P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia*
Science & Technology Asia
Vol.29 No.1 January - March 2024
Page: [29-46]
Original research article

Forward Jump Random Walk on a Cycle Graph and Its Hitting Time

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Received 19 October 2023; Received in revised form 20 February 2024
Accepted 4 March 2024; Available online 31 March 2024

ABSTRACT

This paper presents an investigation into a random walk on a cycle graph with restricted forward movement at most m steps, known as the forward jump random walk. The study derives exact formulas for the probability mass function of the arriving state, the hitting time, and its expected value and variance, where those solutions can be expressed in terms of trigonometric sums. These formulas are obtained using a combinatorial method as an alternative to the eigenvector-based approach commonly used.

Keywords: Cycle graph; Hitting time; Random walk on graph

1. Introduction

A random walk on a graph [1] is a mathematical concept that characterizes the movement of a *walker* between vertices in a graph using random transitions. At each step, the walker moves to a neighboring vertex according to a specific probability distribution. These walks have implications in various fields, such as distributed networks [2], biological networks [3], and protein classification [4]. Moreover, random walks have been extensively applied in the development of important algorithms, including the renowned PageRank algorithm [5, 6]. The mechanism of a random walk is typically defined in a specific manner, depending on the application and the topology of the network. They also have a notable connection to the underlying graph's structure and properties. Numerous intriguing properties are commonly studied in the analysis of dynamic behavior and properties of random walks as stochastic processes such as covering time, commute time, mixing time, and hitting time.

A cycle graph is characterized by a closed loop formation in which each vertex is precisely connected to two adjacent

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doi: 10.14456/scitechasia.2024.3

vertices. A simple random walk on a cycle is a well-known concept covered in textbooks [7]. Furthermore, the random walks on the d-cycle graph, where d is the number of vertices, labelled from 0 to $d-1$, can be considered as a type of random walk on the set of remainders modulo d, taking the form of

$$R_n = (R_{n-1} + X_n) \bmod d. \tag{1.1}$$

In the given expression, let n be a natural number, and let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables, and R_n be the sum of these random variables, i.e., $R_n = (\sum_{j=1}^{n} X_j) \bmod d$. As an example, in the random walk discussed in [7], the choice for $X \in \{-1, 1\}$ is made such that $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = 0.5$. In other words, the walker has an equal probability of moving forward or backward to a neighboring node.

A more intricate random walk on a cycle, beyond what is commonly described in the literature, was employed to study the Hunter vs. Rabbit game on a graph [8], which serves as a model for communication in adhoc mobile networks [9, 10]. The model aids in the efficient transmission of electronic messages using mobile phones, where the expected time for the hunter to catch the rabbit is equivalent to the expected time for the recipient to receive the mail. The strategies used for the movements of the hunter and the rabbit involved the utilization of a random walk, as described in Eq. (1.1), with distinct distributions of the random variable X. The random walk on the cycle graph finds another application in electrical engineering, where resistor networks can be effectively modeled using a cycle graph [11, 12]. The resistance of the circuit can be calculated by the effective resistance using the first passage time, or the hitting time - the anticipated time for a walker, starting from a source node, to reach a target node for the first time. This method effectively circumvents the need to solve the extensive linear system that arises from Kirchhoff's laws, particularly when dealing with a substantial number of nodes.

In this paper, we undertake an investigation into a random walk on an d-cycle graph, where the walker's movement is restricted to a maximum of m forward steps at a time. Considering the random walk in (1.1), let the sequence of arriving states be denoted as $R_0, R_1, R_2, \ldots$, where $R_n \in \{0, 1, \ldots, d-1\}$ for $n \in \mathbb{N}$, and $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed uniform random variables on the set $\{1, 2, \ldots, m\}$, with each value having a probability of $1/m$, i.e. $\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \cdots = \mathbb{P}(X = m) = 1/m$. We derived the probability mass function (PMF) for the arrival state, allowing us to establish a recurrence relation for the hitting time, as well as provide insights into the expected value and variance of the hitting time. By leveraging these formulas, we can explore how changes in these parameters influence the characteristics of the distributions, hitting time, and their expected value and variance, with respect to the parameters m, d, and n. The insights gleaned from our analysis possess significant potential to contribute to forthcoming advancements in electric circuit analysis and other diverse applications. Furthermore, the solution in our work was derived from a combinatorial identity, as described in [13]. This approach presents an alternative method to the previously employed eigenvector-based calculation for determining hitting time, as demonstrated in prior research [11, 12].

