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Generalized Penon Involutive Weak Globular Higher Categories

Paratat Bejrakarbum*, Paolo Bertozzini

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathumthani, Thailand

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ABSTRACT

This research examines a generalization of the concept of an involutive weak globular ω-category through Penon's approach. We first introduce the notion of a globular cone, then equip it with compositions, identities, and self-duality. Next, a free reflexive self-dual globular-cone ω-magma and a free strict involutive globular-cone ω-category along with a specified contraction over a given globular cone are established. An algebra for the monad induced by the free-forgetful adjunction arising from the previously constructed structure naturally defines an involutive weak globular-cone ω-category. We finally provide crucial examples of involutive weak globular-cone ω-categories.

Keywords: Involutive Category; Higher Category; Monad

1. Introduction

Category theory was pioneered by S. Eilenberg and S. Mac Lane [1] in their work in algebraic topology. This branch of mathematics also finds applications in foundation of mathematics, computer science, and mathematical physics (see [2]).

Even though higher category theory was implicitly invented during [1] (since a natural transformation between functors can be seen as a 2-cell in a certain environment), the notion of strict n-categories was formalized in both cubical form [3] and globular form [4] by C. Ehresmann.

Involutions in category theory were primarily studied in [5] and recently in [6]. Higher involutions in higher category theory are studied in [7, 8].

Weak category theory was originally present in the definition of monoidal categories and later that of bicategories [9]. After that the renowned manuscript by A. Grothendieck [10] analyzed strict globular ω-categories and weak ω-groupoids along with their applications.

Among several algebraic definitions

***Corresponding author:** paratat.bejr@dome.tu.ac.th **doi:** 10.14456/scitechasia.2024.42

of weak ω-categories; for example, by M. Batanin [11], J. Penon [12], T. Leinster [13], and C. Kachour [14], we concentrate in this research paper on a generalization of the definition presented by J. Penon in the involutive version (see also [15]).

2. Preliminaries

In order for this paper to be self-contained, we primarily recall fundamental concepts involved in this research. For further discussion, we suggest [16, 17].

2.1 Basic Category Theory

We begin the section by revisiting the notions of a category, a functor beween categories, a natural transformation between functors, a free structure over object, a monad, and an algebra for a monad.

Definition 2.1. A **category** $\mathscr{C}$ comprises a collection $Ob\mathscr{C}$ of **objects**; for each $A, B \in Ob\mathscr{C}$ a set $\mathscr{C}(A, B)$ of **morphisms** from A to B; every object $A \in Ob\mathscr{C}$ admits an **identity morphism** $\iota_A \in \mathscr{C}(A, A)$; for any $A, B, C \in \mathscr{C}$ and $f \in \mathscr{C}(B, C), g \in \mathscr{C}(A, B)$ a **composition** $f \circ g \in \mathscr{C}(A, C)$ such that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever these compositions make sense, and also $\iota_B \circ f = f = f \circ \iota_A$ for all $f \in \mathscr{C}(A, B)$.

Example 2.2. There are categories **Set** of functions between sets, **Mon** of unital homomorphisms of monoids, and **Vect** of linear transformations between vector spaces.

A *(covariant) functor* can be seen as a relationship between two categories.

Definition 2.3. Let $(\mathscr{C}, \circ, \iota)$ and $(\mathscr{D}, \hat{\circ}, \hat{\iota})$ be categories. A **(covariant) functor** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F : Ob\mathscr{C} \to Ob\mathscr{D}$ and, for all $A, B \in Ob\mathscr{C}$ a function $F : \mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$ such that $F(\iota_A) = \hat{\iota}_{F(A)}$ for any object $A \in Ob\mathscr{C}$ and $F(f \circ g) = F(f) \hat{\circ} F(g)$ if they make sense.

Example 2.4. There is a forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ which associates a monoid $(M, *, e)$ with its underlying set M.

As before, a *natural transformation* is a relationship between two functors.

Definition 2.5. Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors between two given categories. A **natural transformation** $\alpha : F \Rightarrow G$ is a map $Ob\mathscr{C} \ni A \mapsto \alpha_A \in \mathscr{D}(F(A), G(A))$ with $\alpha_B \circ F(f) = G(f) \circ \alpha_A$ if $f \in \mathscr{C}(A, B)$.

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha_B]{} & G(B) \end{array}$$

Example 2.6. There is a natural transformation $ev : Id_{\mathbf{Vect}} \Rightarrow (-)^{**}$ associating $ev^V : V \to V^{**}$ sending $v \in V$ to the linear functional $ev^V_v : f \mapsto f(v)$ for any $f \in V^*$.

We may "freely" construct a certain object with more structure from a given object with less structure as follows.

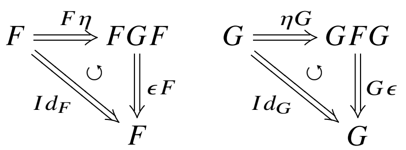
Definition 2.7. Let $U : \mathscr{C} \to \mathscr{D}$ be a forgetful functor. A **free structure** in $\mathscr{C}$ over $D \in Ob\mathscr{D}$ consists of an object $A \in Ob\mathscr{C}$ and a morphism $j \in \mathscr{D}(D, U(A))$ such that, for each $B \in Ob\mathscr{C}$ and $k \in \mathscr{D}(D, U(B))$, there exists a unique morphism $\psi \in \mathscr{C}(A, B)$ satisfying $k = U(\psi) \circ j$.

$$\begin{array}{ccc} U(A) & \xrightarrow{\exists ! U(\psi)} & U(B) \\ {\scriptstyle j}\uparrow & \circlearrowleft & \nearrow{\scriptstyle \forall k} \\ D & & \end{array}$$

Example 2.8. A free monoid over a (nonempty) set S is a set $M := \{()\} \cup \{(x_1, \ldots, x_n) \mid n \in \mathbb{N},\ x_1, \ldots, x_n \in S\}$ with concatenation and a function $j : S \to M$ defined by $j(x) := (x)$ for every $x \in S$.

A significant relationship between functors in the reverse direction is called an *adjunction* which will be crucial later on.

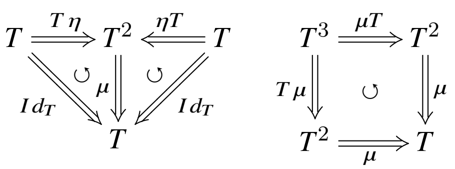
Definition 2.9. Suppose $\mathscr{C}$ and $\mathscr{D}$ are categories, and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors. The functor F is said to be **left adjoint** to the functor G, denoted by $F \dashv G$, if there are natural transformations $\eta : Id_{\mathscr{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow Id_{\mathscr{D}}$ such that $\epsilon F \circ F\eta = Id_F$ and $G\epsilon \circ \eta G = Id_G$.



Example 2.10. The free monoid functor F : **Set** $\to$ **Mon** is left adjoint to the forgetful functor U : **Mon** $\to$ **Set**.

To encapsulate a certain algebraic theory, it is sufficient to require a *monad*.

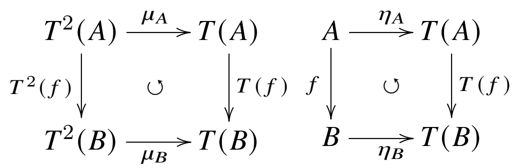
Definition 2.11. A **monad** (T, μ, η) on a category $\mathscr{C}$ comprises a functor $T : \mathscr{C} \to \mathscr{C}$ and natural transformations $\eta : Id_{\mathscr{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ satisfying the properties $\mu_X \circ T\eta_X = \iota_{T(X)} = \mu_X \circ \eta_{T(X)}$ and $\mu_X \circ T\mu_X = \mu_X \circ \mu_X T$.



Example 2.12. Each adjunction provides a monad. Particularly, there is a free-monoid monad $T = U \circ F$ on **Set** arising from the free monoid functor F : **Set** $\to$ **Mon** and the forgetful functor U : **Mon** $\to$ **Set**.

An *algebra* for a monad prescribes how the operations in a certain structure are expected to evaluate.

Definition 2.13. Let $\mathscr{C}$ be a category and (T, μ, η) a monad on $\mathscr{C}$. An **algebra** for the monad T consists of an object $A \in Ob\mathscr{C}$ and a morphism $\theta : T(A) \to A$ such that $\theta \circ \eta_A = \iota_A$ and $\theta \circ T\theta = \theta \circ \mu_A$.



Example 2.14. An algebra for the free monoid monad is given by a monoid.

2.2 Strict Involutive Higher Categories

Next, we recall the formalism of a strict involutive higher category, particularly a strict involutive globular ω-category. We first begin with the base structure of a higher category, called an *ω-globular set*. For further detail, we suggest [15, 18].

Definition 2.15. An **ω-globular set** is an ω-quiver $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ with the globularity condition: $s^{n-1} \circ s^n = s^{n-1} \circ t^n$ and $t^{n-1} \circ s^n = t^{n-1} \circ t^n$ for $n \in \mathbb{N}$. Elememts of Q^n are called **n-cells** of Q.

