. \tag{4.8}$$

By the same method as (4.3) and (4.4) , we obtain

$$\mathcal{P}(gx_{n_k}, gx_{m_k}) \leq \mathcal{P}(gx_{n_k-1}, gx_{m_k-1}). \tag{4.9}$$

From (4.6), (4.9) and $(wt1)$, we obtain

$$\begin{aligned} \varepsilon &\leq \mathcal{P}(gx_{n_k}, gx_{m_k}) \\ &< \mathcal{P}(gx_{n_k-1}, gx_{m_k-1}) \\ &\leq K[\mathcal{P}(gx_{n_k-1}, gx_{n_k}) + \mathcal{P}(gx_{n_k}, gx_{m_k-1})] \\ &\leq K\mathcal{P}(gx_{n_k-1}, gx_{n_k}) + K\varepsilon. \end{aligned}$$

By using (4.2) and above inequality, we have

$$\varepsilon \leq \limsup_{k\to\infty} \mathcal{P}(gx_{n_k-1}, gx_{m_k-1}) \leq K\varepsilon. \tag{4.10}$$

Similarly, we obtain

$$\varepsilon \leq \liminf_{k\to\infty} \mathcal{P}(gx_{n_k-1}, gx_{m_k-1}) \leq K\varepsilon. \tag{4.11}$$

From (4.8) and (4.10), we have

$$\begin{aligned} \varepsilon &< \liminf_{k\to\infty} \mathcal{P}(gx_{n_k}, gx_{m_k}) \\ &\leq K\left(\limsup_{k\to\infty} \mathcal{P}(gx_{n_k-1}, gx_{m_k-1})\right) \\ &\leq K^2\left(\liminf_{k\to\infty} \mathcal{P}(gx_{n_k}, gx_{m_k})\right) \\ &\leq K^3\varepsilon \end{aligned} \tag{4.12}$$

Also, from (4.7) and (4.11), we obtain

$$\begin{aligned} \varepsilon &< \liminf_{n\to\infty} \mathcal{P}(gx_{n_k-1}, gx_{m_k-1}) \\ &\leq K\left(\limsup_{n\to\infty} \mathcal{P}(gx_{n_k}, gx_{m_k})\right) \\ &\leq K^2\left(\liminf_{n\to\infty} \mathcal{P}(gx_{n_k-1}, gx_{m_k-1})\right) \\ &\leq K^3\varepsilon. \end{aligned} \tag{4.13}$$

It follows from $(\theta_K, g)_{\mathcal{P}}$ contractive condition, (4.12), (4.13) and (θ^M_{K2}) with $L = \varepsilon$ that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \theta_K(\mathcal{P}(gx_{n_k-1}, gx_{m_k-1}), \\ &\qquad \mathcal{P}(x_{n_k-1}, x_{m_k-1})) \\ &= \limsup_{k\to\infty} \theta_K(\mathcal{P}(x_{n_k}, x_{m_k}), \\ &\qquad \mathcal{P}(x_{n_k-1}, x_{m_k-1})) \\ &< 0, \end{aligned}$$

which is a contradiction. Therefore (4.5) hold, and $\{gx_n\}$ is a b-Cauchy sequence in X. Next, we show that T, g have at least a coincidence point. Since $(g(X), D)$ is a complete, there exists $u \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = u.$$

Since for each $n \geq 0$, we have $gx_{n+1} = Tx_n \in T(X) \subseteq g(X)$, then

$$\lim_{n\to\infty} Tx_n = u.$$

Now foe every $\epsilon > 0$, from the fact that $\{gx_n\}$ is a b-Cauchy sequence there exits $N_\epsilon \in \mathbb{N}$ such that

$$\mathcal{P}(gx_{N_\epsilon}, gx_n) < \frac{\epsilon}{K} \quad \text{for all} \quad n > N_\epsilon$$

Using $(wt2)$, we have

$$\begin{aligned} \mathcal{P}(gx_{N_\epsilon}, u) &\leq K \liminf_{n\to\infty} \mathcal{P}(gx_{N_\epsilon}, gx_n) \\ &< K \cdot \tfrac{\epsilon}{K} = \epsilon. \end{aligned} \tag{4.14}$$

Setting $\epsilon = \frac{1}{j}$ and $N_\epsilon = n_j$, by (4.14) we obtain that

$$\lim_{j\to\infty} \mathcal{P}(gx_{n_j}, u) = 0.$$

Let v be a point of X such that $gv = u$. Next, we examine that v is a coincidence point of T and g. Suppose that $gv \neq Tv$, then by assumption (iii) we get

$$\begin{aligned} 0 &< \inf\{\mathcal{P}(gx, gv) + \mathcal{P}(gx, Tx) : x \in X\} \\ &\leq \inf\{\mathcal{P}(gx_{n_j}, gv) + \mathcal{P}(gx_{n_j}, Tx_{n_j})\} \\ &\leq \inf\{\mathcal{P}(gx_{n_j}, gv) + \mathcal{P}(gx_{n_j}, gx_{n_j+1})\} \\ &\to 0 \text{ as } j \to \infty : n_j \in \mathbb{N}, \end{aligned}$$