The motivation for this random walk mechanism is derived from the rules governing the popular board game, Monopoly.

This renowned game not only offers entertainment but also serves as an instructive illustration of stochastic processes, extensively explored in various academic literature, including textbooks such as [14–16]. Monopoly is played on a square game board comprising 40 spaces, denoted by $d = 40$. Each space, such as Reading Railroad and Park Place, is assigned a numerical label ranging from 0 (Go) to 39 (Boardwalk), represented by the random variable R. In this case, we will consider the rule where players roll a standard six-sided dice ($m = 6$) and advance a number of spaces equal to the value shown on the dice, represented by the random variable X. This distinct choice of using a single dice distinguishes it from the scenario described in [14]. For simplicity, we will temporarily disregard elements such as Go to Jail, Chance cards, and other squares that introduce complexity to the game, called generally the board game with the circular walk. Our focus is to formulate and analyze the dynamics in a simplified manner.

In addition to proposing the analytic formulas, our focus is on analyzing the random walk on a board game with a circular walk. The game board consists of a square grid with $d = 12$ squares, each labeled from 0 to 11. The objective is to investigate the behavior of players as they engage in the game. A key rule of the game involves rolling an m-sided die, where m can have values of 4, 6, 8, 10, 12, or 20, which are the commonly used physical dice shapes. The number rolled on the die determines the number of spaces a player advances on the board. We chose a board size of $d = 12$ in order to investigate the behavior of PMFs under different conditions. Specifically, our objective is to analyze scenarios where d is less than, equal to, and greater than the number of sides on the die (m). By examining these cases, we can gain insights into how the probability distributions evolve and vary based on the relationship between d and m, in which $d < m$, $d = m$, and $d > m$. The section then outlines the key questions that will be addressed, which include the uniformity of square visits, the change in probability of first visits, the number of steps required to reach a square for the first time, and the evolution of probability distributions and hitting times.

The paper is structured as follows. Firstly, in Section 2, an overview of the notation and background used throughout the subsequent sections is presented. Then, in Section 3, we mathematically introduce the forward jump random walk, which is proposed in this paper. The construction of the probability mass function (PMF) for the arriving state after walking n steps is demonstrated in Section 4. This PMF plays a crucial role in establishing a recurrence relation for the hitting time. We provide three versions of the PMF, namely complex, trigonometric, and combinatorial forms. Moving on to Section 5, we derive the hitting time in the form of a recurrence relation, and also compute its corresponding expected value and variance. To illustrate the application of the theorems developed in the previous sections, an experiment on a board game with circular walk is presented in Section 6. Finally, the paper concludes with a summary of the findings in the last section.

2. Notation and Background

This section provides an overview of the key notation and background information necessary to understand the subsequent analysis. This section establishes a foundation for the rest of the paper by introducing the symbols, terminology, and concepts that will be used throughout. Throughout this article, we determine the integers $d \geq 2$,

$m \geq 2$ and the sets $V = \{0, 1, ..., d-1\}$, $U = \{1, 2, ..., m\}$. Additionally, we define

$$w = e^{\frac{2\pi}{d}i} = \cos\left(\frac{2\pi}{d}\right) + i\sin\left(\frac{2\pi}{d}\right), \quad (2.1)$$

where $i^2 = -1$, and

$$v_k = \frac{w^k}{m}\left(\frac{w^{mk}-1}{w^k-1}\right), \quad k \in V - \{0\}. \quad (2.2)$$

We omit explicitly stating the parameters $d \geq 2$ and $m \geq 2$ for simplicity. For any unspecified constants, we will assume they are integers.