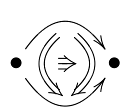


Fig. 1. A globular 3-cell

Definition 2.16. Let Q and R be two given ω-globular sets. A **morphism of ω-globular sets** $f : Q \to R$ is a family of functions $(f^n : Q^n \to R^n)_{n \in \mathbb{N}_0}$ with the properties $s^n_R \circ f^{n+1} = f^n \circ s^n_Q$ and $t^n_R \circ f^{n+1} = f^n \circ t^n_Q$ for every $n \in \mathbb{N}_0$.

$$\begin{array}{ccccccccc} Q^0 & \underset{t_Q^0}{\overset{s_Q^0}{\leftleftarrows}} & Q^1 & \underset{t_Q^1}{\overset{s_Q^1}{\leftleftarrows}} & \cdots & \underset{t_Q^{n-1}}{\overset{s_Q^{n-1}}{\leftleftarrows}} & Q^n & \underset{t_Q^n}{\overset{s_Q^n}{\leftleftarrows}} & \cdots \\ f^0\downarrow & \circlearrowleft & \downarrow f^1 & \circlearrowleft & \cdots & \circlearrowleft & \downarrow f^n & \circlearrowleft & \cdots \\ R^0 & \underset{t_R^0}{\overset{s_R^0}{\leftleftarrows}} & R^1 & \underset{t_R^1}{\overset{s_R^1}{\leftleftarrows}} & \cdots & \underset{t_R^{n-1}}{\overset{s_R^{n-1}}{\leftleftarrows}} & R^n & \underset{t_R^n}{\overset{s_R^n}{\leftleftarrows}} & \cdots \end{array}$$

It can be easily verifies that there exists a category **GSet** of all ω-globular sets.

We then equip an ω-globular set with identity functions at every level.

Definition 2.17. An ω-globular set Q is said to be **reflexive** if it is equipped with functions $Q^0 \xrightarrow{\iota^0} Q^1 \xrightarrow{\iota^1} \cdots \xrightarrow{\iota^{n-1}} Q^n \xrightarrow{\iota^n} \cdots$ with $s^n \circ \iota^n = Id_{Q^n} = t^n \circ \iota^n$ for all $n \in \mathbb{N}_0$.

A morphism of reflexive ω-globular sets is defined as follows.

Definition 2.18. Assume that (Q, ι_Q) and (R, ι_R) are reflexive ω-globular sets. A **morphism of reflexive ω-globular sets** $f : (Q, \iota_Q) \to (R, \iota_R)$ is a morphism of ω-globular sets $f : Q \to R$ along with the assertion $\iota_R^n \circ f^n = f^{n+1} \circ \iota_Q^n$ for all $n \in \mathbb{N}_0$.

We observe that there exists a category of reflexive ω-globular sets.

Compositions in an ω-globular set are defined as follows.

Definition 2.19. A **globular ω-magma** is an ω-globular set Q with compositions $\circ_p^n : Q^n \times_p Q^n \to Q^n$, where $Q^n \times_p Q^n := \{(x, y) \in Q^n \times Q^n \mid t^p t^{p+1} \cdots t^{n-1}(y) = s^p s^{p+1} \cdots s^{n-1}(x)\}$ for all $0 \leq p < n$ such that if $0 \leq p < n$ and $(x, y) \in Q^n \times_p Q^n$,

$$s^q(x \circ_p^n y) = \begin{cases} s^q(x) \circ_p^q s^q(y), & q > p; \\ s^q(x), & q \leq p, \end{cases}$$

$$t^q(x \circ_p^n y) = \begin{cases} t^q(x) \circ_p^q t^q(y), & q > p; \\ t^q(y), & q \leq p. \end{cases}$$

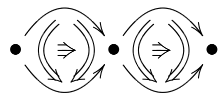


Fig. 2. The composition of 3-cells over 0-cells

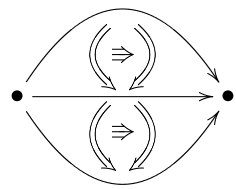


Fig. 3. The composition of 3-cells over 1-cells

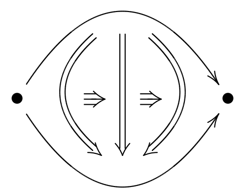


Fig. 4. The composition of 3-cells over 2-cells

Definition 2.20. Suppose that $(Q, \circ)$ and $(R, \hat{\circ})$ are globular ω-magmas. A **morphism of globular ω-magmas** $f : (Q, \circ) \to (R, \hat{\circ})$ is a morphism of ω-globular sets $f : Q \to R$ such that $f^n(x \circ_p^n y) = f^n(x) \hat{\circ}_p^n f^n(y)$ whenever the composition $x \circ_p^n y$ exists.

Obviously, there is a category of globular ω-magmas.

A *strict globular ω-category* is just an ω-globular set with compositions and identities along with certain axioms.

Definition 2.21. A reflexive globular ω-magma $\mathscr{C}$ is a **strict globular ω-category** if it satisfies the following properties:

1. if $0 \leq p < n$ and $x, y, z \in \mathscr{C}^n$ such that $(x, y), (y, z) \in \mathscr{C}^n \times_p \mathscr{C}^n$, then $(x \circ_p^n y) \circ_p^n z = x \circ_p^n (y \circ_p^n z)$,

2. if $0 \leq p < n$ and $x \in \mathscr{C}^n$, then $\iota^{n-1} \cdots \iota^p t^p \cdots t^{n-1}(x) \circ_p^n x = x = x \circ_p^n \iota^{n-1} \cdots \iota^p s^p \cdots s^{n-1}(x)$,
3. if $0 \leq p < n$ and $(x, y) \in \mathscr{C}^n \times_p \mathscr{C}^n$, then $\iota^n(x) \circ_p^{n+1} \iota^n(y) = \iota^n(x \circ_q^n y)$,
4. if $0 \leq q < p < n$ and $x, x', y, y' \in \mathscr{C}^n$ with $(y', y), (x', x) \in \mathscr{C}^n \times_p \mathscr{C}^n$ and $(y', x'), (y, x) \in \mathscr{C}^n \times_p \mathscr{C}^n$, then $(y' \circ_p^n y) \circ_q^n (x' \circ_p^n x) = (y' \circ_q^n x') \circ_p^n (y \circ_q^n x)$.

Definition 2.22. A **strict ω-functor** between strict globular ω-categories is simply a morphism between their underlying reflexive globular ω-magmas.

There exists, as usual, a category **Str-ω-Cat** of strict globular ω-categories.

We then equip a strict globular ω-category with a family of *involutions*.

Definition 2.23. A strict globular ω-category $\mathscr{C}$ is **involutive** if, for each $n \in \mathbb{N}$, there exists a function $*_q^n : \mathscr{C}^n \to \mathscr{C}^n$ for every $q < n$, such that, for each $x, y \in \mathscr{C}^n$,

1. if $x \circ_p^n y$ exists for some $p < n$, then $(x \circ_p^n y)^{*_q^n} = \begin{cases} x^{*_q^n} \circ_p^n y^{*_q^n}, & p \neq q; \\ y^{*_q^n} \circ_p^n x^{*_q^n}, & p = q, \end{cases}$
2. $\iota^n(x)^{*_q^{n+1}} = \iota^n(x^{*_q^n})$,
3. $(x^{*_q^n})^{*_q^n} = x$ for every $q \in \mathbb{N}_0$,
4. $(x^{*_p^n})^{*_q^n} = (x^{*_q^n})^{*_p^n}$ for all $p, q \in \mathbb{N}_0$.

Example 2.24. Strict globular ω-groupoids are strict involutive globular ω-categories such that every morphism is invertible.

Definition 2.25. Assume that $(\mathscr{C}, \circ, \iota, *)$ and $(\mathscr{D}, \hat{\circ}, \hat{\iota}, \hat{*})$ are strict involutive globular ω-categories. A **strict involutive ω-functor** $F : (\mathscr{C}, \circ, \iota, *) \to (\mathscr{D}, \hat{\circ}, \hat{\iota}, \hat{*})$ is a strict ω-functor $F : (\mathscr{C}, \circ, \iota) \to (\mathscr{D}, \hat{\circ}, \hat{\iota})$ on their underlying strict globular ω-categories with the property $F^n(x^{*_q^n}) = F^n(x)^{\hat{*}_q^n}$ for every $0 \leq q < n \in \mathbb{N}$ and $x \in \mathscr{C}^n$.

Proposition 2.26. *There exists a category of strict involutive globular ω-categories.*

2.3 Weak Higher Categories

The concept of weakness in (higher) category theory requires that axioms satisfied in strict situation need not hold true in the weak case, yet to keep track of those relations we need the notion of a *contraction*. Here is the formulation of weak higher categories through J. Penon's method [12].