which is a contradiction and then $gv = Tv$. For the final statement, let $x_\star, y_\star \in X$ are coincidence points of T and g, that is $gx_\star = Tx_\star$ and $gy_\star = Ty_\star$. We now prove that $gx_\star = gy_\star$. Suppose on the contrary, that is $gx_\star \neq gy_\star$. Then by $(wt3)$ we get $\mathcal{P}(gx_\star, gy_\star) > 0$, and also $\mathcal{P}(Tx_\star, Ty_\star) > 0$. From (θ^M_{K1}), we have

$$\begin{aligned} 0 &\leq \theta_K(\mathcal{P}(Tx_\star, Ty_\star), \mathcal{P}(gx_\star, gy_\star)) \\ &< \mathcal{P}(gx_\star, gy_\star) - \mathcal{P}(Tx_\star, Ty_\star), \end{aligned}$$

it follow that

$$\mathcal{P}(gx_\star, gy_\star) = \mathcal{P}(Tx_\star, Ty_\star) < \mathcal{P}(gx_\star, gy_\star) \tag{4.15}$$

which is a contradiction and hence $Tx_\star = gx_\star = gy_\star = Tx_\star$. Further, if g is injective on the set of all coincidence points of T and g then $x_\star = y_\star$ and the proof is complete.

□

Theorem 4.5. *Let $(X, D, \preceq)$ be a $b-$metric space with constant $K \geq 1$ and $\mathcal{P}$ be a wt-distance on X. Suppose that $T, g : X \to X$ is a $(\theta_K, g)_{\mathcal{P}}$ contraction and non-decreasing with $T(X) \subseteq g(X)$ satisfying the following conditions:*

- *(i)* (X, D) *is complete,*
- *(ii)* *there exists* $x_0 \in X$ *such that* $(gx_0, Tx_0) \in X_{\preceq}$,
- *(iii)* T *and* g *are continuous and compatible.*

Then T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and one of the following hold.

- *(a)* *if* $\{gx_n\}$ *contains a coincidence point of* T *and* g,
- *(b)* *the sequence* $\{gx_n\}$ *converges to a coincidence point of* T *and* g.

Then $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if $g(orT)$ is injective on the set of all coincidence points of T and g(or, simply, it is injective), then T and g have a unique coincidence point.

Proof. From Theorem 4.4, we have $gx_n = gx_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$, and $\{gx_n\}$ is a b-Cauchy sequence in X. Since (X, D) is complete, then there exists $u \in X$ such that $\{gx_n\}$ converges to u. That is

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = u.$$

By the continuity of T and g, we obtain that

$$T(gx_n) \to Tu \quad \text{and} \quad g(Tx_n) \to gu. \tag{4.16}$$

Thus, we have

$$\begin{aligned} D(gu, Tu) &\leq K(D(gu, gTx_n) + D(gTx_n, Tu)) \\ &\leq KD(gu, gTx_n) \\ &\quad +K^2 D(gTx_n, Tgx_n) \\ &\quad +K^2 D(Tgx_n, Tu). \end{aligned} \tag{4.17}$$

Taking as $n \to \infty$ in (4.17), and by using the compatibility of T and g, we get $gu = Tu$. The final part follow from Theorem 4.4. □

Theorem 4.6. *Let $(X, D, \preceq)$ be a $b-$metric space with constant $K \geq 1$ and $\mathcal{P}$ be a wt-distance on X. Suppose that $T, g : X \to X$ is a $(\theta_K, g)_{\mathcal{P}}$ contraction and non-decreasing with $T(X) \subseteq g(X)$ satisfying the following conditions:*

- *(i)* (X, D) *is complete,*
- *(ii)* *there exists* $x_0 \in X$ *such that* $(gx_0, Tx_0) \in X_{\preceq}$,
- *(iii)* T *and* g *are continuous and commuting.*

Then T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and one of the following hold.

- *(a)* *if* $\{gx_n\}$ *contains a coincidence point of* T *and* g
- *(b)* *the sequence* $\{gx_n\}$ *converges to a coincidence point of* T *and* g.

Then $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if g (or T) is injective on the set of all coincidence points of T and g (or, simply, it is injective), then $T$and g have a unique coincidence point.

Proof. From the fact that, if T and g are commuting then T and g are compatible. Then we obtain the result. □

5. Consequences

Section 5 give the consequences of the main results from Theorem 4.4 - Theorem 4.6.

Letting $\mathcal{P} = D$ in Theorem 4.4, we have the interesting result as follows:

Corollary 5.1. *Let $(X, D, \preceq)$ be a $b-$metric space with constant $K \geq 1$. Let $T, g : X \to X$ such that*

$$\theta_K(D(Tx, Ty), D(gx, gy)) \geq 0, \quad (5.1)$$

for all $x, y \in X_{\preceq}$ with $gx \neq gy$, and T is g non-decreasing with $T(X) \subseteq g(X)$ satisfying the following conditions:

(i) $(g(X), D)$ (or $(T(X), D)$) is complete,

(ii) there exists $x_0 \in X$ such that $(gx_0, Tx_0) \in X_{\preceq}$.