Kaikeaw and Naenudorn [13] achieved an identity that involves trigonometric power sums. This was accomplished through an enumeration of the solutions to a specific congruence equation,

$$x_1 + x_2 + \cdots + x_n \equiv r \bmod d, \quad (2.3)$$

where $1 \leq x_1, x_2, ..., x_n \leq m$,using two different methods. The first method is based on the inclusion-exclusion principle, which leads to the formulation of Lemma 2.1. The second method utilized a generating function technique, resulting in the derivation of Lemma 2.2. These results can further be utilized to determine the explicit PMF of arriving state in the next section.

Lemma 2.1 ([13]). *For any $n \geq 1$ and r, number of solutions to the congruence (2.3) is equal to*

$$\sum_{k=\lceil\frac{n-r}{d}\rceil}^{\lfloor\frac{mn-r}{d}\rfloor} \sum_{j=0}^{\lfloor\frac{dk+r-n}{m}\rfloor} (-1)^j \binom{n}{j}\binom{dk+r-mj-1}{n-1}.$$

Lemma 2.2 ([13]). *For any $n \geq 1$ and r, the number of solutions to the congruence (2.3) is equal to*

$$\frac{1}{d}\left(m^n + \sum_{k=1}^{d-1}\cos\left(\frac{(nm+n-2r)k\pi}{d}\right) \times \sin^n\left(\frac{mk\pi}{d}\right)\csc^n\left(\frac{k\pi}{d}\right)\right).$$

Furthermore, we can express Lemma 2.2 in a complex number representation as follows:

Lemma 2.3. *For any $n \geq 1$ and r, the number of solutions to the congruence (2.3) is equal to*

$$\frac{m^n}{d} + \frac{m^n}{d}\sum_{k=1}^{d-1} w^{-rk} v_k^n.$$

Proof. From Eq. (4.3) in the proof of Theorem 4.4 in [13], page 35, the number of solutions to the congruence (2.3) is equal to $\frac{1}{d}\sum_{k=0}^{d-1} w^{(n-r)k}\left(\sum_{j=0}^{m-1} w^{jk}\right)^n$ which can be simplified to $\frac{m^n}{d} + \frac{m^n}{d}\sum_{k=1}^{d-1} w^{-rk} v_k^n$. □

3. Forward-Jump Random Walk on Cycle Graph

In this section, we will provide a mathematical explanation of the forward-jump random walk on a cycle graph, which is utilized in this paper.

Definition 3.1 (Forward-Jump Random Walk on Cycle Graph)**.** Let $(X_1, X_2, X_3, ...)$ be a sequence of random variables which are uniformly chosen from the set $U = \{1, 2, ..., m\}$, that is,

$$\mathbb{P}(X_n = x) = \frac{1}{m}, \quad x \in U. \quad (3.1)$$

Let $R_0 = 0$, and we define a sequence of random variables $(R_1, R_2, R_3, ...)$ by

$$R_n = (R_{n-1} + X_n) \bmod d, \quad n \geq 1. \quad (3.2)$$

In other words, we can say that $(R_1, R_2, R_3, ...)$ is a random walk on directed graph G, where G has a set of vertices $V = \{0, 1, ..., d-1\}$, the initial vertex $R_0 = 0$, and transition matrix $P = [p_{r,s}]_{d \times d}$ such that

$$p_{r,s} = \frac{|\{x \in U | x \equiv s - r \bmod d\}|}{m}, \quad r, s \in V, \tag{3.3}$$

where $|A|$ represents the number of elements in any finite set A.

The transition matrix of the random walk is doubly stochastic, with the sum of each row and each column equaling one. This property indicates the existence of a stationary distribution on the Markov transition chain. In Fig. 1, we provide examples of graph topologies and the corresponding walking paths for two scenarios: $d = 5$ and $m = 3$, and $d = 3$ and $m = 4$. Additionally, their transition matrices are as follows:

$$P_{d=5,m=3} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix},$$

and

$$P_{d=3,m=4} = \begin{pmatrix} \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix}.$$

The following remark clarifies the relationship between the summation of random variables, the dynamics of the random walk, and the congruence.