Definition 2.27. Assume M to be a given reflexive globular ω-magma, C a strict globular ω-category, and $\pi : M \to C$ a morphism of reflexive globular ω-magmas. A **contraction on** π is a sequence of functions $([\cdot, \cdot]_n)_{n \in \mathbb{N}}$, where $[\cdot, \cdot]_n : \{(x, y) \in M^n \times M^n \mid s_M^{n-1}(x) = s_M^{n-1}(y),\ t_M^{n-1}(x) = t_M^{n-1}(y),\ \pi^n(x) = \pi^n(y)\} \to M^{n+1}$, for each $n \in \mathbb{N}$, holding true the assertions:

1. $s^n([x, y]_n) = x$ and $t^n([x, y]_n) = y$,
2. $[x, y]_n = \iota_M^n(x) = \iota_M^n(y)$ if $x = y$,
3. $\pi([x, y]_n) = \iota_C^n(\pi(x)) = \iota_C^n(\pi(y))$.

Proposition 2.28. *There is a category $\mathscr{Q}$ whose objects are pairs $(M \xrightarrow{\pi} C, [\cdot, \cdot])$ in which π is a morphism from a reflexive globular ω-magma M to a strict globular ω-category C and $[\cdot, \cdot]$ is a contraction on the morphism $\pi : M \to C$.*

Proposition 2.29. *[12] The forgetful functor $U : \mathscr{Q} \to$ **GSet** admits a left adjoint F.*

Definition 2.30. A **weak ω-category** is an algebra for the monad $T = U \circ F$.

Example 2.31. Every strict globular ω-category is a weak globular ω-category.

Example 2.32. The fundamental ω-groupoid over a given topological space is an example of a weak globular ω-category.

3. Main Results

The objective of this research is to generalize Penon's approach of weakening strict involutive globular ω-categories by adjoining an extra structure dominating the entire given ω-globular set and treating all the involved operations on such setting by demonstrating the existence of a free Penon cone-contraction on a certain morphism over a provided globular cone.

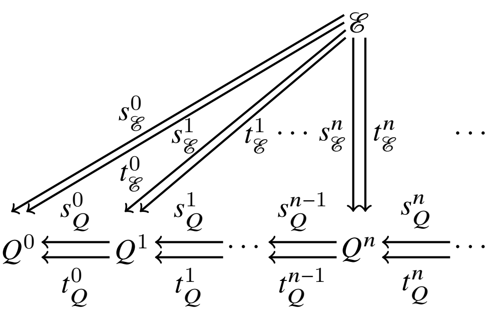
3.1 Globular Cones

We commence with introducing the notion of a generalized ω-globular set.

Definition 3.1. Assume first that

$$Q := Q^0 \underset{t_Q^0}{\overset{s_Q^0}{\leftleftarrows}} Q^1 \underset{t_Q^1}{\overset{s_Q^1}{\leftleftarrows}} \cdots \underset{t_Q^{n-1}}{\overset{s_Q^{n-1}}{\leftleftarrows}} Q^n \underset{t_Q^n}{\overset{s_Q^n}{\leftleftarrows}} \cdots$$

is an ω-globular set. A **globular cone** over Q is a collection $\mathscr{E}$ equipped with source and target functions $s_{\mathscr{E}}^n, t_{\mathscr{E}}^n : \mathscr{E} \to Q^n$ for all $n \in \mathbb{N}_0$ with $s_Q^n \circ s_{\mathscr{E}}^{n+1} = s_{\mathscr{E}}^n = s_Q^n \circ t_{\mathscr{E}}^{n+1}$ and $t_Q^n \circ t_{\mathscr{E}}^{n+1} = t_{\mathscr{E}}^n = t_Q^n \circ s_{\mathscr{E}}^{n+1}$ for any $n \in \mathbb{N}_0$.



Remark 3.2. *The collection $\mathscr{E}$ of a certain globular cone $\mathscr{E}$ over Q can be thought of as a family of infinite cells living on its underlying ω-globular set Q.*

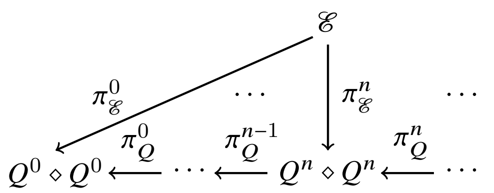
A morphism of globular cones is introduced in the obvious way.

Definition 3.3. Suppose that $\mathscr{E}$ and $\mathscr{F}$ are globular cones over ω-globular sets Q and R, respectively. A **morphism of globular cones** from $\mathscr{E}$ to $\mathscr{F}$ consists of a function $\Phi : \mathscr{E} \to \mathscr{F}$ and a morphism of ω-globular sets $\phi : Q \to R$ such that $\phi^n \circ s_{\mathscr{E}}^n = s_{\mathscr{F}}^n \circ \Phi$ and $\phi^n \circ t_{\mathscr{E}}^n = t_{\mathscr{F}}^n \circ \Phi$ hold for any $n \in \mathbb{N}_0$.

Proposition 3.4. *There exists a category **GCone** of all globular cones.*

Suppose $\mathscr{E}$ is a globular cone over an ω-globular set Q. Set $Q^0 \diamond Q^0 := Q^0 \times Q^0$ and $Q^n \diamond Q^n := \{(x, y) \in Q^n \times Q^n \mid s_Q^n(x) = s_Q^n(y),\ t_Q^n(x) = t_Q^n(y)\}$ for each $n \in \mathbb{N}$. The projection map $\pi_Q^n : Q^{n+1} \diamond Q^{n+1} \to Q^n \diamond Q^n$ is defined by $\pi_Q^n(x, y) := (s_Q^n(x), t_Q^n(y))$ for any $(x, y) \in Q^{n+1} \diamond Q^{n+1}$. Moreover, define $\pi_{\mathscr{E}}^n : \mathscr{E} \to Q^n \diamond Q^n$ by $\pi_{\mathscr{E}}^n(x) := (s_{\mathscr{E}}^n(x), t_{\mathscr{E}}^n(x))$ for each $x \in \mathscr{E}$. Observe that $\pi_{\mathscr{E}}^n = \pi_Q^n \circ \pi_{\mathscr{E}}^{n+1}$ holds for every $n \in \mathbb{N}_0$. As a consequence, what follows is well-defined.

Definition 3.5. Utilizing the terminology as above, the structure $\mathscr{E} \overset{\pi_{\mathscr{E}}}{\to} Q \diamond Q$ is called a **cone over globular products**.



Proposition 3.6. *[18] There is a category **GProd** of cones over globular products.*

Conversely, given a cone over globular products $\mathscr{E} \overset{\pi_{\mathscr{E}}}{\to} Q \diamond Q$, we can construct a globular cone $\mathscr{E}$ over the ω-globular set Q as follows. Note that, for each $n \in \mathbb{N}_0$, there are projections $p_1^n, p_2^n : Q^n \diamond Q^n \to Q^n$ which are defined by $p_1^n(x, y) := x$ and

$p_2^n(x,y) := y$ for every $(x,y) \in Q^n \diamond Q^n$. Hence, we can define $s_{\mathscr{E}}^n, t_{\mathscr{E}}^n : \mathscr{E} \to Q^n$ by $s_{\mathscr{E}}^n := p_1^n \circ \pi_{\mathscr{E}}^n$ and $t_{\mathscr{E}}^n := p_2^n \circ \pi_{\mathscr{E}}^n$. This implies that $\mathscr{E}$ becomes a globular cone over Q. This leads to the following theorem.

Theorem 3.7. *[18] The categories **GCone** and **GProd** are isomorphic; that is, there are functors F : **GCone** $\to$ **GProd** and G : **GProd** $\to$ **GCone** holding the assertions $G \circ F = Id_{\mathbf{GCone}}$ and $F \circ G = Id_{\mathbf{GProd}}$.*

3.2 Reflexive Self-dual Globular-cone Higher Magmas

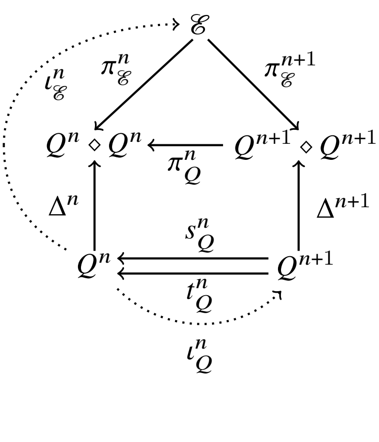
On globular cones we are equipping them with three operations: identities, compositions, and involutions to obtain *reflexive self-dual globular-cone ω-magmas*. These are base structures of strict involutive globular-cone ω-categories which will be carefully discussed later on.

Making use of Theorem 3.7, we can now introduce the following structure.

Definition 3.8. A globular cone $\mathscr{E}$ over an ω-globular set Q is **reflexive** if there exist two sequences $(\iota_Q^n : Q^n \to Q^{n+1})_{n\in\mathbb{N}_0}$ and $(\iota_{\mathscr{E}}^n : Q^n \to \mathscr{E})_{n\in\mathbb{N}_0}$ such that (Q, ι_Q) becomes a reflexive ω-globular set along with

$$\pi_{\mathscr{E}}^{n+k} \circ \iota_{\mathscr{E}}^n = \Delta^{n+k} \circ (\iota_Q^{n+k-1} \circ \cdots \circ \iota_Q^{n+1} \circ \iota_Q^n)$$

for each $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, where the diagonal map $\Delta^n : Q^n \to Q^n \diamond Q^n$ is given by $\Delta^n(x) := (x,x)$ for all $n \in \mathbb{N}_0$ and $x \in Q^n$.