Then T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and one the following hold.

(a) if $\{gx_n\}$ contains a coincidence point of T and g

(b) the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

Then $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if $g\ (orT)$ is injective on the set of all coincidence points of T and 0g (or, simply, it is injective), then $T$and g have a unique coincidence point.

Taking $K = 1$ in Theorem 4.4, we hence obtain the following theorem:

Corollary 5.2. *Let $(X, d, \preceq)$ be a metric space and p be a w-distance on X. Let $T, g : X \to X$ such that*

$$\theta_K(p(Tx, Ty), p(gx, gy)) \geq 0, \quad (5.2)$$

for all $x, y \in X_{\preceq}$ with $gx \neq gy$, and T is g non-decreasing with $T(X) \subseteq g(X)$ satisfying the three conditions as follows

(i) $(g(X), D)$ (or $(T(X), D)$) is complete,

(ii) there exists $x_0 \in X$ such that $(gx_0, Tx_0) \in X_{\preceq}$,

(iii) for every $x \in X$ such that $(x, Tx) \in X_{\preceq}$,

$$\inf\{p(gx, gy) + p(gx, Tx\} > 0,$$

$$\text{for every } \ y \in X \text{ with } gy \neq Ty.$$

Then T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and satisfy at least one of the following conditions.

(a) if $\{gx_n\}$ contains a coincidence point of T and g,

(b) the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

Then $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if $g(orT)$ is injective on the set of all coincidence points of T and g(or, simply, it is injective), then $T$and g have a unique coincidence point.

Letting $p = d$ in Corollary 5.2, and by Proposition 4.2, we get the following result.

Corollary 5.3. *([15]) Let (X,d) be a metric space. Let $T,g : X \to X$ be mapping such that*

$$\theta(d(Tx,Ty), d(gx,gy)) \geq 0, \qquad (5.3)$$

for all $x,y \in X$ with $gx \neq gy$, and $T(X) \subseteq g(X)$ satisfying the two conditions as follows:

(i) $(g(X),d)$ (or $(T(X),d)$) is complete,

(ii) there exists $x_0 \in X$ such that $(gx_0, Tx_0) \in X_{\leq}$,

Then T, g have at least a coincidence point. Moreover, if $x_\star, y_\star \in X$ are coincidence points of T and g, and one the following statements hold.

(a) if $\{gx_n\}$ contains a coincidence point of T and g,

(b) the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point $v \in X$ such that $gv = u$ is a coincidence point of T and g.

Then $Tx_\star = gx_\star = gy_\star = Ty_\star$, and if $g(orT)$ is injective on the set of all coincidence points of T and g(or, simply, it is injective), then $T$and g have a unique coincidence point.

Remark 5.4. .

(1) The statement in Corollary 5.3 is some part of The statement in Theorem 4.8 in [15]. However, by combine The statement in Theorem 4.4, Theorem 4.5 and Theorem 4.6, we obtain all the statement of Theorem 4.8 in [15].

(2) By applying Example 3.2 and the relations of the classes $\mathcal{Z}$, Ξ, Θ, Θ_K and Θ_K^M (see also [21]) with our result, we obtain many famous result in wt-distance, w-distance, b-metric spaces and metric spaces.

6. Numerical Examples

This section provides a few examples to supports our main results. First, give an examples to satisfies modified e_K-simulation function.

Example 6.1. Let $X = [0,\infty)$. Define $\theta_K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\theta_K(\alpha,\beta) = \begin{cases} 2(\beta-\alpha), & \text{if } \beta < \alpha, \\ \beta - 4\alpha - \frac{1}{2}, & \text{otherwise,} \end{cases}$$

We will show that θ_K is in Θ_K^M . Let $\alpha,\beta > 0$. If $\beta < \alpha$, then

$$\theta_K(\alpha,\beta) = 2(\beta-\alpha) < \beta - \alpha.$$

Suppose that $\alpha < \beta$, we have

$$\theta_K(\alpha,\beta) = \beta - 4\alpha - \frac{1}{2} < \beta - \alpha.$$

Thus, (θ_{K1}^M) is satisfied. Furthermore, corresding to (θ_{K2}^M), we suppose that $\{\alpha_n\},\{\beta_n\} \subset (0,\infty)$ are two sequences with with $\alpha_n < \beta_n$, for all $n \in \mathbb{N}$, and $L > 0$ such that

$$\begin{aligned} L \leq \liminf_{n\to\infty} \alpha_n &\leq 2\left(\limsup_{n\to\infty} \beta_n\right) \\ &\leq 4\left(\liminf_{n\to\infty} \alpha_n\right) \\ &\leq 8L, \end{aligned}$$