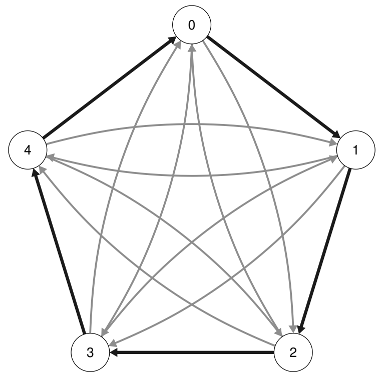
Remark 3.2. *For $n \geq 1$, it is evident from Eq. (3.2) that*

$$X_k \equiv R_k - R_{k-1} \bmod d, \quad k \in \{1, 2, ..., n\}.$$

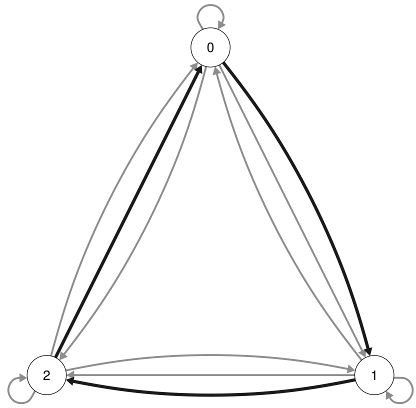
By combining these congruences for all values of $k \in \{1, 2, ..., n\}$, we obtain

$$X_1 + X_2 + \cdots + X_n \equiv R_n \bmod d.$$

Consequently, the event $R_n = r$ occurs if and only if $X_1 + X_2 + \cdots + X_n \equiv r \bmod d$.



(a) Topology of Graph with $d = 5$ and $m = 3$



(b) Topology of Graph with $d = 3$ and $m = 4$

Fig. 1. Topology of Graph with different d and m

Remark 3.3. *For a fixed constant $r \in V$, it is evident from Eq. (3.3) that*

$$p_{j,k} = p_{m,l}, \quad j, k \in V,$$

where $m = (j - r) \bmod d$ and $l = (k - r) \bmod d$.

The remark implies that the probability of transitioning from vertex j to vertex k is equivalent to the probability of transitioning from vertex $(j - r) \bmod d$ to vertex $(k - r) \bmod d$. As a result, we can redefine the vertex r as 0 and represent each vertex $s \in V$ as $(s - r) \bmod d$ without losing generality. In other words, the graph G described in Definition 3.1 exhibits rotational symmetry, where each vertex appears identical from any other vertex.

4. Probability Mass Function of Arriving State

In this section, we shall ascertain the PMF of R_n, i.e. the probability that the vertex r will be visited at n^{th} movement, through the utilization of the results from [13]. The sequence of vertices generated by the forward jump random walk on a cycle graph follows the congruence given in Eq. (2.3). Consequently, the PMF of the arriving vertex or state, under equal probability of jumping, is equivalent to the set of all possible solutions of the congruence in Eq. (2.3) divided by the total number of solutions to the general congruence, m^n. The PMF of R_n is then expressed as follows.

Theorem 4.1 (PMF of R_n). *For $n \geq 1$ and $r \in V$, the random variable R_n satisfies the following PMF:*

$$\mathbb{P}(R_n = r) = \frac{1}{d} + \frac{1}{d}\sum_{k=1}^{d-1} w^{-rk} v_k^n. \quad (4.1)$$

Proof. We know that $(X_1, X_2, ..., X_n)$ is uniformly selected from a set of m^n possible outcomes. Additionally, the event $R_n = r$ occurs if and only if $X_1 + X_2 + \cdots + X_n \equiv r \bmod d$. According to Lemma 2.3, the number of outcomes for such an event is $\frac{m^n}{d} + \frac{m^n}{d}\sum_{k=1}^{d-1} w^{-rk} v_k^n$. Dividing this quantity by m^n, we obtain the desired result. □

The complexity of the function in Eq. (4.1) may pose difficulties in computation. To address this, we propose two alternative versions of the PMF expressed using combinatorial identities and trigonometric functions. These alternative formulations provide representations that may be computationally more tractable.

