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Convergence of Best Proximity Pair for Noncyclic Suzuki's Relatively Nonexpansive with Numerical Simulation

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ABSTRACT

The goal of this research is to examine a Thakur's iterative approach for a noncyclic relatively Suzuki's nonexpansive with a projection mapping in the famework of convex uniformly Banach space. Using this iteration as a base, we offer a few sufficient conditions and useful lemma to ensure the convergence of a best proximity pair for a mapping. We also provide a case study to illustrate the main results with numerical simulation for this algorithm.

Keywords: Best proximity pair ; Noncyclic mapping; Suzuki's relatively nonexpansive ; Uniformly convex Banach space.

1. Introduction

Since a large number of problems can be transformed to fixed point problems, fixed point theory is a useful technique for problem resolution many in real-world situations. This can be expressed in the form of equations $\Gamma x = x$, $\Gamma : \Omega \to \Omega$, where is topological space Ω. However, in situation E and K are nonempty disjoint subsets of a metric space Ω. There may not always be a solution to the aforementioned equation. It is crucial in this case to find a solution u

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that minimizes the error $d(u, \Gamma u)$. This is the concept behind, Fan proposed the best approximation theory [1]. Afterward, several authors have developed numerous expansions of Fan's Theorem, one can refer to [2–8]. In 2003, Banach's contraction concept was expanded to apply to cyclic mappings by Kirk et al., that is $\Upsilon : \Lambda \cup \Phi \to \Lambda \cup \Phi$ with the properties $\Upsilon(\Lambda) \subseteq \Phi$ and $\Upsilon(\Phi) \subseteq \Lambda$. They show that a mapping's fixed point solution implies that Λ must intersect Φ, which leads to the conclusion that Banach's principle of contractions is implied. In 2005, Eldred et al. [9] created a mapping $\Gamma : E \cup K \to E \cup K$ with properties $\Gamma(E) \subseteq E$ and $\Gamma(K) \subseteq K$. It is known as a noncyclic mapping, which investigates whether the following minimization problem existed to determine $s \in E$ and $h \in K$ satisfying

$$\min_{s \in E} d(s, \Gamma s), \quad \min_{h \in K} d(h, \Gamma h)$$
$$\text{and} \quad \min_{(s,h) \in E \times K} d(s, h), \qquad (1.1)$$

and a solution of Eq. (1.1) is an element $(s, h) \in E \times K$ with the property

$$s = \Gamma s, \;\; h = \Gamma h \;\text{ and }\; d(s, h) = dist(E, K).$$

which is called a *best proximal pair*. Following, other authors have investigated and developed the existence of a solution of Eq. (1.1), see for instance[10–12].

On the other hand, for the approximation of nonexpansive and generalized nonexpansive mappings, many iterative methods were introduced and studied Suzuki [13] presented the condition C in 2008, which is weaker than nonexpansiveness, later we know that as the Suzuki's type or Suzuki's generalized nonexpansive mapping. Thakur et al. [14] proposed the iteration approach in 2017. For an arbitrary chosen element u_1 in subset of Banach space Ω and sequence $\{u_n\}$ generated by :

$$\begin{cases} u_{n+1} = (1 - \sigma_n)\Gamma w_n + \sigma_n \Gamma v_n, \\ v_n = (1 - \varrho_n) w_n + \varrho_n \Gamma w_n, \\ w_n = (1 - \rho_n) u_n + \rho_n \Gamma u_n, \end{cases} \qquad (1.2)$$

where $\{\sigma_n\}$, $\{\varrho_n\}$ and $\{\rho_n\}$ are sequence in $(0, 1)$. In 2019,Suparatulatorn and Suantai [14], suggested a new technique for global optimum proximity minimization points in a classes proximally nonexpansive mapping. Recently, Gabeleh [11] used the concept of proximity pairs to examine iterations of Ishikawa for the best proximity pairs utilizing the concept of noncyclic relatively nonexpansive mapping.

Motivated by the preceding, this paper examines Thakur's iterative approach for a noncyclic Suzuki's relatively nonexpansive with a projection mapping in the famework of convex uniformly Banach space. First, we provide an algorithm with sufficient conditions and useful lemma. Second, we show the weak and strong convergence of a best proximity pair. Eventually, we provide an example todemonstrate the outcomes with numerical simulation for this algorithm. Also, we show that the conclusion of Gabeleh [12] is not applicable of our main result.

2. Preliminaries

In this part, we provide definitions and attributes for our important results. As nonempty subsets of Ω, that E, K. (Ω, d) to be a metric space. Define

$$d(E, K) = \inf\{d(s, w) : s \in E, w \in K\},$$

$$Fix(\Gamma) \;:=\; \{s \in E \times K : s = \Gamma s\},\; Fix_E(\Gamma) \;:=\; \{s \in E : s = \Gamma s\},$$

$$E_0 \;:=\; \{s \in E : d(s, w) = dist(E, K) \text{ for some } w \in K\},$$

$$K_0 := \{w \in K : d(s,w) = dist(E,K) \text{ for some } s \in E\},$$

$$Prox_{E\times K}(\Gamma) := \{(s,w) \in E \times K : s = \Gamma s, w = \Gamma w \text{ and } d(s,w) = dist(E,K)\}.$$

Note that, if $s \in E$ and $w \in K$ s.t. $d(s,w) = d(E,K)$, then w is proximal point of s (s is proximal point of w). Also, the proximal pair of (E,K) denote by (E_0,K_0).

Definition 2.1. Let $\Gamma : E \cup K \to E \cup K$. A *best proximity point* of Γ in E is defined as a point $s \in E$ where $d(s,\Gamma s) = d(E,K)$.

Definition 2.2. A pair (E,K) is said to be have P-property if and only if $d(s_1,w_1) = d(E,K)$ and $d(s_2,w_2) = d(E,K)$ implies $d(s_1,s_2) = d(w_1,w_2)$.

Lemma 2.3 ([10])**.** *The P-property exists in every bounded, nonempty, closed, and convex pair in a uniformly convex Banach space* Ω.

Definition 2.4 ([15])**.** If $\Gamma : E\cup K \to E\cup K$ is a cyclic mapping, $\exists \xi \in [0,1)$ s.t. said to be if

$$d(\Gamma r,\Gamma s) \le \xi d(r,s) + (1-\xi)d(E,K), \tag{2.1}$$

$\forall r \in E$ and $\forall s \in K$. Then Γ is considered a *cyclic contraction*.

If $d(E,K) = 0$, it is obvious that the mapping of cyclic contractions is reduced to mapping of contractions.

Definition 2.5. A noncyclic mapping $\Gamma : E \cup K \to E \cup K$ is called *relatively nonexpansive* if and only if

$$d(\Gamma r,\Gamma s) \le d(r,s), \tag{2.2}$$

$\forall r \in E$ and $\forall s \in K$.

Definition 2.6 ([16])**.** A noncyclic mapping $\Gamma : E\cup K \to E\cup K$ is called *quasi-noncyclic relatively nonexpansive* if and only if

(i) $Fix(\Gamma) \cap E_0$ and $Fix(\Gamma) \cap K_0$ are nonempty.

(ii) for each $e \in Fix(\Gamma) \cap K_0$, $g \in Fix(\Gamma) \cap E_0$.

$$\begin{cases} d(\Gamma u,e) \le d(u,e), & u \in E \\ d(g,\Gamma v) \le d(g,v), & v \in K. \end{cases} \tag{2.3}$$

Note that, if $E = K$ then the quasi-noncyclic relatively nonexpansive becomes the quasi-nonexpansive.