$$\begin{aligned} L \leq \liminf_{n\to\infty} \beta_n &\leq 2\left(\limsup_{n\to\infty} \alpha_n\right) \\ &\leq 4\left(\liminf_{n\to\infty} \beta_n\right) \\ &\leq 8L \end{aligned}$$

and $\beta_n > L.$

We then obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \theta_K(\alpha_n,\beta_n)\\
&= \limsup_{n\to\infty} \left(\beta_n - 4\alpha_n - \frac{1}{2}\right)\\
&\le 2\limsup_{n\to\infty} \beta_n - 4\liminf_{n\to\infty} \alpha_n - \frac{1}{2}\\
&\le 4\liminf_{n\to\infty} \alpha_n - 4\liminf_{n\to\infty} \alpha_n - \frac{1}{2}\\
&= -\frac{1}{2}\\
&< 0.
\end{aligned}$$

It yields that (θ^M_{K2}) holds. Therefore, $\theta_K \in \Theta^M_K$. Next, we will prove that θ_K is not an e-simulation function. Let $\{\alpha_n\} \subset (0,\infty)$ and $L \ge 0$. Now we assume

$$\lim_{n\to\infty} \alpha_n = L \quad \text{and} \quad \theta(\alpha_n, L) \ge 0.$$

We thus obtain

$$0 \le \theta(\alpha_n, L) = L - 4\alpha_n - \frac{1}{2}.$$

As $n \to \infty$ in above inequality, we have

$$L \le -\frac{(6L+1)}{2}.$$

This follows that (θ_3) is not satisfied and so θ_K is not an e-simulation function. Furthermore, if we take $\alpha_n = 1$ and $\beta_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} \theta_K(\alpha_n,\beta_n) &= \limsup_{n\to\infty} 2(1 - \frac{1}{n} - 1)\\
&= \limsup_{n\to\infty} (-\frac{2}{n}) = 0.
\end{aligned}$$

Thus, θ_K is also not an e_K-simulation function in sense of Yamaod and Sintunavarat [21]. Therefore the result of Roldán López de Hierro and Samet [16] and Yamaod and Sintunavarat [21] are not applicable in this example.

Next, we consider an instance to support Theorem 4.4

Example 6.2. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with $\le$ is the usual ordering, and $D = (x-y)^2$ for all $x, y \in X$, and consider the wt−distance $\mathcal{P}$ on X defined by $\mathcal{P}(x,y) =| y |^2$ for every $x, y \in X$. Define mapping $T, g : X \to X$ given by

$$Tx = \begin{cases} \dfrac{x^2}{4}, & \text{if } x \in \{\frac{1}{2^n} : n \in \mathbb{N}\},\\[2ex] 0, & \text{otherwise,}\end{cases}$$

and

$$gx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in \{\frac{1}{2^n} : n \in \mathbb{N}\},\\[2ex] 0, & \text{otherwise.}\end{cases}$$

Now, we consider the modified e_K-simulation function define by

$$\theta_K(\alpha,\beta) = \beta - 4\alpha - \frac{1}{2} \quad \text{for all } \alpha, \beta > 0.$$

By similarly as an Example 6.1, we can see that $\theta_K \in \Theta^M_K$. We next prove that T and g satisfies inequality (4.1).

Case I: For $x, y \in \{\frac{1}{n} : n \in \mathbb{N}\}$ with $x \ne y$, we have

$$\begin{aligned}
&\theta_K(\mathcal{P}(Tx,Ty), \mathcal{P}(gx,gy))\\
&= \theta_K\big(\mathcal{P}(\frac{x^2}{4}, \frac{y^2}{4}), \mathcal{P}(\frac{x}{2}, \frac{y}{2})\big)\\
&= \theta_K\big(\frac{y^2}{4}, \frac{y}{2}\big)\\
&= \frac{y}{2} - 4\big(\frac{y^2}{4}\big) - \frac{1}{2}\\
&> \frac{y}{2} - y^2\\
&\ge 0.
\end{aligned}$$

Case II: For $x = 0$ and $y \in \{\frac{1}{n} : n \in \mathbb{N}\}$, we have

$$\begin{aligned}
&\theta_K(\mathcal{P}(Tx,Ty),\mathcal{P}(gx,gy))\\
&= \theta_K\big(\mathcal{P}(0,\frac{y^2}{4}),\mathcal{P}(0,\frac{y}{2})\big)\\
&= \theta_K\big(\frac{y^2}{4},\frac{y}{2}\big)\\
&= \frac{y}{2} - 4(\frac{y^2}{4}) - \frac{1}{2}\\
&> \frac{y}{2} - y^2\\
&\geq 0.
\end{aligned}$$

Case III: For $x = 0 = y$ is obvious.