Proposition 4.2. *For $n \geq 1$ and $r \in V$, we have*

$$\mathbb{P}(R_n = r) = \frac{1}{m^n} \sum_{k=\lceil \frac{n-r}{d} \rceil}^{\lfloor \frac{mn-r}{d} \rfloor} \sum_{j=0}^{\lfloor \frac{dk+r-n}{m} \rfloor} (-1)^j \binom{n}{j} \binom{dk + r - mj - 1}{n - 1} \quad (4.2)$$

and

$$\mathbb{P}(R_n = r) = \frac{1}{d}\Bigg(1 + \frac{1}{m^n} \times \sum_{k=1}^{d-1} \cos\left(\frac{(nm + n - 2r)k\pi}{d}\right) \times \sin^n\left(\frac{mk\pi}{d}\right) \csc^n\left(\frac{k\pi}{d}\right)\Bigg). \quad (4.3)$$

Proof. Following a similar approach to the proof of Theorem 4.1, we divide the number of solutions to congruence (2.3) obtained from Lemma 2.1 and Lemma 2.2 by m^n, respectively. □

5. Hitting Time

The hitting time represents the minimum number of steps needed for a random walk to reach a specific vertex r starting from an initial vertex. In this section, we utilize the results obtained in the previous section to establish the hitting time as well as its expected value and variance. First, we provide the mathematical definition of the hitting time used in this study.

Definition 5.1. Let $N_0, N_1, ..., N_{d-1}$ be random variables which are defined by

$$N_r = \min\{n \geq 1 : R_n = r\}, \quad r \in V = \{0, 1, ..., d-1\}. \quad (5.1)$$

In other words, we can say that N_r is the hitting time from initial vertex $R_0 = 0$ to vertex r.

Remark 5.2. *For $n \geq 1$ and $r \in V$, it is evident from Eq. (5.1) that the event $N_r = n$ occurs if and only if $R_1, R_2, ..., R_{n-1} \neq r$, but $R_n = r$. Furthermore, if $R_n = r$, then $N_r \leq n$.*

During the movement of the random walk, the label of the reference vertex changes. In fact, we can represent the current state by the initial label without losing generality. The details will be provided in the following remark.

Remark 5.3. *To determine the hitting time from vertex r to vertex s, denoted as $H_{r,s}$, we can follow the same procedure described in Remark 3.3. Consequently, the hitting time can be represented as $H_{r,s} = N_l$, where $l = (s - r) \bmod d$.*

5.1 Probability mass function

The hitting time is intricately linked to the PMF of arriving state, manifesting in the form of a recurrence relation as outlined below.

Theorem 5.4 (PMF of N_r)**.** *For $n \geq 1$ and $r \in V$, random variables N_r and R_n satisfy the relation :*

$$\sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_{n-k} = 0) = \mathbb{P}(R_n = r). \tag{5.2}$$

Proof. There are exactly n disjoint cases that occur when $R_n = r$: case $N_r = 1$, case $N_r = 2$, ..., and case $N_r = n$. Then we have

$$\begin{aligned}
&\mathbb{P}(R_n = r) \\
&= \sum_{k=1}^{n} \mathbb{P}(R_n = r, N_r = k) \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_n = r | N_r = k) \\
&\quad ;\text{Conditional probability} \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_n = r | \\
&\qquad R_1, R_2, ..., R_{k-1} \neq r \text{ but } R_k = r) \\
&\quad ;\text{Remark 5.2} \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_n = r | R_k = r) \\
&\quad ;\text{Markov property} \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_n = 0 | R_k = 0) \\
&\quad ;\text{Remark 3.3} \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_{n-k} = 0 | R_0 = 0) \\
&\quad ;\text{Time homogenous property} \\
&= \sum_{k=1}^{n} \mathbb{P}(N_r = k)\mathbb{P}(R_{n-k} = 0) \\
&\quad ;\mathbb{P}(R_0 = 0) = 1.
\end{aligned}$$

□

The forthcoming proposition represents a significant property of N_r, which will be instrumental in evaluating both the expected value and variance of the hitting time in the next section.