Definition 2.7. If a function that is strictly increasing exists, $\phi : (0,2] \to [0,1]$ in a Banach space Ω, it is said to be uniformly convex, such that, $\forall r,s,\varsigma \in \Omega, \sigma > 0$ and $l \in [0,2\sigma]$,

$$\left.\begin{array}{l} \|r-\varsigma\| \le \sigma \\ \|s-\varsigma\| \le \sigma \\ \|r-s\| \ge l \end{array}\right\} \implies \left\|\frac{r+s}{2} - \varsigma\right\| \le \left(1-\phi\left(\frac{l}{\sigma}\right)\right)\sigma.$$

Lemma 2.8. ([17, 18]) *Let E,K be a pair of a reflexive and strictly convex Banach space Ω that is nonempty closed, convex and proximal. Define $\mathrm{P} : E_0 \cup K_0 \to E_0 \cup K_0$ as*

$$\mathrm{P}(s) = \begin{cases} \mathrm{P}_{E_0}(s), & s \in K_0 \\ \mathrm{P}_{K_0}(s), & s \in E_0. \end{cases} \tag{2.4}$$

Then

(i) $\|s-\mathrm{P}s\| = d(E,K)$ *for any* $s \in E_0 \cup K_0$ *and* $\mathrm{P}(E_0) \subseteq K_0, \mathrm{P}(K_0) \subseteq E_0$.

(ii) P *is isometry, that is*

$$\|\mathrm{P}s-\mathrm{P}\bar{s}\| = \|s-\bar{s}\| \ \textit{for all}\ s,\bar{s} \in E_0,$$

and

$$\|\mathrm{P}w-\mathrm{P}\bar{w}\| = \|w-\bar{w}\| \ \textit{for all}\ w,\bar{w} \in K_0.$$

(iii) P *is affine.*

Remark 2.9. *Well know that, if* P *is projection on E, when* $E \subseteq \Omega$*,*

$$\|\mathrm{P}s - \mathrm{P}w\| \le \|s - w\|,$$

for all $s, w \in E$.

Lemma 2.10 ([13])**.** *Assume that a normed space* Ω*, has nonempty closed and convex subsets, E and K. Then* $\|u - \mathrm{P}_K u\| = d(E, K)\ \forall u \in E_0$.

Definition 2.11 ([11])**.** Assume that the pair (E, K) in a uniformly convex Banach space Ω, is nonempty, closed, convex, and proximal. Then (E, K) satisfying proximal Opail's condition, if every $\{u_n\}$ in E(respectively in K) with $\{u_n\} \rightharpoonup u \in E$ (respectively $\{u_n\} \rightharpoonup u \in K$), then

$$\limsup_{n\to\infty} \|u_n - \mathrm{P}u\| < \limsup_{n\to\infty} \|u_n - w\|,$$

$\forall w \neq \mathrm{P}u \in K$ (respectively for all $w \neq \mathrm{P}u \in E$.

Lemma 2.12 ([19])**.** *Let* $\{h_n\}$ *and* $\{w_n\}$ *be sequences in uniformly convex Banach space* Ω *s.t.* $\limsup_{n\to\infty} \|h_n\| \le \rho$, $\limsup_{n\to\infty} \|w_n\| \le \rho$ *and*

$$\lim_{n\to\infty} \|\varpi_n h_n + (1 - \varpi_n) w_n\| = \rho$$

for some $\rho \ge 0$ *where* $\{\varpi_n\} \in \mathbb{R}$, $0 < c \le \varpi_n < d < 1, \forall n \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} \|h_n - w_n\| = 0.$$

Definition 2.13 ([11])**.** A noncyclic mapping $\Gamma : E \cup K \to E \cup K$ is called *demi-closed at zero*, if $\{(u_n, \bar{u}_n)\}$ in $(E_0 \times K_0)$ with $\|u_n - \bar{u}_n\| = d(E, K)$ $n \in \mathbb{N}$, if $\{(u_n, \bar{u}_n)\} \rightharpoonup (u_\star, \bar{u}_*)$ and $\{(I - \Gamma u_n, I - \Gamma \bar{u}_n)\} \to (0, 0)$ then $(u_\star, \bar{u}_*) \in Prox_{E\times K}(\Gamma)$.

3. Main Results

This section presents the new idea of a noncyclic relatively Suzuki's nonexpansive and apply this idea to establish a best proximity pair's convergence in uniformly convex Banach space. First, sequence $(t_n, \bar{t}_n)$ is defined using the concept of Thakur's iteration scheme with the control sequences as follows :

Algorithm I

Initialization : Set $n = 1$ and choose $t_1 \in E_0$ and $\bar{t}_1 := \mathrm{P}(t_1) \in K_0$.

Iterative step. Compute $(t_{n+1}, \bar{t}_{n+1})$ by using

$$\begin{cases} t_{n+1} = (1 - \phi_n)\Gamma s_n + \phi_n \Gamma r_n, \\ r_n = (1 - \vartheta_n) s_n + \vartheta_n \Gamma s_n, \\ s_n = (1 - \zeta_n) t_n + \zeta_n \Gamma t_n, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \bar{t}_{n+1} = (1 - \phi_n)\bar{t}_n + \phi_n \Gamma \bar{r}_n, \\ \bar{r}_n = (1 - \vartheta_n) \bar{s}_n + \vartheta_n \Gamma \bar{s}_n, \\ \bar{s}_n = (1 - \zeta_n) \bar{t}_n + \zeta_n \Gamma \bar{t}_n, \end{cases} \tag{3.2}$$

where $\{\phi_n\}$, $\{\vartheta_n\} \subset [0, 1]$ and $\{\zeta_n\} \subset [\frac{1}{2}, 1]$ s.t.

$$\lim_{n\to\infty} (1 - 2\zeta_n) \neq 0. \tag{3.3}$$

After setting n to $n + 1$, the iterative step is performed.

Definition 3.1. Assume that E and K be nonempty subsets of a normed spaces Ω. $\Gamma : E \cup K \to E \cup K$ referred to as a *noncyclic relatively Suzuki's nonexpansive*, if Γ is noncyclic mapping and

$$\frac{1}{2} \| r - \Gamma r \| \le \| r - g \| \Rightarrow \|\Gamma r - \Gamma g\| \le \| r - g \|, \tag{3.4}$$

$\forall (r, g) \in E \times K$.

Proposition 3.2. *Every noncyclic relatively nonexpansive mapping is a noncyclic relatively Suzuki's nonexpansive.*

Proof. Clearly, from definition of noncyclic relatively nonexpansive and noncyclic relatively Suzuki's nonexpansive. □

Proposition 3.3. *Assume that a mapping $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively Suzuki's nonexpansive has a best proximity pair. Then Γ is a quasi-noncyclic relatively nonexpansive mapping.*

Proof. Let (s, r) best proximity pair of Γ. Then $s = \Gamma s$ and $r = \Gamma r$. Since,

$$\frac{1}{2}\|r - \Gamma r\| = 0 \leq \|q - r\| \quad \text{for all } q \in E,$$

we have

$$\|r - \Gamma q\| = \|\Gamma r - \Gamma q\| \leq \|r - q\|.$$

Similarly, $\|s - \Gamma z\| = \|\Gamma s - \Gamma z\| \leq \|s - z\|$ for all $z \in K$. Then Γ is a quasi-noncyclic relatively nonexpansive mapping. □

Lemma 3.4. *Let $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively Suzuki's nonexpansive mapping. be a mapping on a subset of a Banach space. Then*

$$\|s - \Gamma r\| \leq 3\|s - \Gamma s\| + \|s - r\|,$$

for all $(s, r) \in E \times K$.

Proof. By applying the proof of Lemma 7 in [20], we obtain the result of this Lemma. □

Lemma 3.5. *Assume that (E, K) be a closed, nonempty convex pair of uniformly convex Banach space Ω. and let $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively Suzuki's nonexpansive mapping. Let $\{t_n\}$, $\{\bar{t}_n\}$ be generated by* Algorithm I.