Therefore, (4.1) is satisfied. Also, T and g satisfies assumption (iii) of Theorem 4.4. Indeed for any $n \in \mathbb{N}$ with $y := \frac{1}{2^n}$, we get $g(y) = g\big(\frac{1}{2^n}\big) = \frac{1}{2^{n+1}} \neq \frac{1}{2^{2n+2}} = \frac{x^2}{4} = T(y)$, and for every $x \in X$ with $(x,Tx) \in X_{\leq}$, we have

$$\begin{aligned}
&\inf\big\{\mathcal{P}(gx,gy)+\mathcal{P}(gx,Tx)\big\}\\
&= \inf\big\{\mathcal{P}(\frac{x}{2},\frac{y}{2})+\mathcal{P}(\frac{x}{2},\frac{x^2}{4})\big\}\\
&= \inf\big\{\frac{y}{2}+\frac{x^2}{4}\big\}\\
&= \inf\big\{\frac{1}{2^{n+1}}+\frac{1}{2^{2m+2}} \ : \ \text{for some } m \in \mathbb{N}\big\}\\
&\geq \frac{1}{2^{n+1}}\\
&> 0.
\end{aligned}$$

The rest is obvious. Thus all hypothesis of Theorem 4.4 are satisfied. Consequently, in this case 0 is a coincidence point of T and g.

7. An Application to Matrix Equation

In this section, we apply our result to prove the existence of Hermitian positive definite solutions for nonlinear matrix equations. First, we denote

- the set of all $n \times n$ complex matrices by $M(n)$,
- the family of all $n \times n$ Hermitian matrices $H(n)$ (i.e. $H(n) \subseteq M(n)$),
- the set of all $n \times n$ positive defi3nite matrices $P(n)$ (i.e. $P(n) \subseteq H(n)$),
- the set of all $n \times n$ positive semi-definite matrices $P(n)^+$ (i.e. $P(n)^+ \subseteq H(n)$).

For any $X \in P(n)$ and $X \in P(n)^+$, we write

$X \succ 0$ and $X \succeq 0$, respectively.

Furthermore, $X - Y \succeq 0$ and $X - Y \succ 0$ mean that

$X \succeq Y$ and $X \succ Y$, respectively.

Also, for any $X, Y \in H(n)$, there is a greatest lower bound and a least upper bound (see [14]). Now, we denote $\|\cdot\|$ is the spectral norm of a matrix A, that is,

$$\|A\| = \sqrt{\lambda^+(A^*A)},$$

where A^* is the conjugate transpose of A and $\lambda^+(A^*A)$ is the largest eigenvalue of the matrix A^*A. The Ky Fan norm is defined by

$$\|A\|_1 = \sum_{i=1}^{m} s_i(A),$$

where $s_i(A)$ for each $i = 1, 2, \cdots, m$ is the singular values of $A \in M(n)$. Also, we have

$$\|A\|_1 = tr((A^*A)^{1/2}),$$

which is $tr(A)$ (trace norm) for (Hermitian) nonnegative matrices. Define b-metric on $H(n)$ as follows:

$$D(X,Y) = (\|X-Y\|_{tr})^{\lambda}, \text{ for all } X,Y \in H(n), \tag{7.1}$$

where $\lambda \geq 1$ and the notion $\|\cdot\|_{tr}$ denote the trace norm, that is $\|\cdot\|_{tr} = \|\cdot\|_1$. Then $(H(n), D)$ is a complete b-metric space. Moreover, $H(n)$ is a partially ordered set with partial order $\preceq$, where $X \preceq Y \Longleftrightarrow Y \succeq X$. Now we consider the following matrix equation, defined

$$X = Q + \sum_{k=1}^{m} A_i^* \mathcal{G}(X) A_i. \tag{7.2}$$

We now assume that $\mathcal{G}$ is an order-preserving and continuous mapping from $H(n)$ to $P(n)$ and let $F : H(n) \to H(n)$ be the mapping defined by

$$F(X) = Q + \sum_{k=1}^{m} A_i^* \mathcal{G}(X) A_i, \tag{7.3}$$

for all $X \in H(n)$, where Q is in □ $P(n)$ and A_i is an arbitrary $n \times n$ matrix for each $i = 1, 2, 3, \cdots, m$.

It this section, we next prove some results.

Theorem 7.1. *Consider the matrix equation (7.2). Let $F : H(n) \to H(n)$ be an order-preserving mapping and $Q \in P(n)$ be defined by (7.3). Suppose that there is a positive number μ and $\lambda \geq 1$ and the following hold:*

(i) $\sum_{i=1}^{m} A_i A_i^* < \mu \cdot I_n$ *and* $\sum_{k=0}^{m} A_i^* \mathcal{G}(Q) A_i \succ 0$*;*

(ii) for all $X, Y \in H(n)$ such that $X \preceq Y$,
$$(\|\mathcal{G}(X) - \mathcal{G}(Y))\|_1)^{\lambda} \leq \frac{1}{4\mu}(\|X-Y\|_1)^{\lambda} - \frac{1}{8\mu}.$$

Then we have the following a statements :

(1) *The matrix equation* (7.2) *has a solution.*

(2) *Moreover, for any $X_0 \in H(n)$ such that $X_0 \preceq Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_0) A_i$, the iteration $\{X_n\}$ defined by*

$$X_n = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_{n-1}) A_i, \tag{7.4}$$

converges to a solution of the matrix equation (7.2) in the sense of the trace norm $\|\cdot\|_{tr}$.