Proposition 5.5. *For $n \geq d + 1$ and $r \in V$, the following inequality holds:*

$$\mathbb{P}(N_r = n) < \left(\frac{m^d - 1}{m^d}\right)^{\frac{n-1}{d} - 1}. \tag{5.3}$$

Proof. Let E denote the following event:

$$\forall k \in \left\{0, 1, ..., \left\lfloor \frac{n-1}{d} \right\rfloor - 1\right\}, X_{dk+1} \neq 1 \text{ or } X_{dk+2} \neq 1 \text{ or } \ldots \text{ or } X_{dk+d} \neq 1.$$

First, we will show that if the event $N_r = n$ occurs, then E occurs. This claim can conclude that $\mathbb{P}(N_r = n) \leq \mathbb{P}(E)$. Using a contrapositive proof, suppose that E does not occur, or in other words:

$$\exists k \in \left\{0, 1, ..., \left\lfloor \frac{n-1}{d} \right\rfloor - 1\right\},$$
$$X_{dk+1} = X_{dk+2} = \cdots = X_{dk+d} = 1.$$

Consequently, there exists $s \in \{1, 2, ..., d\}$ such that $R_{dk+s} = r$. Hence, $N_r \leq dk + s < n$, and the event $N_r = n$ does not occur. Thus, the claim is proven. Next, we will show that $\mathbb{P}(E) = \left(\frac{m^d-1}{m^d}\right)^{\lfloor \frac{n-1}{d} \rfloor}$. Consider each $k \in \{0, 1, ..., \lfloor \frac{n-1}{d} \rfloor - 1\}$, there are m^d possible outcomes for

$$X_{dk+1}, X_{dk+2}, ..., X_{dk+d} \in \{1, 2, ..., m\}.$$

Among these outcomes, there are $m^d - 1$ possibilities such that

$$X_{dk+1} \neq 1 \text{ or } X_{dk+2} \neq 1 \text{ or}$$
$$\ldots \text{ or } X_{dk+d} \neq 1.$$

Since there are $\lfloor \frac{n-1}{d} \rfloor$ values of $k \in \{0, 1, ..., \lfloor \frac{n-1}{d} \rfloor - 1\}$, we obtain a total of $m^{d\lfloor \frac{n-1}{d} \rfloor}$ possible outcomes in the sample space, and $(m^d - 1)^{\lfloor \frac{n-1}{d} \rfloor}$ outcomes in the event E . Since all X_j are uniformly chosen, we can establish that

$$\mathbb{P}(E) = \left(\frac{m^d - 1}{m^d}\right)^{\lfloor \frac{n-1}{d} \rfloor} < \left(\frac{m^d - 1}{m^d}\right)^{\frac{n-1}{d} - 1}.$$

Finally, we can infer that

$$\mathbb{P}(N_r = n) \leq \mathbb{P}(E) < \left(\frac{m^d - 1}{m^d}\right)^{\frac{n-1}{d} - 1}.$$

□

5.2 Expected value and variance

The expected value of the hitting time corresponds to the average number of steps required for a random walk to reach a specific state or vertex. Specifically, in the context of the forward jump random walk on a cycle graph, it denotes the average number of steps needed for the random walk to reach a particular vertex for the first time. On the other hand, the variance of the hitting time measures the dispersion or spread of the hitting time distribution. The general approach for computing the expected value and variance of the hitting time involves solving a system of linear equations. However, the methodology employed in this article revolves around utilizing generating functions and complex numbers.

Definition 5.6. For $r \in V$, we define the generating functions $G_r(z)$ and $F_r(z)$ as follows:

$$G_r(z) = 1 + \sum_{n=1}^{\infty} \mathbb{P}(R_n = r) z^n,$$
$$\text{and} \quad F_r(z) = \sum_{n=1}^{\infty} \mathbb{P}(N_r = n) z^n. \quad (5.4)$$

These generating functions correspond to the probability sequences

$$\{\mathbb{P}(R_n = r)\}_{n\geq 1} \text{ and } \{\mathbb{P}(N_r = n)\}_{n\geq 1},$$

respectively.