(i) $\lim_{n\to\infty} \|t_n - \theta\|$ *and* $\lim_{n\to\infty} \|\bar{t}_n - q\|$ *exists for any $\theta \in Fix(\Gamma) \cap K_0$, $q \in Fix(\Gamma) \cap E_0$,*

(ii) sequence $\{t_n\}$ and $\{\bar{t}_n\}$ are bounded.

Proof. Note that, since every noncyclic contraction is relatively nonexpansive, and every noncyclic relatively nonexpansive is a noncyclic relatively Suzuki's nonexpansive mapping, and uniformly convex is strictly convex, then by Theorem 3.5 in [21], we have Γ has best proximity pair and (E_0, K_0) is nonempty closed and convex pair of Ω. By *Algorithm I* with any $\theta \in Fix(\Gamma) \cap K_0$, we have

$$\begin{aligned}
\|t_{n+1} - \theta\| &= \|(1-\phi_n)\Gamma s_n + \phi_n \Gamma r_n - \theta\| \\
&= \|\phi_n(\Gamma r_n - \theta) \\
&\qquad + (1-\phi_n)(\Gamma s_n - \theta)\| \\
&\leq \phi_n\|\Gamma r_n - \theta\| + (1-\phi_n)\|\Gamma s_n - \theta\| \\
&\leq \phi_n\|r_n - \theta\| + (1-\phi_n)\|s_n - \theta\| \\
&\leq \phi_n\|(1-\vartheta_n)s_n + \vartheta_n\Gamma s_n - \theta\| \\
&\quad + (1-\phi_n)\|(1-\zeta_n)t_n \\
&\qquad + \zeta_n\Gamma t_n - \theta\| \\
&= \phi_n\|\vartheta_n(\Gamma s_n - \theta) \\
&\qquad + (1-\vartheta_n)(s_n - \theta)\| \\
&\quad + (1-\phi_n)\|\zeta_n(\Gamma t_n - \theta) \\
&\qquad + (1-\zeta_n)(t_n - \theta\| \\
&\leq \phi_n\big(\vartheta_n\|\Gamma s_n - \theta\| \\
&\qquad + (1-\vartheta_n)\|s_n - \theta)\|\big) \\
&\quad + (1-\phi_n)\big(\zeta_n\|\Gamma t_n - \theta\| \\
&\qquad + (1-\zeta_n)\|t_n - \theta\|\big) \\
&\leq \phi_n\vartheta_n\|s_n - \theta\| \\
&\quad + \phi_n(1-\vartheta_n)\|s_n - \theta)\| \\
&\quad + (1-\phi_n)\zeta_n\|t_n - \theta\| \\
&\quad + (1-\phi_n)(1-\zeta_n)\|t_n - \theta\| \\
&= \phi_n\|s_n - \theta\| \\
&\quad + (1-\phi_n)\zeta_n\|t_n - \theta\| \\
&\quad + (1-\phi_n)(1-\zeta_n)\|t_n - \theta\| \\
&= \phi_n\|(1-\zeta_n)t_n + \zeta_n\Gamma t_n - \theta\| \\
&\quad + (1-\phi_n)\zeta_n\|t_n - \theta\| \\
&\quad + (1-\phi_n)(1-\zeta_n)\|t_n - \theta\| \\
&= \phi_n\|\zeta_n(\Gamma t_n - \theta) \\
&\quad + (1-\zeta_n)(t_n - \theta)\| \\
&\quad + (1-\phi_n)\zeta_n\|t_n - \theta\| \\
&\quad + (1-\phi_n)(1-\zeta_n)\|t_n - \theta\|
\end{aligned}$$

$$\begin{aligned}
&\leq \phi_n\zeta_n\|t_n-\theta\| \\
&\quad +\phi_n(1-\zeta_n)\|t_n-\theta\| \\
&\quad +(1-\phi_n)\zeta_n\|t_n-\theta\| \\
&\quad +(1-\phi_n)(1-\zeta_n)\|t_n-\theta\| \\
&\leq \phi_n\|t_n-\theta\| \\
&\quad +(1-\phi_n)\zeta_n\|t_n-\theta\| \\
&\quad +(1-\phi_n)(1-\zeta_n)\|t_n-\theta\| \\
&= (\phi_n-\zeta_n-\phi_n\zeta_n+1 \\
&\quad -\zeta_n-\phi_n+\phi_n\zeta_n)\|t_n-\theta\| \\
&\leq \|t_n-\theta\|.
\end{aligned}$$

This means that $\{\|t_n-\theta\|\}$ is non-increasing and bounded. Hence $\lim_{n\to\infty}\|t_n-\theta\|$ exists. Since $\{\|t_n-\theta\|\}$ bounded, $M>0$ exists such that $\|t_n-\theta\|\leq M$ for all $n\in\mathbb{N}$. Further

$$\begin{aligned}
\|t_n\| = \|t_n-\theta+\theta\| &\leq \|t_n-\theta\|+\|\theta\| \\
&\leq M+\|\theta\|,
\end{aligned}$$

for all $n\in\mathbb{N}$. Then $\{t_n\}$ is also bounded. Similarly, we can show that $\lim_{n\to\infty}\|\bar{t}_n-\theta\|$ exists and $\{\bar{t}_n\}$ is bounded.

□

Lemma 3.6. *Let (E,K) be a nonempty closed and convex pair of uniformly convex Banach space Ω such that E or K bounded, and let $\Gamma: E\cup K\to E\cup K$ be a noncyclic relatively Suzuki's nonexpansive mapping. Let $\{t_n\}$, $\{\bar{t}_n\}$, $\{r_n\}$, $\{\bar{r}_n\}$, $\{s_n\}$ and $\{\bar{s}_n\}$ be created by* Algorithm I. *Then*

$$\lim_{n\to\infty}\|s_n-\Gamma s_n\|=0,\quad \lim_{n\to\infty}\|t_n-\Gamma t_n\|=0$$

and

$$\lim_{n\to\infty}\|\bar{s}_n-\Gamma\bar{s}_n\|=0,\quad \lim_{n\to\infty}\|\bar{t}_n-\Gamma\bar{t}_n\|=0.$$

Further,

$$\lim_{n\to\infty}\|\Gamma r_n-\Gamma s_n\|=0,\quad \lim_{n\to\infty}\|\Gamma\bar{r}_n-\Gamma\bar{s}_n\|=0.$$

Proof. Let $(\bar{\theta},\theta)\in Prox_{E\times K}(\Gamma)$,then $\theta=\Gamma\theta$, $\bar{\theta}=\Gamma\bar{\theta}$ and $\|\theta-\bar{\theta}\|=d(E,K)$. By Lemma 3.5, we have $\lim_{n\to\infty}\|t_n-\theta\|$ exists. Suppose that

$$\lim_{n\to\infty}\|t_n-\theta\|=k. \tag{3.5}$$

Using Proposition 3.3, we have

$$\begin{aligned}
\|\Gamma s_n-\theta\| &\leq \|s_n-\theta\| \\
&\leq \|(1-\zeta_n)t_n+\zeta_n\Gamma t_n-\theta\| \\
&= \|\zeta_n(\Gamma t_n-\theta)+(1-\zeta_n)(t_n-\theta)\| \\
&\leq \zeta_n\|t_n-\theta\|+(1-\zeta_n)\|t_n-\theta\| \\
&= \|t_n-\theta\|,
\end{aligned}$$

and

$$\begin{aligned}
\|\Gamma r_n-\theta\| &\leq \|r_n-\theta\| \\
&\leq \|(1-\vartheta_n)s_n+\vartheta_n\Gamma s_n-\theta\| \\
&= \|\vartheta_n(\Gamma s_n-\theta) \\
&\quad +(1-\vartheta_n)(s_n-\theta)) \\
&\leq \vartheta_n\|s_n-\theta\| \\
&\quad +(1-\vartheta_n)\|s_n-\theta\| \\
&= \|s_n-\theta\| \\
&\leq \|t_n-\theta\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\limsup_{n\to\infty}\|\Gamma s_n-\theta\| &\leq \limsup_{n\to\infty}\|s_n-\theta\| \\
&\leq \limsup_{n\to\infty}\|t_n-\theta\| \\
&= k,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\limsup_{n\to\infty}\|\Gamma r_n-\theta\| &\leq \limsup_{n\to\infty}\|r_n-\theta\| \\
&\leq \limsup_{n\to\infty}\|t_n-\theta\| \\
&= k.
\end{aligned} \tag{3.7}$$