Proposition 3.9. *There exists a category of reflexive globular cones.*

Next, we consider the notion of involutions on a globular cone as follows.

Definition 3.10. A globular cone $\mathscr{E}$ over an ω-globular set Q is said to be **self-dual** if there exist functions $*_\alpha : \mathscr{E} \to \mathscr{E}$ for each $\alpha \subseteq \mathbb{N}_0$ and $(*_\alpha^n : Q^n \to Q^n)_{\alpha\subseteq\mathbb{N}_0\ni n}$ such that $(Q, *)$ is a self-dual ω-globular set and

1. $s_{\mathscr{E}}^n(f^{*_\alpha}) = \begin{cases} s_{\mathscr{E}}^n(f), & n \notin \alpha; \\ t_{\mathscr{E}}^n(f), & n \in \alpha, \end{cases}$

2. $t_{\mathscr{E}}^n(f^{*_\alpha}) = \begin{cases} t_{\mathscr{E}}^n(f), & n \notin \alpha; \\ s_{\mathscr{E}}^n(f), & n \in \alpha, \end{cases}$

for any $f \in \mathscr{E}$ and $n \in \mathbb{N}_0$.

Proposition 3.11. *There exists a category of self-dual globular cones.*

The final operations of composition are established in the following definition.

Definition 3.12. A **globular-cone** ω-**magma** is a globular cone $\mathscr{E}$ over an ω-globular set Q equipped with compositions $(\circ_p : \mathscr{E} \times_p \mathscr{E} \to \mathscr{E})_{p\in\mathbb{N}_0}$, where $\mathscr{E} \times_p \mathscr{E} := \{(x,y) \in \mathscr{E} \times \mathscr{E} \mid s_{\mathscr{E}}^p(x) = t_{\mathscr{E}}^p(y)\}$, and $(\circ_p^n : Q^n \times_p Q^n \to Q^n)_{0\le p<n\in\mathbb{N}}$ such that $(Q, \circ)$ is a globular ω-magma and

$$s_{\mathscr{E}}^q(x \circ_p y) = \begin{cases} s_{\mathscr{E}}^q(x) \circ_p^q s_{\mathscr{E}}^q(y), & q > p; \\ s_{\mathscr{E}}^q(y), & q \le p, \end{cases}$$

$$t_{\mathscr{E}}^q(x \circ_p y) = \begin{cases} t_{\mathscr{E}}^q(x) \circ_p^q t_{\mathscr{E}}^q(y), & q > p; \\ t_{\mathscr{E}}^q(x), & q \le p, \end{cases}$$

for any $p, q \in \mathbb{N}_0$ and $(x,y) \in \mathscr{E} \times_p \mathscr{E}$.

Proposition 3.13. *There exists a category of globular-cone ω-magmas.*

Combining all these operations with a globular cone, we obtain the following:

Definition 3.14. A globular cone $\mathscr{E}$ is said to be a **reflexive self-dual globular-cone ω-magma** whenever it is equipped with identities $(\iota^n_{\mathscr{E}})_{n\in\mathbb{N}_0}$, self-duality $(*_\alpha)_{\alpha\subseteq\mathbb{N}_0}$, and compositions $(\circ_p)_{p\in\mathbb{N}_0}$. For our convenience, we may denote it by $(\mathscr{E}, \circ, \iota, *)$.

Definition 3.15. Suppose that $(\mathscr{E}, \circ, \iota, *)$ and $(\hat{\mathscr{E}}, \hat{\circ}, \hat{\iota}, \hat{*})$ are given reflexive self-dual globular-cone ω-magmas over ω-globular sets Q and $\hat{Q}$, respectively. A **morphism of reflexive self-dual globular-cone ω-magmas** from $(\mathscr{E}, \circ, \iota, *)$ to $(\hat{\mathscr{E}}, \hat{\circ}, \hat{\iota}, \hat{*})$ consists of a morphism of globular cones $\Phi : \mathscr{E} \to \hat{\mathscr{E}}$ and a morphism of reflexive self-dual globular ω-magmas $\phi : Q \to \hat{Q}$ such that the following assertions are satisfied:

1. $\Phi\big(\iota^n_{\mathscr{E}}(w)\big) = \iota^n_{\hat{\mathscr{E}}}\big(\phi^n(w)\big)$,
2. $\Phi(z^{*_\alpha}) = [\Phi(z)]^{\hat{*}_\alpha}$, and
3. $\Phi(x \circ_p y) = \Phi(x)\hat{\circ}_p\Phi(y)$

for every $n \in \mathbb{N}_0$, $w \in Q^n$, $\alpha \subseteq \mathbb{N}_0$, $p \in \mathbb{N}_0$, and $x, y, z \in \mathscr{E}$ with $(x, y) \in \mathscr{E} \times_p \mathscr{E}$.

Proposition 3.16. *There is a category of reflexive self-dual globular-cone ω-magmas.*

The next theorem assures that we can construct a reflexive self-dual globular-cone ω-magma from a given globular cone.

Theorem 3.17. *A free reflexive self-dual globular-cone ω-magma over a globular cone exists.*

Proof. Let $\mathscr{E}$ be a globular cone over an ω-globular set Q. We proceed by constructing a new ω-globular set M from Q. We denote $\Gamma := \{\varnothing\} \cup \{(\alpha_1, \dots, \alpha_k) \mid k \in \mathbb{N},\ \alpha_1, \dots, \alpha_k \subseteq \mathbb{N}_0\}$ and write $\triangle\alpha := \alpha_1 \triangle \cdots \triangle \alpha_k$ for any $\alpha = (\alpha_1, \dots, \alpha_k) \in \Gamma$.

Set $M^0 := \{(x, \alpha) \mid x \in Q^0,\ \alpha \in \Gamma\}$, let $\bar{M}^0 := \{(y, 0) \mid y \in M^0\}$, and also write $M^1[1] := \{(z, \alpha) \mid z \in Q^1 \cup \bar{M}^0,\ \alpha \in \Gamma\}$. We define $s^0[1], t^0[1] : M^1[1] \to M^0$ by

$$s^0[1]/t^0[1]\big((y,\alpha)\big) := \begin{cases} \big(s^0_Q/t^0_Q(y), \alpha\big), & y \in Q^1, 0 \notin \triangle\alpha; \\ \big(t^0_Q/s^0_Q(y), \alpha\big), & y \in Q^1, 0 \in \triangle\alpha; \\ (x, \alpha), & y = (x, 0) \in \bar{M}^0, \end{cases}$$

for every $(y, \alpha) \in M^1[1]$. Let $M^1[2] := \big\{((x, 0, y), \alpha) \,\big|\, x, y \in M^1[1],\ s^0[1](x) = t^0[1](y),\ \alpha \in \Gamma\big\}$. We define the source and target $s^0[2], t^0[2] : M^1[2] \to M^0$ by

$$s^0[2]/t^0[2]\big(((x,0,y),\alpha)\big) := \begin{cases} \big(s^0[1](y)/t^0[1](x), \alpha\big), & 0 \notin \triangle\alpha; \\ \big(t^0[1](y)/s^0[1](x), \alpha\big), & 0 \in \triangle\alpha, \end{cases}$$

for every $((x, 0, y), \alpha) \in M^1[2]$. Next, suppose, by induction, that $M^1[h]$ and $s^0[h], t^0[h] : M^1[h] \to M^0$ are well-defined for all $h = 1, \dots, k-1$. Set $M^1[k] := \{((x, 0, y), \alpha) \mid x \in M^1[i],\ y \in M^1[j],\ i + j = k,\ s^0[i](x) = t^0[j](y),\ \alpha \in \Gamma\}$. We define $s^0[k], t^0[k] : M^1[k] \to M^0$ by

$$s^0[k]/t^0[k]\big(((x,0,y),\alpha)\big) := \begin{cases} \big(s^0[j](y)/t^0[i](x), \alpha\big), & 0 \notin \triangle\alpha; \\ \big(t^0[j](y)/s^0[i](x), \alpha\big), & 0 \in \triangle\alpha, \end{cases}$$

for all $((x, 0, y), \alpha) \in M^1[k]$ with $i + j = k$. By induction, we can set $M^1 := \bigcup_{k\in\mathbb{N}} M^1[k]$, $s^0_M := \bigcup_{k\in\mathbb{N}} s^0[k]$, and $t^0_M := \bigcup_{k\in\mathbb{N}} t^0[k]$.