Proof. Consider $wt-$distance $\mathcal{P} = D$, and let $D : H(n) \times H(n) \to [0, \infty)$ be defined as in (7.1). Let $X, Y \in H(n)$ such that $X \preceq Y$. Then we have

$$\begin{aligned}
(\|F(X) - F(Y)\|_1)^{\lambda} &= (tr(F(X) - F(Y)))^{\lambda} \\
&= \Big[\sum_{i=1}^{m} tr(A_i^*(\mathcal{G}(X) - \mathcal{G}(Y))A_i\Big]^{\lambda} \\
&= \Big[\sum_{i=1}^{m} tr(A_i A_i^*(\mathcal{G}(X) - \mathcal{G}(Y))\Big]^{\lambda} \\
&= \Big[tr\big((\sum_{i=1}^{m}(A_i A_i^*)\Big]^{\lambda}(\mathcal{G}(X) - \mathcal{G}(Y))\big)\Big]^{\lambda} \\
&\leq \Big[\Big\|\sum_{i=1}^{m}(A_i A_i^*)\Big\|\Big]^{\lambda}\big(\mathcal{G}(X) - \mathcal{G}(Y)\|_1\big)^{\lambda} \\
&\leq \frac{\|\sum_{i=1}^{m}(A_i A_i^*)\|}{\mu}\left(\frac{1}{4}(\|X-Y\|_1)^{\lambda} - \frac{1}{8}\right) \\
&\leq \frac{1}{4}(\|X-Y\|_1)^{\lambda} - \frac{1}{8},
\end{aligned}$$

it follow that

$$(\|X-Y\|_1)^{\lambda} - 4(\|F(X)-F(Y)\|_1)^{\lambda} - \frac{1}{2} \geq 0.$$

Now, we take $\theta_K(\alpha, \beta) = \beta - 4\alpha - \frac{1}{2}$ for all $\alpha, \beta \geq 0$. Then

$$\begin{aligned}
\theta_K(\mathcal{P}(F(X), F(Y)), \mathcal{P}(X,Y)) &= \theta_K(D(F(X), F(Y)), D(X,Y)) \\
&= \theta_K((\|F(X) - F(Y)\|_1)^{\lambda}, (\|X-Y\|_1)^{\lambda})
\end{aligned}$$

$$
\begin{aligned}
&= (\|X-Y\|_1)^\lambda - 4(\|F(X)-F(Y)\|_1)^\lambda \\
&\quad -\frac{1}{2} \\
&\geq 0.
\end{aligned}
$$

and hence F satisfies the $(\theta_K, g)_{\mathcal{P}}$ contractive condition with $g := I_n$. Since $\sum_{k=0}^{m} A_i^* \mathcal{G}(Q) A_i \succ 0$, we have

$$F(Q) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(Q) A_i \geq Q.$$

This mean that the condition (ii) of Theorem 4.5 is satisfied. Since $\mathcal{G}$ is continuous, then F is also continuous. Furthermore, F and $I_n := g$ are compatible. Hence the condition (iii). of Theorem 4.5 hold. Therefore all the hypothesis of Theorem 4.5 is satisfied. By Theorem 4.5, the solution of the matrix equation (7.2) exists. This completes the proof. □

Theorem 7.2. *Consider the matrix equations (7.2) . Let $F : H(n) \to H(n)$ be an order-preserving mapping and $Q \in P(n)$ be defined by (7.3). Suppose that there is a positive number μ and the following hold:*

(i) $\sum_{i=1}^{m} A_i A_i^* < \mu \cdot I_n$ *and* $\sum_{k=0}^{m} A_i^* \mathcal{G}(Q) A_i \succ 0$;

(ii) *for all $X, Y \in H(n)$ such that $X \preceq Y$,* $(\|\mathcal{G}(X) - \mathcal{G}(Y))\|_1)^\lambda \leq \frac{1}{\mu} \ln\left(1 + \frac{(\|X-Y\|_1)^\lambda}{2e^{(\|X-Y\|_1)^\lambda}}\right)$.

Then we have the following:

(1) The matrix equation (7.2) *has a solution.*

(2) In addition, for any $X_0 \in H(n)$ such that $X_0 \preceq Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_0) A_i$, the iteration $\{X_n\}$ defined by

$$X_n = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_{n-1}) A_i, \qquad (7.5)$$

converges to a solution of the matrix equation (7.2) in the sense of the trace norm.