Furthermore,
$$\begin{aligned}
&\lim_{n\to\infty}\|\phi_n(\Gamma r_n-\theta)+(1-\phi_n)(\Gamma s_n-\theta)\| \\
&\quad = \lim_{n\to\infty}\|(1-\phi_n)\Gamma s_n+\phi_n\Gamma r_n-\theta\| \\
&\quad = \lim_{n\to\infty}\|t_{n+1}-\theta\|
\end{aligned}$$

$$= \quad k.$$

By Lemma 2.12, we have

$$\lim_{n\to\infty} \|\Gamma r_n - \Gamma s_n\| = 0. \qquad (3.8)$$

Moreover, by Eq. (3.1), we have

$$\lim_{n\to\infty} \|t_{n+1} - \Gamma s_n\| = \lim_{n\to\infty} \phi_n \|\Gamma r_n - \Gamma s_n\| = 0.$$

Next, we will show that $\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0$. Since

$$\begin{aligned} \|t_{n+1} - \theta\| &= \|(1-\phi_n)\Gamma s_n + \phi_n \Gamma r_n - \theta\| \\ &\le \phi_n \|\Gamma r_n - \Gamma s_n\| + \|\Gamma s_n - \theta\|. \end{aligned}$$

It follows that Eqs. (3.5) and (3.8) combine to give

$$k \le \liminf_{n\to\infty} \|\Gamma s_n - \theta\|. \qquad (3.9)$$

Hence, by Eqs. (3.6) and (3.9)

$$\lim_{n\to\infty} \|\Gamma s_n - \theta\| = k. \qquad (3.10)$$

Further,

$$\begin{aligned} \|\Gamma s_n - \theta\| &\le \|\Gamma s_n - \Gamma r_n\| + \|\Gamma r_n - \theta\| \\ &\le \|\Gamma s_n - \Gamma r_n\| \qquad (3.11) \\ &+ \|r_n - \theta\|. \end{aligned}$$

Then, by Eqs. (3.10)-(3.11)

$$k \le \liminf_{n\to\infty} \|r_n - \theta\|. \qquad (3.12)$$

Therefore, by Eqs. (3.1), (3.5), and (3.12) combine to give

$$\begin{aligned} &\lim_{n\to\infty} \|(1-\vartheta_n)(s_n - \theta) + \vartheta_n(\Gamma s_n - \theta)\| \\ &= \lim_{n\to\infty} \|(1-\vartheta_n)s_n + \vartheta_n \Gamma s_n - \theta\| \\ &= \lim_{n\to\infty} \|r_n - \theta\| \\ &= k. \qquad (3.13) \end{aligned}$$

Again, by Lemma 2.12, give us

$$\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0. \qquad (3.14)$$

On the other hand, by Eq. (3.1), we have $s_n = (1-\zeta_n)t_n + \zeta_n \Gamma t_n$, then $s_n - t_n = \zeta_n(\Gamma t_n - t_n)$, $\forall n \in \mathbb{N}$. Since, $\zeta_n \in [1/2, 1]$, then

$$\begin{aligned} \frac{1}{2}\|t_n - \Gamma t_n\| &\le \zeta_n \|t_n - \Gamma t_n\| \\ &= \|t_n - s_n\|, \quad \forall n \in \mathbb{N}. \end{aligned}$$

By Eq. (3.4), we have

$$\|\Gamma t_n - \Gamma s_n\| \le \|t_n - s_n,\| \qquad \forall n \in \mathbb{N}. \qquad (3.15)$$

Since

$$\begin{aligned} \|t_n - \Gamma t_n\| &\le \|t_n - s_n\| + \|s_n - \Gamma s_n\| \\ &\quad + \|\Gamma s_n - \Gamma t_n\| \\ &\le \|t_n - s_n\| + \|s_n - \Gamma s_n\| \\ &\quad + \|s_n - t_n\| \\ &= 2\|s_n - t_n\| + \|s_n - \Gamma s_n\| \\ &= 2\|(1-\zeta_n)t_n + \zeta_n \Gamma t_n - t_n\| \\ &\quad + \|s_n - \Gamma s_n\| \\ &= 2\zeta_n \|\Gamma t_n - t_n\| \\ &\quad + \|s_n - \Gamma s_n\|. \end{aligned}$$

Then, we have

$$\begin{aligned} \|t_n - \Gamma t_n\| &\le \frac{1}{1-2\zeta_n}\|s_n - \Gamma s_n\| \\ &\le \frac{1}{1-2b}\|s_n - \Gamma s_n\|. \end{aligned}$$

Taking $n \to \infty$, by using Eqs. (3.3) and (3.14), we have

$$\lim_{n\to\infty} \|t_n - \Gamma t_n\| = 0.$$

In the same way, we may demonstrate that

$$\lim_{n\to\infty} \|\bar{s}_n - \Gamma \bar{s}_n\| = 0,$$

and

$$\lim_{n\to\infty} \|\bar{t}_n - \Gamma \bar{t}_n\| = 0.$$

□

Theorem 3.7. *Let (E, K) be a nonempty, bounded, closed and convex pair of uniformly convex Banach space Ω. Let $\Gamma : E \cup K \to E \cup K$ is a noncyclic Suzuki's relatively nonexpansive and $\mathrm{P} : E_0 \cup K_0 \to E_0 \cup K_0$ is a projection mapping defined as Eq. (2.4). Assume that $\{t_n\}$ and $\{\bar{t}_n\}$ are a sequence generated by* Algorithm I. *Then there exist $(t_\star, \bar{t}_*) \in E_0 \times K_0$ such that*

(i) $T\mathrm{P}(t_\star) = \mathrm{P}(t_\star)$ *and* $\Gamma\mathrm{P}(t_*) = \mathrm{P}(t_*)$,

(ii) $t_\star = \Gamma t_\star$ *and* $\bar{t}_* = \Gamma\bar{t}_*$,

(iii) $\mathrm{P}\Gamma t_\star = \Gamma\mathrm{P} t_\star$ *and* $\mathrm{P}\Gamma\bar{t}_* = \Gamma\mathrm{P}\bar{t}_*$.

Furthermore, if (E_0, K_0) satisfying proximal Opail's condition then $\{(t_n, \bar{t}_n)\}$ converges weakly to $(t_\star, \bar{t}_*)$.