Assume that we have established M^{m+1} and $s^m_M, t^m_M : M^{m+1} \to M^m$ for every $m = 1, 2, \dots, n-1$. We let $M^{n+1}[1] := \{(x, \alpha) \mid x \in Q^{n+1} \cup \bar{M}^n,\ \alpha \in \Gamma\}$, where $\bar{M}^n := \{(x, n) \mid x \in M^n\}$. Then we can define $s^n[1], t^n[1] : M^{n+1}[1] \to M^n$ by

$$s^n[1]/t^n[1]\big((y,\alpha)\big) := \begin{cases} \big(s^n_Q/t^n_Q(y), \alpha\big), & y \in Q^{n+1}, n \notin \triangle\alpha; \\ \big(t^n_Q/s^n_Q(y), \alpha\big), & y \in Q^{n+1}, n \in \triangle\alpha; \\ (x, \alpha), & y = (x, n) \in \bar{M}^n, \end{cases}$$

for each $(y, \alpha) \in M^{n+1}[1]$. Set $M^{n+1}[2] := \bigcup_{p=0}^{n} \{((x, p, y), \alpha) \mid \alpha \in \Gamma,\ x, y \in M^{n+1}[1],\ s_Q^p \circ \cdots \circ s^n[1](x) = t_Q^p \circ \cdots \circ t^n[1](y)\}$. Define $s^n[2], t^n[2] : M^{n+1}[2] \to M^n$ by

$$s^n[2]/t^n[2]\big(((x, n, y), \alpha)\big) := \begin{cases} \big(s^n[1](y)/t^n[1](x), \alpha\big), & n \notin \triangle\alpha; \\ \big(t^n[1](y)/s^n[1](x), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

$$s^n[2]\big(((x, p, y), \alpha)\big) := \begin{cases} \big(s^n[1](x), p, s^n[1](y), \alpha\big), & n \notin \triangle\alpha; \\ \big(t^n[1](x), p, t^n[1](y), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

$$t^n[2]\big(((x, p, y), \alpha)\big) := \begin{cases} \big(t^n[1](x), p, t^n[1](y), \alpha\big), & n \notin \triangle\alpha; \\ \big(s^n[1](x), p, s^n[1](y), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

for $0 \leq p < n$. Then suppose $M^{n+1}[h]$ and $s^n[h], t^n[h] : M^{n+1}[h] \to M^n$ are all constructed for every $h = 1, 2, \ldots, k-1$. Let $M^{n+1}[k] := \bigcup_{p=0}^{n} \{((x, p, y), \alpha) \mid \alpha \in \Gamma,\ x \in M^{n+1}[i],\ y \in M^{n+1}[j],\ i + j = k,\ s^p[i] \cdots s^n[i](x) = t^p[j] \cdots t^n[j](y)\}$. Define $s^n[k], t^n[k] : M^{n+1}[k] \to M^n$ by

$$s^n[k]/t^n[k]\big(((x, n, y), \alpha)\big) := \begin{cases} \big(s^n[i](y)/t^n[j](x), \alpha\big), & n \notin \triangle\alpha; \\ \big(t^n[i](y)/s^n[j](x), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

$$s^n[k]\big(((x, p, y), \alpha)\big) := \begin{cases} \big(s^n[i](x), p, s^n[j](y), \alpha\big), & n \notin \triangle\alpha; \\ \big(t^n[i](x), p, t^n[j](y), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

$$t^n[k]\big(((x, p, y), \alpha)\big) := \begin{cases} \big(t^n[i](x), p, t^n[j](y), \alpha\big), & n \notin \triangle\alpha; \\ \big(s^n[i](x), p, s^n[j](y), \alpha\big), & n \in \triangle\alpha, \end{cases}$$

for every $0 \leq p < n$ and $((x, p, y), \alpha) \in M^{n+1}[k]$. Then set $M^{n+1} := \bigcup_{k \in \mathbb{N}} M^{n+1}[k]$, $s_M^n := \bigcup_{k \in \mathbb{N}} s^n[k]$, and $t_M^n := \bigcup_{k \in \mathbb{N}} t^n[k]$. So

$$M := M^0 \underset{t_M^0}{\overset{s_M^0}{\leftleftarrows}} M^1 \underset{t_M^1}{\overset{s_M^1}{\leftleftarrows}} \cdots \underset{t_M^{n-1}}{\overset{s_M^{n-1}}{\leftleftarrows}} M^n \underset{t_M^n}{\overset{s_M^n}{\leftleftarrows}} \cdots$$

becomes an ω-globular set. Furthermore, we define the following operations

1. $\iota_M^n : M^n \to M^{n+1}$ by $\iota_M^n(x) := (x, n)$ for each $n \in \mathbb{N}_0$ and $x \in M^n$,
2. $\circ_p^n : M^n \times_p M^n \to M^n$ by $x \circ_p^n y := (x, p, y)$ for any $0 \leq p < n \in \mathbb{N}$ and $(x, y) \in M^n \times_p M^n$, and
3. $*_\alpha^n : Q^n \to Q^n$ by $x \mapsto (x, \alpha)$ for every $n \in \mathbb{N}_0$ and $\alpha \subseteq \mathbb{N}_0$

so that the quadruple $(M, \circ, \iota, *)$ becomes a reflexive self-dual globular ω-magma.

Next, we construct a new globular cone $\mathscr{M}$ from $\mathscr{E}$ as follows. We first let $\hat{\mathscr{E}} := \{(w, \alpha) \mid w \in \mathscr{E},\ \alpha \in \Gamma\}$ and define $\hat{\mathscr{E}}^n := \{(n, (x, \alpha)) \mid x \in M^n,\ \alpha \in \Gamma\}$ for any $n \in \mathbb{N}_0$. Then we set $\mathscr{M}[1] := \hat{\mathscr{E}} \cup \bigcup_{n \in \mathbb{N}_0} \hat{\mathscr{E}}^n$. For each $q \in \mathbb{N}_0$, we can define the source $\hat{s}^q[1] : \mathscr{M}[1] \to M^q$ by $\hat{s}^q[1]\big((x, \alpha)\big) :=$

$$\begin{cases} \big(s_{\mathscr{E}}^q(x), \alpha\big), & x \in \mathscr{E}, q \notin \triangle\alpha; \\ \big(t_{\mathscr{E}}^q(x), \alpha\big), & x \in \mathscr{E}, q \in \triangle\alpha; \\ \big(s_M^q \cdots s_M^{k-1}(y), \alpha\big), (x, \alpha) = \big(k, (y, \alpha)\big), & y \in M^k, k \geq q \notin \triangle\alpha; \\ \big(t_M^q \cdots t_M^{k-1}(y), \alpha\big),\ (x, \alpha) = \big(k, (y, \alpha)\big), & y \in M^k, k \geq q \in \triangle\alpha; \\ ((((z, k), \beta), q-1), \gamma),\ z \in M^k, k < q, & (x, \alpha) = (q-1, ((z, k), \beta), \gamma), \end{cases}$$

while the target $\hat{t}^q[1] : \mathscr{M}[1] \to M^q$ can be defined in a similar fashion. Assume $\mathscr{M}[k]$ and, $\hat{s}^q[k], \hat{t}^q[k] : \mathscr{M}[k] \to M^q$, for all $q \in \mathbb{N}_0$, are constructed for any $k = 1, 2, \ldots, n-1$. Next, we let $\mathscr{M}[n] := \bigcup_{p=0}^{\infty} \{((x, p, y), \alpha) \mid \alpha \in \Gamma,\ x \in \mathscr{M}[i],\ y \in \mathscr{M}[j],\ i + j = n,\ \hat{s}^p[i](x) = \hat{t}^p[j](y)\}$ and the source function $\hat{s}^q[n] : \mathscr{M}[n] \to M^q$ is then defined by $\hat{s}^q[n]\big((x, p, y), \alpha\big) :=$

$$\begin{cases} \big(\hat{s}^q[i](x), p, \hat{s}^q[j](y), \alpha\big), & p < q \notin \triangle\alpha; \\ \big(\hat{t}^q[i](x), p, \hat{t}^q[j](y), \alpha\big), & p < q \in \triangle\alpha; \\ \big(\hat{s}^q[j](y), \alpha\big), & p \geq q \notin \triangle\alpha; \\ \big(\hat{t}^q[i](x), \alpha\big), & p \geq q \in \triangle\alpha, \end{cases}$$

for every $\big((x, p, y), \alpha\big) \in \mathscr{M}[n]$ such that

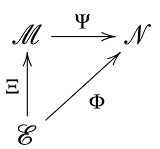
$x \in \mathscr{M}[i]$, $y \in \mathscr{M}[j]$ with $i + j = n$. The target $\hat{s}^q[n] : \mathscr{M}[n] \to M^q$ can be similarly constructed. In this way, we can define $\mathscr{M} := \bigcup_{n\in\mathbb{N}} \mathscr{M}[n]$, $s^q_{\mathscr{M}} := \bigcup_{n\in\mathbb{N}} \hat{s}^q[n]$, and $t^q_{\mathscr{M}} := \bigcup_{n\in\mathbb{N}} \hat{t}^q[n]$. This implies that $\mathscr{M}$ is a globular cone over M. Additionally, we define the operations

1. $\iota^n_{\mathscr{M}} : M^n \to \mathscr{M}$ by $x \mapsto (n, (x, \varnothing))$ for every $n \in \mathbb{N}_0$ and $x \in Q^n$,
2. $\circ_p : \mathscr{M} \times_p \mathscr{M} \to \mathscr{M}$ by $x \circ_p y := (x, p, y)$ for all $p \in \mathbb{N}_0$ and $(x, y) \in \mathscr{M} \times_p \mathscr{M}$, and
3. $*_\alpha : \mathscr{M} \to \mathscr{M}$ by $x \mapsto (x, \alpha)$ for each $x \in \mathscr{M}$ and $\alpha \in \Gamma$.