Proof. Putting $wt-$distance $\mathcal{P} = D$, and let $D : H(n) \times H(n) \to [0, \infty)$ be defined as in (7.1). Consider be defined as Example 3.2(a), that is $\theta_K(\alpha, \beta) = \psi(s) - \phi(t)$ for all $s, t \in [0, \infty)$ with $\psi(t) = \ln(1 + \frac{t}{2e^t})$ and $\phi(t) = t$ for all $t \in [0, \infty)$. By applying the relation of various kinds of simulation (see [21]), we get $\theta_K \in \Theta_K^M$. According to the proof of Theorem 7.1, It follows all the hypothesis of Theorem 4.5 hold. Thus the conclusions of this theorem follow from Theorem 4.5. The proof is completed. □

Theorem 7.3. *Consider the matrix equations (7.2). Let $F : H(n) \to H(n)$ be an order-preserving mapping and $Q \in P(n)$ be defined by (7.3). Suppose that there is a positive number μ and the following statements hold:*

(i) $\sum_{i=1}^{m} A_i A_i^* < \mu \cdot I_n$ *and* $\sum_{k=0}^{m} A_i^* \mathcal{G}(Q) A_i \succ 0$;

(ii) *for all $X, Y \in H(n)$ such that $X \preceq Y$,*

$$(\|\mathcal{G}(X) - \mathcal{G}(Y))\|_1)^\lambda \leq \frac{k}{\mu} (\|X-Y\|_1)^\lambda$$

for some $k \in [0, 1)$. Then we have the following statements:

(1) The matrix equation (7.2) *has a solution.*

(2) Furthermore, for any $X_0 \in H(n)$ such that $X_0 \preceq Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_0) A_i$, the iteration $\{X_n\}$ defined by

$$X_n = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(X_{n-1}) A_i, \qquad (7.6)$$

converges to a solution of the matrix equation (7.2) *in the sense of the trace norm* $\|\cdot\|_{tr}$.

Proof. Consider $wt-$distance $\mathcal{P} = D$, and let $D : H(n) \times H(n) \to [0, \infty)$ be defined as in (7.1). Consider $\theta_K \in \Theta_K^M$ defined by

$\theta_K(t,s) = ks - t$, for all $t, s \in [0,\infty)$, where $k \in [0,1)$. In the same line of Theorem 7.3, we obtain so the conclusions of this theorem. □

8. Numerical Experiments

We now provide a few instances to support the results in the section 4 and section 7 with the numerical method to approximate a solution of the matrix equation (7.2).

Example 8.1. Let $\mathbb{X} \subset H(4)$ Hermitian matrices defined by

$$\mathbb{X} = \left\{ \begin{pmatrix} a & a & b & c \\ a & a & a & b \\ b & a & a & a \\ c & b & a & a \end{pmatrix} : \begin{array}{c} 0 < a \leq 1, \\ 0 \leq b \leq 1, \\ 0 \leq c \leq 1 \end{array} \right\},$$

and let

$$Q = \begin{pmatrix} 0.2 & 0.02 & 0.03 & 0.01 \\ 0.02 & 0.2 & 0.02 & 0.03 \\ 0.03 & 0.02 & 0.2 & 0.02 \\ 0.01 & 0.03 & 0.02 & 0.2 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.02 & -0.021 & 0.35 & 0.12 \\ 0.1 & 0.6 & 0 & 0.25 \\ 0.06 & 0.1 & 0.07 & 0 \\ 0.17 & 0.06 & 0.01 & 0.022 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.05 & 0.1 & 0.02 & -0.24 \\ 0.01 & 0.11 & 0.4 & 0 \\ 0.12 & 0.01 & 0.1 & 0.02 \\ 0.18 & 0.3 & -0.08 & 0.26 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.41 & 0.01 & 0.40 & -0.02 \\ 0.07 & 0.12 & 0.25 & 0.51 \\ 0.18 & -0.23 & 0.14 & 0.05 \\ 0.06 & 0.15 & 0.04 & 0.06 \end{pmatrix}.$$

Then $Q \in P(4) \subseteq \mathbb{X}$ and $A_i \in M(4)$ for each $i = 1, 2, 3$. Further,

$$\sum_{i=1}^{3} A_i A_i^* =$$

$$\begin{pmatrix} 1.3573 & 0.0050 & 0.4097 & 0.4223 \\ 0.0050 & 1.3655 & 0.2008 & 0.0765 \\ 0.4097 & 0.2008 & 0.4435 & 0.1253 \\ 0.4223 & 0.0765 & 0.1253 & 0.3464 \end{pmatrix},$$

and hence

$$\sum_{i=1}^{3} A_i A_i^* < 1.3573 \cdot I_4 < \sqrt{2} \cdot I_4,$$

and

$$\sum_{k=0}^{3} A_i^* \mathcal{G}(Q) A_i =$$

$$\begin{pmatrix} 0.0179 & 0.0056 & 0.0202 & 0.0118 \\ 0.0056 & 0.0441 & 0.0094 & 0.0197 \\ 0.0202 & 0.0094 & 0.0445 & 0.0209 \\ 0.0118 & 0.0197 & 0.0209 & 0.0291 \end{pmatrix} \succ 0.$$

where $\mathcal{G}(X) = \frac{1}{3\sqrt{2}} X$. Consider the matrix equation (7.2) with $\mathcal{G}(X) = \frac{1}{3\sqrt{2}} X$, that is,

$$\begin{aligned} X \;=\; & Q + A_1^* \left(\tfrac{1}{3\sqrt{2}} X \right) A_1 \\ & + A_2^* \left(\tfrac{1}{3\sqrt{2}} X \right) A_2 \\ & + A_3^* \left(\tfrac{1}{3\sqrt{2}} X \right) A_3. \end{aligned} \tag{8.1}$$