Proof. Let $\{t_n\}$ and $\{\bar{t}_n\}$ are a sequence generated by *Algorithm I*. By Lemma 3.5 (ii), sequence $\{t_n\}$ and $\{\bar{t}_n\}$ are bounded. Then there exist $\{t_{n_k}\} \subseteq \{t_n\}$ with $\{t_{n_k}\} \rightharpoonup t_\star \in E_0$, and a subsequence $\{\bar{t}_{n_k}\}$ of $\{\bar{t}_n\}$ with converges weakly to $\bar{t}_* \in K_0$. Since $t_\star \in E_0$, then

$$\|t_\star - \mathrm{P}t_\star\| = d(E, K). \tag{3.16}$$

By Lemma 3.4, we have

$$\|t_{n_k} - \Gamma\mathrm{P}(t_\star)\| \leq 3\|t_{n_k} - \Gamma t_{n_k}\| + \|t_{n_k} - \mathrm{P}(t_\star)\|.$$

Letting $k \to \infty$, we get

$$\|t_\star - \Gamma\mathrm{P}(t_\star)\| \leq \|t_\star - \mathrm{P}(t_\star)\| = d(E, K),$$

and this yields that

$$\|t_\star - \Gamma\mathrm{P}(t_\star)\| = d(E, K). \tag{3.17}$$

By Eqs. (3.16), (3.17) and Lemma 2.3, give us $\Gamma\mathrm{P}(t_\star) = \mathrm{P}(t_\star)$. Hence, by Lemma 2.8 (see also Remark 2.9)

$$\|\Gamma\mathrm{P}(t_\star) - \mathrm{P}\Gamma(t_\star)\| = \|\mathrm{P}(t_\star) - \mathrm{P}\Gamma(t_\star)\| \leq \|t_\star - \Gamma t_\star\|. \tag{3.18}$$

Next, we will show that $t_\star = \Gamma t_\star$. By Lemma 3.4, we have

$$\|t_{n_k} - \Gamma t_\star\| \leq 3\|t_{n_k} - \Gamma t_{n_k}\| + \|t_{n_k} - t_\star\|,$$

$\forall n \in \mathbb{N}$. Letting $n \to \infty$, we get

$$\|t_\star - \Gamma t_\star\| = 0,$$

and hence $t_\star = \Gamma t_\star$. Therefore, by Eq. (3.18), we have

$$\|\Gamma\mathrm{P}(t_\star) - \mathrm{P}\Gamma(t_\star)\| = 0,$$

that is $\Gamma\mathrm{P}(t_\star) = \mathrm{P}\Gamma(t_\star)$.

Similarly, we can show that $\bar{t}_* = \Gamma\bar{t}_*$ and $\mathrm{P}\Gamma\bar{t}_* = \Gamma\mathrm{P}\bar{t}_*$. Therefore, we obtain (i), (ii), (iii) and (iii).

Next, we will show that $\{t_n\}$ converges weakly to $t_\star$. Let $\{t_{n_j}\}$ be another subsequence of $\{t_n\}$ which converges weakly to $z \in E_0$. Suppose that $t_\star \neq z$. Then, by Lemma 3.5(i) $\lim_{n\to\infty} \|t_n - \mathrm{P}t_\star\|$ and $\lim_{n\to\infty} \|t_n - \mathrm{P}z\|$ exist. From the fact that (E_0, K_0) satisfies proximal Opail's condition, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|t_n - \mathrm{P}t_\star\| &= \limsup_{k\to\infty} \|t_{n_k} - \mathrm{P}t_\star\| \\
&< \limsup_{k\to\infty} \|t_{n_k} - \mathrm{P}z\| \\
&= \lim_{n\to\infty} \|t_n - \mathrm{P}z\| \\
&= \lim_{j\to\infty} \|t_{n_j} - \mathrm{P}z\| \\
&< \limsup_{j\to\infty} \|t_{n_j} - \mathrm{P}t_\star\| \\
&= \limsup_{n\to\infty} \|t_n - \mathrm{P}t_\star\|,
\end{aligned}$$

which is a contradiction and thus $t_\star = z$. Hence $\{t_n\} \rightharpoonup t_\star \in Fix(\Gamma) \cap E_0$. Likewise, we can demonstrate that $\{\bar{t}_n\}$ converges weakly to some element $\bar{t}_* \in Fix(\Gamma) \cap K_0$. □

Theorem 3.8. *Let (E, K) be a nonempty, bounded, closed and convex pair of* *uniformly convex Banach space Ω with (E_0, K_0) satisfying proximal Opail's condition. Let $\Gamma : E \cup K \to E \cup K$ is a noncyclic Suzuki's relatively nonexpansive, and $\{u_n\}$, $\{\bar{u}_n\}$ are sequence generated by Algorithm I. Then $I - \Gamma$ is demi-closed at zero.*

Proof. By Lemma 3.7, we have $\{(t_n, \bar{t}_n)\}$ converges weakly to $(t_\star, \bar{t}_*)$. Now, we claim that

$$\|t_n - \bar{t}_n\| = d(E, K) \quad \forall n \in \mathbb{N}. \tag{3.19}$$

Since $\{t_n\}$, $\{\bar{t}_n\}$ be a sequence generated by *Algorithm I*, then by assumption

$$\|t_1 - \bar{t}_1\| = d(E, K).$$

Suppose that, for each $k \in \mathbb{N}$, we have

$$\|t_k - \bar{t}_k\| = d(E, K), \tag{3.20}$$

and Lemma 3.6, for any $\epsilon > 0, \exists N \in \mathbb{N}$such that

$$\zeta_k \|t_k - \Gamma t_k\| < \frac{\epsilon}{6},$$

$$\zeta_k \|\bar{t}_k - \Gamma \bar{t}_k\| < \frac{\epsilon}{6},$$

$$\|\Gamma s_k - s_k\| < \frac{\epsilon}{6},$$

$$\|\bar{s}_k - \Gamma \bar{s}_k\| < \frac{\epsilon}{6},$$

$$\phi_k \|\Gamma r_k - \Gamma s_k\| < \frac{\epsilon}{6},$$

$$\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| < \frac{\epsilon}{6}.$$

for all $k \geq N$. Then, we have

$$\begin{aligned}
\|t_{k+1} - \bar{t}_{k+1}\| &= \|((1-\phi_k)\Gamma s_k + \phi_k \Gamma r_k) \\
&\quad -((1-\phi_k)\Gamma \bar{s}_k + \phi_n \Gamma \bar{r}_k)\| \\
&= \|\Gamma s_k + \phi_n(\Gamma r_k - \Gamma s_k) \\
&\quad -\Gamma \bar{s}_k + (\phi_k(\Gamma \bar{s}_k - \Gamma \bar{r}_k)\| \\
&\leq \|\Gamma s_k - \Gamma \bar{s}_k\| \\
&\quad +\phi_k \|\Gamma r_k - \Gamma s_k\| \\
&\quad +\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| \\
&= \|(\Gamma s_k - s_k) + (s_k - \bar{s}_k) \\
&\quad +(\bar{s}_k - \Gamma \bar{s}_k)\| \\
&\quad +\phi_k \|\Gamma r_k - \Gamma s_k\| \\
&\quad +\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| \\
&\leq \|s_k - \bar{s}_k\| + \|\Gamma s_k - s_k\| \\
&\quad +\|\bar{s}_k - \Gamma \bar{s}_k\| \\
&\quad +\phi_k \|\Gamma r_k - \Gamma s_k\| \\
&\quad +\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| \\
&= \|((1-\zeta_k)t_k + \zeta_k \Gamma t_k) \\
&\quad -((1-\zeta_k)\bar{t}_k + \zeta_k \Gamma \bar{t}_k)\| \\
&\quad +\|\Gamma s_k - s_k\| + \|\bar{s}_k - \Gamma \bar{s}_k\| \\
&\quad +\phi_k \|\Gamma r_k - \Gamma s_k\| \\
&\quad +\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| \\
&\leq \|t_k - \bar{t}_k\| + \zeta_k \|t_k - \Gamma t_k\| \\
&\quad +\zeta_k \|\bar{t}_k - \Gamma \bar{t}_k\| \\
&\quad +\|\Gamma s_k - s_k\| + \|\bar{s}_k - \Gamma \bar{s}_k\| \\
&\quad +\phi_k \|\Gamma r_k - \Gamma s_k\| \\
&\quad +\phi_k \|\Gamma \bar{s}_k - \Gamma \bar{r}_k\| \\
&< \|t_k - \bar{t}_k\| + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\
&\quad +\frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\
&= \|t_k - \bar{t}_k\| + \epsilon \\
&= d(E, K) + \epsilon.
\end{aligned}$$