It follows that $(\mathscr{M}, \circ, \iota, *)$ is a reflexive self-dual globular-cone ω-magma over M.

Ultimately, we verify its universal factorization property. Observe that there exists a morphism $\Xi : \mathscr{E} \to \mathscr{M}$ given by $x \mapsto (x, \varnothing)$ for every $x \in \mathscr{E}$. Suppose that $(\mathscr{N}, \bar{\circ}, \bar{\iota}, \bar{*})$ is a reflexive self-dual globular-cone ω-magma and $\Phi : \mathscr{E} \to \mathscr{N}$ is a morphism of globular cones. There exists a unique morphism of reflexive self-dual globular-cone ω-magmas $\Psi : \mathscr{M} \to \mathscr{N}$ satisfying the property $\Phi = \Psi \circ \Xi$ which can be defined on simple cells as follows: for all $w \in \mathscr{E}$, $n \in \mathbb{N}_0$, $x \in Q^n$, $0 \le p < n$, $((y, p, z), \alpha) \in \mathscr{M} \times_p \mathscr{M}$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \Gamma$,

$$\begin{aligned}
\Psi(w, \varnothing) &:= \Phi(w)^{\bar{*}_\varnothing},\\
\Psi(x, n) &:= \bar{\iota}^n_N(\phi^n(x)),\\
\Psi(n, x) &:= \bar{\iota}^n_{\mathscr{N}}(\phi^n(x)),\\
\Psi(w, \alpha) &:= \Phi(w)^{\bar{*}_{\alpha_1}\cdots\bar{*}_{\alpha_m}},\\
\Psi(y, p, z) &:= \Phi(y)\bar{\circ}_p\Phi(z).
\end{aligned}$$



As a consequence, the pair $((\mathscr{M}, \circ, \iota, *), \Xi)$ is a free reflexive self-dual globular-cone ω-magma over the globular cone $\mathscr{E}$. □

3.3 Strict Involutive Globular-cone Higher Categories

The cone version of a strict involutive globular ω-category is introduced through a reflexive self-dual globular-cone ω-magma satisfying a number of analogous axioms. This section aims to prove the existence of a free strict involutive globular-cone ω-category over a provided globular cone.

Definition 3.18. A **strict involutive globular-cone ω-category** $(\mathscr{C}, \circ, \iota, *)$ is a reflexive self-dual globular-cone ω-magma $(\mathscr{C}, \circ, \iota, *)$ satisfying the properties:

1. $x \circ_p (y \circ_p z) = (x \circ_p y) \circ_p z$ when $(x, y), (y, z) \in \mathscr{C} \times_p \mathscr{C}$ and $p \in \mathbb{N}_0$,
2. $\iota^{n-1}_{\mathscr{C}} \cdots \iota^p_Q t^p_Q \cdots t^{n-1}_{\mathscr{C}}(x) \circ_p x = x$ and $x \circ_p \iota^{n-1}_{\mathscr{C}} \cdots \iota^p_Q s^p_Q \cdots s^{n-1}_{\mathscr{C}}(x) = x$ for every $0 \le p < n$ and $x \in \mathscr{C}$,
3. $\iota^n_{\mathscr{C}}(x) \circ_p \iota^n_{\mathscr{C}}(y) = \iota^n_{\mathscr{C}}(x \circ^n_p y)$ for each $0 \le p < n$ with $(x, y) \in Q^n \times_p Q^n$,
4. $(x \circ_p y) \circ_q (w \circ_p z) = (x \circ_q w) \circ_p (y \circ_q z)$ for any $(x, y), (w, z) \in \mathscr{C} \times_p \mathscr{C}$, $(x, w), (y, z) \in \mathscr{C} \times_q \mathscr{C}$, $0 \le q < p$,
5. $(x^{*_\alpha})^{*_\alpha} = x$ and $(x^{*_\alpha})^{*_\beta} = (x^{*_\beta})^{*_\alpha}$ for every $x \in \mathscr{C}$ and $\alpha, \beta \subseteq \mathbb{N}_0$,
6. $(x \circ_p y)^{*_\alpha} = \begin{cases} x^{*_\alpha} \circ_p y^{*_\alpha}, & p \notin \alpha; \\ y^{*_\alpha} \circ_p x^{*_\alpha}, & p \in \alpha, \end{cases}$ if $\alpha \subseteq \mathbb{N}_0 \ni p$ and $(x, y) \in \mathscr{C} \times_p \mathscr{C}$,
7. $\iota^n_{\mathscr{M}}(x^{*^n_\alpha}) = \iota^n_{\mathscr{M}}(x)^{*_\alpha}$ for each $n \in \mathbb{N}_0$, $\alpha \subseteq \mathbb{N}_0$, and $x \in Q^n$.

Proposition 3.19. *There exists a category of strict involutive globular-cone ω-categories.*

The upcoming theorem ensures that a strict involutive globular-cone ω-category can be "freely" constructed from a given globular cone. Before doing so, we introduce the notion of a *congruence* in a reflexive self-dual globular-cone ω-magma.

Definition 3.20. A **congruence** in a reflexive self-dual globular-cone ω-magma $(\mathscr{M}, \circ, \iota_{\mathscr{M}}, *)$ is a reflexive self-dual globular-cone ω-magma $(\mathscr{R}, \diamond, \iota_{\mathscr{R}}, \star)$ such that $\mathscr{R}^n \subseteq \mathscr{M}^n \times \mathscr{M}^n$ is an equivalence relation for each $n \in \mathbb{N}$ with the properties:

1. $(x, y) \in \mathscr{R}$ implies $\big(s^n_{\mathscr{R}}(x), s^n_{\mathscr{R}}(y)\big)$, $\big(t^n_{\mathscr{R}}(x), t^n_{\mathscr{R}}(y)\big) \in \mathscr{R}$ for any $n \in \mathbb{N}_0$,
2. $(x_1, y_1) \diamond_p (x_2, y_2) = \big(x_1 \circ_p x_2, y_1 \circ_p y_2\big)$ whenever these composites exist,
3. $(x, y)^{\star_\alpha} = (x^{*_\alpha}, y^{*_\alpha})$ for every $\alpha \in \Gamma$ and $(x, y) \in \mathscr{R}$, and
4. $\iota^n_{\mathscr{R}}(x, y) = (\iota^n_{\mathscr{M}}(x), \iota^n_{\mathscr{M}}(y))$ for each $n \in \mathbb{N}_0$ and $(x, y) \in R^n \times R^n$.

Next proposition asserts that a quotient of a reflexive self-dual globular-cone ω-magma by its congruence is a reflexive self-dual globular-cone ω-magma and the quotient map becomes a morphism of reflexive self-dual globular-cone ω-magmas.

Proposition 3.21. *Let $\mathscr{R}$ be a congruence in a reflexive self-dual globular-cone ω-magma $(\mathscr{M}, \circ, \iota_{\mathscr{M}}, *)$. Then the family of quotient sets $\mathscr{M}/\mathscr{R} := (\mathscr{M}^n/\mathscr{R}^n)_{n\in\mathbb{N}_0}$ becomes a reflexive self-dual globular-cone ω-magma $(\mathscr{M}/\mathscr{R}, \diamond, \iota_{\mathscr{M}/\mathscr{R}}, \star)$ in which*

$$\begin{aligned} s_{\mathscr{M}/\mathscr{R}}([x]) &:= [s_{\mathscr{M}}(x)] \\ t_{\mathscr{M}/\mathscr{R}}([x]) &:= [t_{\mathscr{M}}(x)] \\ [x] \diamond_p [y] &:= [x \circ_p y] \\ \iota_{\mathscr{M}/\mathscr{R}}([x]) &:= [\iota_{\mathscr{M}}(x)] \\ [x]^{\star_\alpha} &:= [x^{*_\alpha}] \end{aligned}$$

whenever the operations can be performed. Furthermore, the quotient map $\Pi : \mathscr{M} \to \mathscr{M}/\mathscr{R}$, defined by $x \mapsto [x]$ for all $x \in \mathscr{M}$, is a morphism of reflexive self-dual globular-cone ω-magmas.