Consider wt−distance $\mathcal{P} = D$, and let $D : H(n) \times H(n) \to [0,\infty)$ be defined as in (7.1). Letting $\theta_K \in \Theta_K^M$ defined by $\theta_K(t,s) = \psi(s) - \phi(t)$ for all $s, t \in [0,\infty)$ with $\psi(t) = \frac{t}{3^\lambda}$ and $\phi(t) = t$ for all $t \in [0,\infty)$. Let $X, Y \in \mathbb{X}$ be such that $X \preceq Y$. Then we have

$$
\begin{aligned}
&(\|F(X)-F(Y)\|_1)^\lambda \\
&= (tr(F(X)-F(Y)))^\lambda \\
&= \Big[\sum_{i=1}^{m} tr(A_i^*(\mathcal{G}(X)-\mathcal{G}(Y))A_i\Big]^\lambda \\
&= \Big[\sum_{i=1}^{m} tr(A_iA_i^*(\mathcal{G}(X)-\mathcal{G}(Y))\Big]^\lambda \\
&= \Big[tr\Big(\Big(\sum_{i=1}^{m}(A_iA_i^*)\Big]^\lambda(\mathcal{G}(X)-\mathcal{G}(Y))\Big)\Big]^\lambda \\
&\leq \Big[\Big\|\sum_{i=1}^{m}(A_iA_i^*)\Big\|\Big\|\Big]^\lambda(\mathcal{G}(X)-\mathcal{G}(Y)\|_1)^\lambda \\
&< (\sqrt{2})^\lambda\Big(\frac{1}{3\sqrt{2}}\|X-Y\|_1\Big)^\lambda \\
&= \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{3}\|X-Y\|_1\right) \\
&\leq \frac{1}{3^\lambda}\cdot(\|X-Y\|_1)^\lambda,
\end{aligned}
$$

it follow that

$$
\begin{aligned}
&\theta_K(\mathcal{P}(F(X),F(y)),\mathcal{P}(x,y)) \\
&= \theta_K(D(F(X),F(y)),D(X,X)) \\
&= \theta_K(\|F(X)-F(Y)\|_1)^\lambda,(\|X-Y\|_1)^\lambda) \\
&= \psi(\|X-Y\|_1)^\lambda)-\phi(\|F(X)-F(Y)\|_1)^\lambda) \\
&= \frac{1}{3^\lambda}\cdot(\|X-Y\|_1)^\lambda-(\|F(X)-F(Y)\|_1)^\lambda \\
&\geq 0.
\end{aligned}
$$

Thus, F satisfies the $(\theta_K, g)_{\mathcal{P}}$ contractive condition with $g := I_n$. Similar to the proof of Theorem 7.3, notice that, all hypotheses of Theorem 4.5 are satisfied. Next, we approximate a solution of the equation 8.1 by considering the iteration $\{X_n\}$ defined by

$$X_n = Q+A_1^*X_{n-1}A_1+A_2^*X_{n-1}A_2+A_3^*X_{n-1}A_3, \tag{8.2}$$

where $X_0 = Q$, and the error $E_n := (\|X_n - X_{n-1}\|_1)^\lambda$, with $\lambda = 1.5$. Finally, a solution of the equation (5.1) can be approximated at iteration number of 9.

$$
X_* \approx X_9 = \begin{pmatrix} 0.2199 & 0.0288 & 0.0472 & 0.0194 \\ 0.0288 & 0.2336 & 0.0258 & 0.0457 \\ 0.0472 & 0.0258 & 0.2331 & 0.0321 \\ 0.0194 & 0.0457 & 0.0321 & 0.2263 \end{pmatrix},
$$

with $E_9 = 1.5865x10^{12}$ (see in Fig. 1).

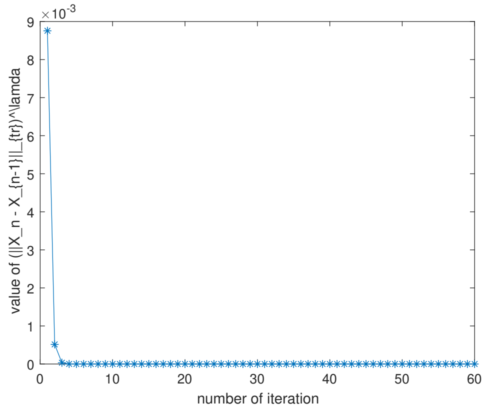

Fig. 1. The error of iteration process (8.2) for the Equation (7.2).

9. Conclusion

A conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

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