Hence, we can conclude that $\|t_{k+1} - \bar{t}_{k+1}\| \leq d(E, K)$. By mathematical induction, we can imply that the claim Eq. (3.19) holds. Then

$$\begin{aligned}
d(E, K) &\leq \|t_\star - \bar{t}_*\| \\
&\leq \liminf_{n \to \infty} \|t_n - \bar{t}_n\| \\
&= d(E, K).
\end{aligned}$$

Therefore $(t_\star, \bar{t}_*) \in Prox_{E \times K}(\Gamma)$ and hence the proof is complete. □

Theorem 3.9. *Let (E, K) be a nonempty, bounded, closed and convex pair of uniformly convex Banach space Ω with (E_0, K_0) satisfying proximal Opail's condition. Let $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively Suzuki's nonexpansive mapping,*

$\{t_n\}$ and $\{\bar{t}_n\}$ be generated by Algorithm I. Then $\{(t_n, \bar{t}_n)\}$ converges weakly to a best proximity pair of Γ.

Proof. By Theorem 3.7, we now have the sequence $\{(t_n, \bar{t}_n)\}$ converges weakly to $(t_\star, \bar{t}_*) \in E_0 \times K_0$, $t_\star = \Gamma t_\star$ and $\bar{t}_* = \Gamma \bar{t}_*$. By Lemma 3.6, we have $\{(I - \Gamma t_n, I - \Gamma \bar{t}_n)\}$ converges to $(0, 0)$, by Theorem 3.8, the sequence $\{(t_n, \bar{t}_n)\}$ converges weakly to $(t_\star, \bar{t}_*) \in Prox_{E\times K}(\Gamma)$. Hence the sequence $\{(t_n, \bar{t}_n)\}$ converges weakly to a best proximity pair of Γ. □

Corollary 3.10. *Assume that (E, K) be a , bounded, closed, nonempty and convex pair of uniformly convex Banach space Ω. Assume that $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively nonexpansive mapping, $\{t_n\}$ and $\{\bar{t}_n\}$ generated by Algorithm I. $\{(t_n, \bar{t}_n)\}$ converges weakly to a best proximity pair of Γ.*

Next, we give a strong convergence of *Algorithm I* with noncyclic relatively Suzuki's nonexpansive mapping.

Theorem 3.11. *Let (E, K) be a nonempty, compact and convex pair of uniformly convex Banach space Ω. Let $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively Suzuki's nonexpansive mapping, $\{t_n\}$ and $\{\bar{t}_n\}$ be generated by Algorithm I. Then $\{(t_n, \bar{t}_n)\}$ converges strongly to a best proximity pair of Γ.*

Proof. By Lemma 3.6, we have $\{(I - \Gamma t_n, I - \Gamma \bar{t}_n)\}$ converges to $(0, 0)$. Since (E, K) is a compact subset of Ω and $\{(t_n, \bar{t}_n\}$ is a sequences in $E_0 \times K_0$. Then there exist $\{t_{n_k}\} \subseteq \{t_n\}$ with converges strongly to $t_\star \in E_0$, and $\{\bar{t}_{n_k}\} \subseteq \{\bar{t}_n\}$ with converges strongly to $\bar{t}_* \in K_0$. Since $t_\star \in E_0$, then

$$\|t_\star - \mathrm{P}t_\star\| = d(E, K). \tag{3.21}$$

By Lemma 3.4, we have

$$\|t_{n_k} - \Gamma t_\star\| \le 3\|t_{n_k} - \Gamma t_{n_k}\| + \|t_{n_k} - t_\star\|,$$

for all $k \in \mathbb{N}$. Letting $n \to \infty$, we get

$$\|t_\star - \Gamma t_\star\| = 0,$$

and hence $t_\star = \Gamma t_\star$. Similarly, $\bar{t}_* = \Gamma \bar{t}_*$. Then $t_\star \in Fix(\Gamma) \cap E_0$ and $\bar{t}_* \in Fix(\Gamma) \cap K_0$. Therefore, by Lemma 3.5(i) $\lim_{n\to\infty} \|t_n - \mathrm{P}t_\star\|$ and $\lim_{n\to\infty} \|\bar{t}_n - \mathrm{P}t_*\|$ exist. Hence

$$\begin{aligned} \lim_{n\to\infty} \|t_n - \mathrm{P}t_\star\| &= \lim_{k\to\infty} \|t_{n_k} - \mathrm{P}t_\star\| \\ &= \|t_\star - \mathrm{P}t_\star\| = d(E, K). \end{aligned}$$

Therefore $\{t_n\}$ with converges strongly to $t_\star \in E_0$. Similarly, $\{\bar{t}_n\}$ with converges strongly to $\bar{t}_* \in K_0$. Furthermore, by the same argument as the proof of Theorem 3.8, we can show that

$$\|t_n - \bar{t}_n\| = d(E, K) \quad \forall n \in \mathbb{N}. \tag{3.22}$$

Hence,

$$\|t_\star - \bar{t}_*\| = \lim_{n\to\infty} \|t_n - \bar{t}_n\| = d(A, B),$$

and the proof is complete. □

Corollary 3.12. *Let (E, K) be a nonempty, compact and convex pair of uniformly convex Banach space Ω. Let $\Gamma : E \cup K \to E \cup K$ be a noncyclic relatively nonexpansive mapping, $\{t_n\}$ and $\{\bar{t}_n\}$ be generated by Algorithm I. Then $\{(t_n, \bar{t}_n)\}$ converges strongly to a best proximity pair of Γ.*

4. An Example and Numerical Simulation

We present in this section, an illustrative example of noncyclic relatively Suzuki's nonexpansive mapping for support our main results with numerical experiment via Algroithm I. Also we show that the conclusion of Gabeleh [12] is not applicable for our main result.

Example 4.1. Let $\Omega = \mathbb{R}^2$ with the Euclidian norm, $E = \{(\psi,\mu) \in \mathbb{R}^2 : 0 \leq \psi \leq 1, 1 \leq \mu \leq 2\}$ and $D = \{(\psi,\mu) \in \mathbb{R}^2 : 0 \leq \psi \leq 1, -2 \leq \mu \leq -1\}$. Then $C_0 = \{(\psi,1) \in \mathbb{R}^2 : 0 \leq \psi \leq 1\}$, $D = \{(\psi,-1) \in \mathbb{R}^2 : 0 \leq \psi \leq 1\}$ and $d(C,D) = 2$. Define the mapping $\Gamma : C \cup D \to C \cup D$ by

$$\Gamma(\psi,\mu) = \begin{cases} (1-\psi,\mu) & \text{if } \psi \in [0,\frac{1}{6}); \\ (\frac{\psi+5}{6},\mu) & \text{if } \psi \in [\frac{1}{6},1], \end{cases}$$

for all $\psi,\mu \in \Omega$. Then Γ is noncyclic mapping. First, we will show that, Γ is a noncyclic relatively Suzuki's nonexpansive mapping.

case I Let $(\psi_1,\mu_1) \in C$ such that $\psi_1 \in [0,\frac{1}{6})$. Then

$$\frac{1}{2}\|(\psi_1,\mu_1) - \Gamma(\psi_1,\mu_1)\| = \frac{1}{2}\sqrt{(2\psi_1-1)^2} = \frac{1}{2}\|2\psi_1 - 1\|.$$