Theorem 3.22. *A free strict involutive globular-cone ω-category over a globular cone exists.*

Proof. Assume that $(\mathscr{M}, \Xi)$ is the free reflexive self-dual globular-cone ω-magma over the given globular cone $\mathscr{E}$ established in Theorem 3.17. Construct the smallest congruence $\mathscr{R}$ in $\mathscr{M}$ generated by all the pairs of terms contained in the expressions in Definition 3.18 as follows. Let us denote by $\mathscr{X}$ the following union

$$\begin{aligned} &\big\{\big((x \circ_p y) \circ_p z, x \circ_p (y \circ_p z)\big) \,\big|\, (x, y), (y, z) \in \mathscr{M} \times_p \mathscr{M}, p \in \mathbb{N}_0\big\} \\ \cup\ &\big\{(\iota^{n-1}_{\mathscr{M}} \cdots \iota^p_Q t^p_Q \cdots t^{n-1}_{\mathscr{M}}(x) \circ_p x, x) \,\big|\, x \in \mathscr{M},\ 0 \le p < n \in \mathbb{N}\big\} \\ \cup\ &\big\{(x \circ_p \iota^{n-1}_{\mathscr{M}} \cdots \iota^p_Q s^p_Q \cdots s^{n-1}_{\mathscr{M}}(x), x) \,\big|\, x \in \mathscr{M},\ 0 \le p < n \in \mathbb{N}\big\} \\ \cup\ &\big\{\big(\iota^n_{\mathscr{M}}(x \circ^n_p y), \iota^n_{\mathscr{M}}(x) \circ_p \iota^n_{\mathscr{M}}(y)\big) \,\big|\, (x, y) \in Q^n \times_p Q^n,\ 0 \le p < n\big\} \\ \cup\ &\big\{\big((y' \circ_p y) \circ_q (x' \circ_p x), (y' \circ_q x') \circ_p (y \circ_q x)\big) \,\big|\, 0 \le q < p \\ &\quad (y', y), (x', x) \in \mathscr{M} \times_p \mathscr{M}, (y', x'), (y, x) \in \mathscr{M} \times_q \mathscr{M}\big\} \\ \cup\ &\big\{\big((x^{*_\alpha})^{*_\alpha}, x\big) \,\big|\, x \in \mathscr{M},\ \alpha \subseteq \mathbb{N}_0\big\} \\ \cup\ &\big\{\big((x^{*_\alpha})^{*_\beta}, (x^{*_\beta})^{*_\alpha}\big) \,\big|\, \alpha, \beta \subseteq \mathbb{N}_0, x \in \mathscr{M}\big\} \cup \big\{\big(\iota^n_{\mathscr{M}}(x^{*^n_\alpha}), (\iota^n_{\mathscr{M}}(x))^{*_\alpha}\big) \,\big|\, n \in \mathbb{N}_0,\ x \in Q^n,\ \alpha \subseteq \mathbb{N}_0\big\} \\ \cup\ &\big\{\big((x \circ_p y)^{*_\alpha}, x^{*_\alpha} \circ_p y^{*_\alpha}\big) \,\big|\, p \in \mathbb{N}_0, (x, y) \in \mathscr{M} \times_p \mathscr{M},\ p \notin \alpha \subseteq \mathbb{N}_0\big\} \\ \cup\ &\big\{\big((x \circ_p y)^{*_\alpha}, y^{*_\alpha} \circ_p x^{*_\alpha}\big) \,\big|\, p \in \mathbb{N}_0, (x, y) \in \mathscr{M} \times_p \mathscr{M},\ p \in \alpha \subseteq \mathbb{N}_0\big\} \\ \cup\ &\big\{\big(\iota^n_{\mathscr{M}}(x) \circ_p \iota^n_{\mathscr{M}}(y), \iota^n_{\mathscr{M}}(x \circ^n_p y)\big) \,\big|\, \end{aligned}$$

$$0 \leq p < n \in \mathbb{N},\ (x, y) \in Q^n \times_p Q^n\}$$

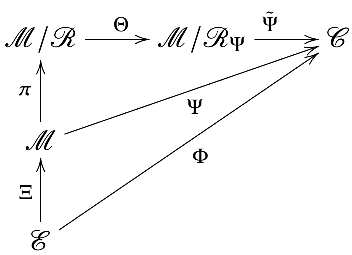
Then we take the quotient reflexive self-dual globular-cone ω-magma $\mathscr{M}/\mathscr{R}$. Since $\mathscr{X} \subseteq \mathscr{R}$, $\mathscr{M}/\mathscr{R}$ becomes a strict involutive globular-cone ω-category.

Finally, the universal factorization property is verified: suppose that $\mathscr{C}$ is a strict involutive globular-cone ω-category and $\Phi : \mathscr{E} \to \mathscr{C}$ is a morphism of globular cones. Since $(\mathscr{M}, \Xi)$ is the free reflexive self-dual globular-cone ω-magmas, there exitsts a unique morphism of reflexive self-dual globular-cone ω-magmas $\Psi : \mathscr{M} \to \mathscr{C}$ such that $\Phi = \Psi \circ \Xi$. Observe that

$$\mathscr{R}_\Psi := \{(x, y) \in \mathscr{M} \times \mathscr{M} \mid \Psi(x) = \Psi(y)\}$$

is naturally a congruence in $\mathscr{M}$. Holding true $\mathscr{X} \subseteq \mathscr{R}_\Psi$ implies that $\mathscr{M}/\mathscr{R}_\Psi$ is a strict involutive globular-cone ω-category. The map $\tilde{\Psi} : \mathscr{M}/\mathscr{R}_\Psi \to \mathscr{C}$, defined by $\tilde{\Psi}([x]_\Psi) := \Psi(x)$ for every $x \in \mathscr{M}$, is the unique morphism such that $\tilde{\Psi} \circ \pi_\Psi = \Psi$, where $\pi_\Psi : \mathscr{M} \to \mathscr{M}/\mathscr{R}_\Psi$ is defined by $\pi_\Psi(x) := [x]_\Psi$ for any $x \in \mathscr{M}$. Since $\mathscr{R}$ is the smallest congruence containing $\mathscr{X}$, $\mathscr{R} \subseteq \mathscr{R}_\Psi$ and so $\Theta : \mathscr{M}/\mathscr{R} \to \mathscr{M}/\mathscr{R}_\Psi$, defined by $\Theta([x]) := [x]_\Psi$ for all $x \in \mathscr{M}$, is the unique morphism such that $\pi_\Psi = \Theta \circ \pi$. Therefore, $\tilde{\Psi} \circ \Theta : \mathscr{M}/\mathscr{R} \to \mathscr{C}$ is the unique morphism of strict involutive globular-cone ω-categories satisfying the assertions:

$$\Phi = \Psi \circ \Xi = \tilde{\Psi} \circ \pi_\Psi \circ \Xi = \tilde{\Psi} \circ \Theta \circ \pi \circ \Xi.$$



As a consequence, $(\mathscr{M}/\mathscr{R}, \pi \circ \Xi)$ becomes a free strict involutive globular-cone ω-category over the globular cone $\mathscr{E}$. □

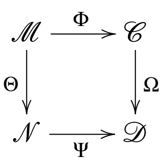
3.4 Involutive Weak Globular-cone Higher Categories

Having organized all the ingredients involved, we are now in a fit state to define involutive weak globular-cone ω-categories. We begin with the concept of a Penon contraction in our situation.

Definition 3.23. Let $\mathscr{M}$ be a reflexive self-dual globular-cone ω-magma, $\mathscr{C}$ a strict involutive globular-cone ω-category, and $\Phi : \mathscr{M} \to \mathscr{C}$ a morphism of reflexive self-dual globular-cone ω-magmas. Consider the set $Par_\Phi := \{(x, y) \in \mathscr{M} \times \mathscr{M} \mid \Phi(x) = \Phi(y),$ $s^n_{\mathscr{M}}(x) = s^n_{\mathscr{M}}(y), t^n_{\mathscr{M}}(x) = t^n_{\mathscr{M}}(y) \forall n \in \mathbb{N}_0\}$. We define a **Penon cone-contraction** on Φ as a map $[\cdot, \cdot] : Par_\Phi \to \mathscr{M}$ with the following assertions: for each $(x, y) \in Par_\Phi$,

1. $s^n_{\mathscr{M}}([x, y]) = s^n_{\mathscr{M}}(x)$ for all $n \in \mathbb{N}_0$,
2. $t^n_{\mathscr{M}}([x, y]) = t^n_{\mathscr{M}}(y)$ for any $n \in \mathbb{N}_0$,
3. $\Phi([x, y]) = \Phi(x) = \Phi(y)$.

Proposition 3.24. *There is a category $\mathscr{Q}^*$ whose objects are pairs $(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot, \cdot])$, where $\mathscr{M}$ is a reflexive self-dual globular-cone ω-magma, $\mathscr{C}$ is a strict involutive globular-cone ω-category, $\Phi : \mathscr{M} \to \mathscr{C}$ is a morphism of reflexive self-dual globular-cone ω-magmas, and $[\cdot, \cdot]$ is a Penon cone-contraction on Φ, whereas its morphism $(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot, \cdot]^1) \to (\mathscr{N} \xrightarrow{\Psi} \mathscr{D}, [\cdot, \cdot]^2)$ consists of a morphism of reflexive self-dual globular-cone ω-magmas $\Theta : \mathscr{M} \to \mathscr{N}$ and a morphism of strict involutive globular-cone ω-categories $\Omega : \mathscr{C} \to \mathscr{D}$ such that $\Omega \circ \Phi = \Psi \circ \Theta$ and $\Phi([x, y]^1) = [\Phi(x), \Phi(y)]^2$ for any $(x, y) \in Par_\Phi$.*



Provided a globular cone, we can construct from it a Penon cone-contraction on a morphism from a reflexive self-dual globular-cone ω-magma to a strict involutive globular-cone ω-category as follows.