For

$$\frac{1}{2}\|(\psi_1,\mu_1) - \Gamma(\psi_1,\mu_1)\| \leq \|(\psi_1,\mu_1)-(\psi_2,\mu_2)\|,$$

that is

$$\frac{1-2\psi_1}{2} \leq \sqrt{(\psi_1-\psi_2)^2 + (\mu_1-\mu_2)^2}. \tag{4.1}$$

Since $(\mu_1-\mu_2)^2 \geq 0$. If $(\mu_1-\mu_2)^2 = 0$, then

$$\begin{aligned} \frac{1-2\psi_1}{2} &\leq \sqrt{(\psi_1-\psi_2)^2 + (\mu_1-\mu_2)^2} \\ &= \sqrt{(\psi_1-\psi_2)^2} \\ &= \|\psi_1 - \psi_2\|. \end{aligned} \tag{4.2}$$

Thus, to show that Eq. (4.1) holds, without generality we will consider an element $\psi_2 \in [0,1]$ which satisfies Eq. (4.2). Now, we divided four case.

(i) If $\frac{1-2\psi_1}{2} \leq \|\psi_1 - \psi_2\|$ and $\psi_1 \leq \psi_2$, then

$$\frac{1-2\psi_1}{2} \leq \psi_2 - \psi_1, \quad \text{i.e,} \quad \frac{1}{2} \leq \psi_2 \leq 1.$$

Thus, for $\psi_1 \in [0,\frac{1}{6})$ and $\psi_2 \in (\frac{1}{2},1]$, we now have

$$\begin{aligned} &\|\Gamma(\psi_1,\mu_1) - \Gamma(\psi_2,\mu_2)\| \\ &= \|(1-\psi_1,\mu_1) - (\frac{\psi_2+5}{6},\mu_2)\| \\ &= \sqrt{(1-\psi_1-\frac{\psi_2+5}{6})^2 + (\mu_1-\mu_2)^2} \\ &\leq \|1-\psi_1-\frac{\psi_2+5}{6}\| + \|\mu_1-\mu_2\| \\ &= \|\frac{1-6\psi_1-\psi_2}{6}\| + \|\mu_1-\mu_2\| \\ &\leq \|\frac{1-12\psi_1}{12}\| + \|\mu_1-\mu_2\| \\ &\leq \frac{1}{12} + \|\mu_1-\mu_2\| \\ &\leq \frac{1}{3} + \|\mu_1-\mu_2\| \\ &\leq \sqrt{(\psi_1-\psi_2)^2 + (\mu_1-\mu_2)^2} \\ &= \|(\psi_1,\mu_1) - (\psi_2,\mu_2)\|. \end{aligned}$$

(ii) If $\frac{1-2\psi_1}{2} \leq \|\psi_1 - \psi_2\|$ and $\psi_1 > \psi_2$, then

$$\frac{1-2\psi_1}{2} \leq \psi_1 - \psi_2,$$

it follows that

$$\psi_2 < 2\psi_1 - \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} < 2\psi_1 - \frac{1}{2} < -\frac{1}{6},$$

which is impossible in this case.

(iii) If $\frac{1-2\psi_1}{2} \geq \|\psi_1 - \psi_2\|$ and $\psi_1 \leq \psi_2$, then $\frac{1-2\psi_1}{2} \geq \psi_2 - \psi_1$ it follows that $\psi_2 \geq \frac{1}{2}$, and hence we must have $\psi_2 \in [0,\frac{1}{2}]$. If $\psi_2 \in [\frac{1}{6},\frac{1}{2}]$, we can show Γ is a noncyclic relatively Suzuki's nonexpansive mapping by similarly (i). Suppose $\psi_2 \in [0,\frac{1}{6})$, then we have

$$\begin{aligned} &\|\Gamma(\psi_1,\mu_1) - \Gamma(\psi_2,\mu_2)\| \\ &= \|(1-\psi_1,\mu_1) - (1-\psi_2,\mu_2)\| \\ &= \|(\psi_2-\psi_1,\mu_1-\mu_2)\| \\ &= \sqrt{(\psi_1-\psi_2)^2 + (\mu_1-\mu_2)^2} \\ &= \|(\psi_1,y_1) - (\psi_2,\mu_2)\|. \end{aligned}$$

(iv) If $\frac{1-2\psi_1}{2} \geq \|\psi_1 - \psi_2\|$ and $\psi_1 > \psi_2$, then $\frac{1-2\psi_1}{2} \geq \psi_1 - \psi_2$ it follows that

$$\psi_2 \geq 2\psi_1 - \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} < 2\psi_1 - \frac{1}{2} < -\frac{1}{3}.$$

Thus, we must have $\psi_2 \in [\frac{1}{6}, 1]$, we obtain Γ is a noncyclic relatively Suzuki's nonexpansive mapping by similarly in (i).

case II Let $(\psi_1, \mu_1) \in C$ such that $\psi_1 \in [\frac{1}{6}, 1]$. Then

$$\begin{aligned}\frac{1}{2}\|(\psi_1, \mu_1) - \Gamma(\psi_1, \mu_1)\| &= \frac{1}{2}\sqrt{(\psi_1 - \frac{\psi_1 + 5}{6})^2} \\ &= \frac{5 - 5\psi_1}{12}.\end{aligned}$$

For

$$\frac{1}{2}\|(\psi_1, \mu_1) - \Gamma(\psi_1, \mu_1)\| \leq \|(\psi_1, \mu_1) - (\psi_2, \mu_2)\|,$$

that is

$$\frac{5 - 5\psi_1}{12} \leq \sqrt{(\psi_1 - \psi_2)^2 + (\mu_1 - \mu_2)^2}.$$

We will consider four case.

(i) If $\frac{5-5\psi_1}{12} \leq \|\psi_1 - \psi_2\|$ and $\psi_1 \leq \psi_2$, then $\frac{5-5\psi_1}{12} \leq \psi_2 - \psi_1$. which implies that $\frac{5+7\psi_1}{12} \leq \psi_2$ and $\psi_2 \in [\frac{37}{72}, 1]$. Hence, for $\psi_1 \in [\frac{1}{6}, 1]$ and $\psi_2 \in [\frac{37}{72}, 1]$, we have

$$\begin{aligned}\|\Gamma(\psi_1, \mu_1) - \Gamma(\psi_2, \mu_2)\| &= \|(\frac{\psi_1 + 5}{6}, \mu_1) - (\frac{\psi_2 + 5}{6}, \mu_2)\| \\ &= \sqrt{(\frac{\psi_1 + 5}{6} - \frac{\psi_2 + 5}{6})^2 + (\mu_1 - \mu_2)^2} \\ &= \sqrt{(\frac{\psi_1 - \psi_2}{6})^2 + (\mu_1 - \mu_2)^2} \\ &\leq \sqrt{(\psi_1 - \psi_2)^2 + (\mu_1 - \mu_2)^2} \\ &= \|(\psi_1, \mu_1) - (\psi_2, \mu_2)\|.\end{aligned}$$

(ii) If $\frac{5-5\psi_1}{12} \leq \|\psi_1 - \psi_2\|$ and $\psi_1 > \psi_2$, then $\frac{5-5\psi_1}{12} < \psi_1 - \psi_2$. it follows that

$$\psi_2 < \frac{17\psi_1 - 5}{72},$$

and

$$-\frac{13}{12} < \frac{17\psi_1 - 5}{72} < 1,$$

which is impossible in this cases.