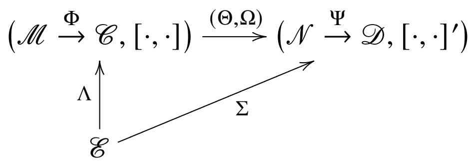
Theorem 3.25. *A free Penon cone-contraction on a morphism from a reflexive self-dual globular-cone ω-magma to a strict involutive globular-cone ω-category over a given globular cone exists.*

Proof. Let $\mathscr{E}$ be a globular cone over an ω-globular set Q. By Theorem 3.17, we have a free reflexive self-dual globular-cone ω-magma $\mathscr{M}$ over $\mathscr{E}$. Adopting the notation in Theorem 3.22, we construct a Penon cone-contraction $(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot,\cdot])$ as follows.

Set $\mathscr{M}[1] := \hat{\mathscr{E}} \cup \left(\bigcup_{n\in\mathbb{N}_0} \hat{\mathscr{E}}^n\right) \cup \mathscr{X}[1]$, where $\mathscr{X}[1] := \mathscr{X}$ is the domain of the map $[\cdot,\cdot]_1$. For each $n \in \mathbb{N}_0$, define $s^n_{\mathscr{M}}(x,y) := s^n_{\mathscr{M}}(x)$ and $t^n_{\mathscr{M}}(x,y) := t^n_{\mathscr{M}}(y)$ for every $(x,y) \in \mathscr{X}[1]$. The contraction map $[\cdot,\cdot]_1 : \mathscr{X}[1] \to \mathscr{M}[1]$ is defined as the inclusion $\mathscr{X}[1] \subseteq \mathscr{M}[1]$. Let $\mathscr{R}[1] \subseteq \mathscr{M}[1] \times \mathscr{M}[1]$ be the congruence generated by all algebraic axioms in $\mathscr{X}[1]$, set $\mathscr{C}[1] := \mathscr{M}[1]/\mathscr{R}[1]$, and $\Phi[1] : \mathscr{M}[1] \to \mathscr{C}[1]$ is the quotient map. Moreover, we define $\Phi[1](x,y) := \Phi(x) = \Phi(y)$ for any $(x,y) \in \mathscr{X}[1]$.

Next, we assume that the family $\mathscr{M}[k]$, the source and target functions $\hat{s}^m[k], \hat{t}^m[k] : \mathscr{M}[k] \to M^m$ and the contraction map $[\cdot,\cdot]_k : \mathscr{X}[k] \to \mathscr{M}[k]$ have been defined for any $k = 1, 2, \ldots, n-1$ and $m \in \mathbb{N}_0$. We define the family $\mathscr{M}[n] := \bigcup_{p=0}^{\infty} \left\{((x,p,y),\alpha) \;\middle|\; \alpha \in \Gamma,\ x \in \mathscr{M}[i],\ y \in \mathscr{M}[j],\ i+j=n,\ \hat{s}^p[i](x) = \hat{t}^p[j](y)\right\}$. The source and target functions can be defined as in Theorem 3.17. Let $\mathscr{X}[n] \subseteq \mathscr{M}[n] \times \mathscr{M}[n]$ denote the set of algebraic axioms between arrows in $\mathscr{M}[n]$. The contraction map $[\cdot,\cdot]_n : \mathscr{X}[n] \to \mathscr{M}[n]$ is, as before, the inclusion map. Let $\mathscr{R}[n] \subseteq \mathscr{M}[n] \times \mathscr{M}[n]$ be the congruence generated by all algebraic axioms in $\mathscr{X}[n]$, set $\mathscr{C}[n] := \mathscr{M}[n]/\mathscr{R}[n]$, and $\Phi[n] : \mathscr{M}[n] \to \mathscr{C}[n]$ is the quotient map. Then define $\mathscr{M} := \bigcup_{k=1}^{\infty} \mathscr{M}[k]$, $\mathscr{C} := \bigcup_{k=1}^{\infty} \mathscr{C}[k]$, $\Phi := \bigcup_{k=1}^{\infty} \Phi[k]$, and $[\cdot,\cdot] := \bigcup_{k=1}^{\infty} [\cdot,\cdot]_k$. Note that $\mathscr{C}$ becomes a strict involutive globular-cone ω-category.

Observe that Proposition 3.21 assures that the quotient map $\Phi : \mathscr{M} \to \mathscr{C}$ immediately becomes a morphism of reflexive self-dual globular-cone ω-magmas. From this construction, the pair $(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot,\cdot])$ immediately becomes a Penon cone-contraction. Notice also that there is a structural morphism $\Lambda : \mathscr{E} \to \left(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot,\cdot]\right)$ which is defined by $\Lambda(x) := (x, \varnothing)$ for all $x \in \mathscr{E}$. Ultimately, it remains to examine its universal factorization property.



Assume that $\left(\mathscr{N} \xrightarrow{\Psi} \mathscr{D}, [\cdot,\cdot]'\right)$ is a Penon cone-contraction and $\Sigma : \mathscr{E} \to \left(\mathscr{N} \xrightarrow{\Psi} \mathscr{D}, [\cdot,\cdot]'\right)$ is a morphism of globular cones. We must define a unique morphism of Penon cone-contractions $(\Theta,\Omega) : \left(\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot,\cdot]\right) \to \left(\mathscr{N} \xrightarrow{\Psi} \mathscr{D}, [\cdot,\cdot]'\right)$ satisfying the equation $\Sigma = \Theta \circ \Lambda$. Necessarily, we define by induction, for each $u, v, w \in \mathscr{E}$, $n \in \mathbb{N}_0$, $x \in Q^n$, $0 \leq p < n$, and $((y,p,z),\alpha) \in \mathscr{M} \times_p \mathscr{M}$ with $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Gamma$, by

$$\begin{aligned}
\Sigma(w,\varnothing) &:= \Theta(w)^{\bar{*}_\varnothing}, \\
\Sigma(x,n) &:= \vec{\iota}^{\,n}_N(\theta^n(x)), \\
\Sigma(n,x) &:= \vec{\iota}^{\,n}_{\mathscr{N}}(\theta^n(x)), \\
\Sigma(w,\alpha) &:= \Theta(w)^{\bar{*}_{\alpha_1}\cdots\bar{*}_{\alpha_m}},
\end{aligned}$$

$$\Sigma(y,p,z) := \Theta(y)\bar{\circ}_p\Theta(z),$$
$$\Sigma([u,v]) := [\Sigma(u),\Sigma(v)]'.$$

The construction thus defined immediately becomes the desired morphism. As a consequence, the pair $\left((\mathscr{M} \xrightarrow{\Phi} \mathscr{C}, [\cdot,\cdot]), \Lambda\right)$ is indeed a free Penon cone-contraction over the provided globular cone $\mathscr{E}$. □

As a consequence, the free functor $\mathscr{F}$: **GCone** $\to \mathscr{Q}^*$ constructed in Theorem 3.25 is left adjoint to the obvious forgetful functor $\mathscr{U} : \mathscr{Q}^* \to$ **GCone**. This gives rise to the monad $\mathscr{T} = \mathscr{U} \circ \mathscr{F}$ on **GCone**, making sense the desired definition.

Definition 3.26. An **involutive weak globular-cone ω-category** is an algebra for the monad $\mathscr{T} = \mathscr{U} \circ \mathscr{F}$ on **GCone**.

Example 3.27. Strict involutive globular-cone ω-categories are a particular case of an involutive weak globular-cone ω-category of which contraction map is the identity.

Example 3.28. Globular ω-spans together with compositions (pullbacks over common sets), identities (lifting sets), and involutions (permuting source and target sets) described in [7] are an example of involutive weak globular-cone ω-categories.

Example 3.29. Fundamental ω-groupoids $\Pi_\omega(X)$ over a topological space X discussed in [18] are a typical example of involutive weak globular-cone ω-categories.

4. Outlook

This paper focuses on the formulation of a generalization of involutive weak globular ω-categories together with the usual exchange property. We may further relax such a property to the non-commutative one to be convenient for the environment of categorical non-commutative geometry.

Regarding the shape of cells in weak higher categories, we can investigate the cone-version concepts of involutive weak cubical/hybrid higher categories using Penon's method as well as other perspectives described in the introduction.

We can further scrutinize the possible notions of involutive weak ω-algebroids and thus of weak ω-C*-categories in either globular, cubical, hybrid, or even globular-cone setting whenever we can equip the former with an appropriate uniform structure and completeness which will be a generalization of the notion of strict higher C*-categories investigated in [8].

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