(iii) If $\frac{5-5\psi_1}{12} \geq \|\psi_1 - \psi_2\|$ and $\psi_1 \leq \psi_2$, then $\frac{5-5\psi_1}{12} > \psi_2 - \psi_1$ implies that

$$\psi_2 < \frac{7\psi_1 + 5}{12},$$

and

$$\frac{37}{72} < \frac{17\psi_1 - 5}{72} < 1.$$

Thus $\psi_2 \in [0, \frac{1}{6})$, then we have

$$\begin{aligned}\|\Gamma(\psi_1, \mu_1) - \Gamma(\psi_2, \mu_2)\| &= \|(\frac{\psi_1 + 5}{6}, \mu_1) - (1 - \psi_2, \mu_2)\| \\ &= \sqrt{(\frac{\psi_1 + 5}{6} - (1 - \psi_2))^2 + (\mu_1 - \mu_2)^2} \\ &= \sqrt{(\frac{\psi_1 - 1 + 6\psi_2}{6})^2 + (\mu_1 - \mu_2)^2} \\ &< \sqrt{(\frac{\psi_1 - 6\psi_2 + 6\psi_2}{6})^2 + (\mu_1 - \mu_2)^2}, \\ &\quad (\because \ 0 < \psi_2 < 1/6) \\ &< \sqrt{(\frac{\psi_1}{6})^2 + (\mu_1 - \mu_2)^2} \\ &\leq \sqrt{(\psi_1 - \psi_2)^2 + (\mu_1 - \mu_2)^2} \\ &= \|(\psi_1, \mu_1) - (\psi_2, \mu_2)\|.\end{aligned}$$

(iv) If $\frac{5-5\psi_1}{12} > \|\psi_1 - \psi_2\|$ and $\psi_1 > \psi_2$, then $\frac{5-5\psi_1}{12} < \psi_1 - \psi_2$ implies that

$$\psi_2 < \frac{17\psi_1 - 5}{12},$$

and

$$\frac{-13}{72} \leq \frac{17\psi_1 - 5}{12} \leq 1,$$

which is impossible in this case. Then T is a noncyclic relatively Suzuki's nonexpansive mapping for the case considered. Figs. 1-2 show that Case I and Case II hold.

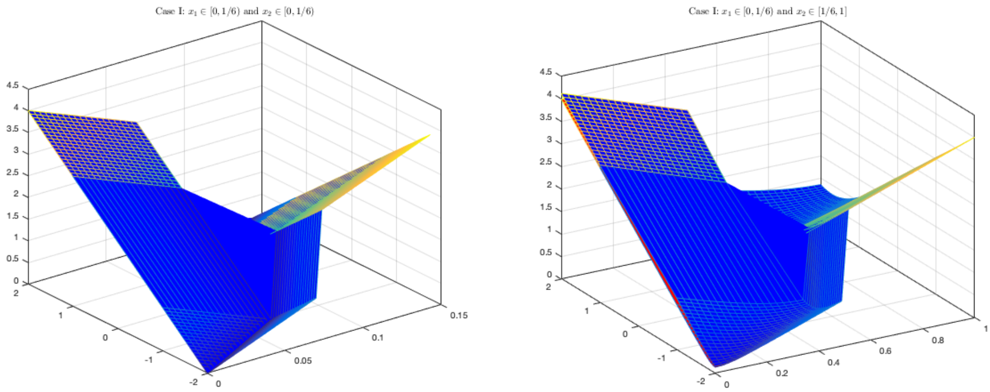

Fig. 1. The Red and Blue surfaces represent Case II's LHS and RHS, respectively.

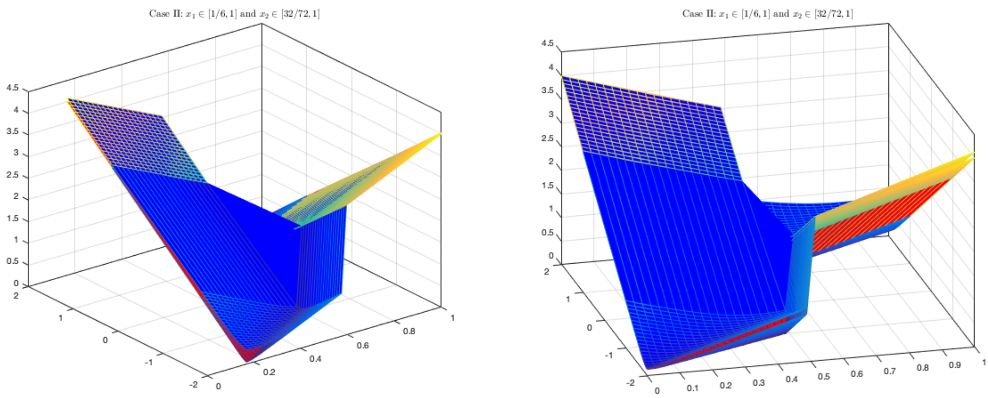

Fig. 2. The Red and Blue surfaces represent Case II's LHS and RHS, respectively.

Therefore, by both cases Γ is a noncyclic relatively Suzuki's nonexpansive mapping. Further, there is $(1,1) \in C_0$ and $(1,-1) \in D_0$ such that

$$\Gamma(1,1) = (\frac{\psi+5}{6}, 1) = (1,1),$$

$$\Gamma(1,-1) = (\frac{\psi+5}{6}, -1) = (1,-1),$$

and

$$\|(1,1)-(1,-1)\| = \sqrt{(1-1)^2 + (1-(-1))^2} = 2.$$

That is $((1,1),(1,-1)) \in Prox_{C\times D}$. However, Γ is not noncyclic relatively nonexpansive mapping, by setting $(\psi_1,\mu_1) = (\frac{1}{30}, 1) \in C$ and $(\psi_2,\mu_2) = (\frac{1}{6}, -1) \in D$, we have

$$\|\Gamma(\psi_1,\mu_1) - \Gamma(\psi_2,\mu_2)\| = \sqrt{(\frac{126}{30})^2 + 2^2} > \sqrt{(\frac{4}{30})^2 + 2^2} = \|\psi - \mu\|.$$

So, the conclusion of Gabeleh [12] cannot applied this example.

Now, we shall give a numerical experiment and convergence behavior of *Algrorithm I* involving noncyclic relatively Suzuki's nonexpansive mapping. We set $\phi_n = \vartheta_n = \zeta_n = 0.5 + \frac{1}{n+2}$. With initial point $t_1 = (0,1)$ and $t_1 = (0.5,1)$. Tables 1-2 shows the numerical approximations for Γ's best proximity pair ((1, 1); (1, 1)).

5. Conclusions

In this work we presented a new concept of a noncyclic relatively Suzuki's nonexpansive and used this concept for proving the convergence of a best proximity pair in uniformly convex Banach space and an illustrative example of noncyclic relatively Suzuki's nonexpansive mapping for support our main results. Also we show that the conclusion of Gabeleh [12] is not applicable for our main result.

Table 1. Convergence of sequence for $t_1 = (0,1)$.

n	t_n	$\bar{t}_n$
1	(0,1)	(0,-1)
2	(0.978545,1)	(0.978545,-1)
3	(0.999288,1)	(0.999288,-1)
4	(0.999971,1)	(0.999971,-1)
5	(0.999999,1)	(0.999999,-1)
6	(1,1)	(1,-1)
7	(1,1)	(1,-1)

Table 2. Convergence of sequence for $t_1 = (0.5,1)$.

n	t_n	$\bar{t}_n$
1	(0.5,1)	(0.5,-1)
2	(0.989273,1)	(0.989273,-1)
3	(0.999644,1)	(0.999644,-1)
4	(0.999985,1)	(0.999985,-1)
5	(0.999999,1)	(0.999999,-1)
6	(1,1)	(1,-1)
7	(1,1)	(1,-1)

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