The expected value of N_r is equal to $F_r'(1)$. In order to ascertain the convergence of the power series $F_r'(1)$, it is imperative to establish that the radius of convergence of $F(z)$ is greater than 1. The application of Proposition 5.5 presented in the previous section leads us to the following proposition of the radius of converge.

Proposition 5.7. *For $r \in V$, the series $F_r(z)$ converges if* $|z| < \left(\frac{m^d}{m^d - 1}\right)^{\frac{1}{d}}$.

Proof. It follows from Proposition 5.5 that

$$0 \leq |\mathbb{P}(N_r = n) z^n| < \left| \left(\frac{m^d - 1}{m^d}\right)^{\frac{n-1}{d} - 1} z^n \right|,$$

where $n \geq d+1$. If $|z| < \left(\frac{m^d}{m^d-1}\right)^{\frac{1}{d}}$, then $\left|\left(\frac{m^d-1}{m^d}\right)^{\frac{1}{d}} z\right| < 1$ and

$$\begin{aligned}&\sum_{n=1}^{\infty}\left|\left(\frac{m^d-1}{m^d}\right)^{\frac{n-1}{d}-1} z^n\right| \\ &= \left(\frac{m^d-1}{m^d}\right)^{-\frac{1}{d}-1}\sum_{n=1}^{\infty}\left|\left(\frac{m^d-1}{m^d}\right)^{\frac{1}{d}} z\right|^n\end{aligned}$$

converges. Hence, $\sum_{n=1}^{\infty}|\mathbb{P}(N_r = n)z^n|$ converges by the comparison test, and $F_r(z) = \sum_{n=1}^{\infty}\mathbb{P}(N_r = n)z^n$ also converges. □

Remark 5.8. *It is well-known that the radius of convergence of a power series is equal to the radius of convergence of its derivative. Additionally, power series are continuous within their radius of convergence. It follows straightforwardly from the proposition that $z = 1$ lies within the radius of convergence of $F_r(z)$. Consequently, we conclude that*

$$\lim_{z\to 1} F_r(z) = F_r(1), \lim_{z\to 1} F_r'(z) = F_r'(1),$$

and

$$\lim_{z\to 1} F_r''(z) = F_r''(1).$$

The subsequent step entails the transformation of the power series $G_r(z)$ and $F_r(z)$ into a finite sum representation. This undertaking will commence with the proof of the necessary lemmas.

Lemma 5.9. *Let $z_1, z_2, ..., z_m \in \mathbb{C}$.*

If $|z_1| = |z_2| = \cdots = |z_m| = 1$
and $|z_1 + z_2 + \cdots + z_m| = m$,
then $z_1 = z_2 = \cdots = z_m$.

Proof. The lemma follows immediately from the Cauchy-Schwarz inequality's equality condition. □

Lemma 5.10. *For $k \in V - \{0\}$, the inequality $|v_k| < 1$ holds.*

Proof. Since $k \in V - \{0\}$, we have $w^k \neq 1$ and

$$\begin{aligned}v_k &= \frac{w^k}{m}\left(\frac{w^{mk}-1}{w^k-1}\right)\\ &= \frac{w^k}{m}(1 + w^k + w^{2k} + \cdots + w^{(m-1)k}).\end{aligned}$$

It follows that

$$\begin{aligned}|v_k| &= \frac{|1 + w^k + w^{2k} + \cdots + w^{(m-1)k}|}{m}\\ &\leq \frac{|1| + |w^k| + |w^{2k}| + \cdots + |w^{(m-1)k}|}{m}\\ &= 1.\end{aligned}$$

Hence, either $|v_k| = 1$ or $|v_k| < 1$. Assuming that $|v_k| = 1$, which is equivalent to

$$|1 + w^k + w^{2k} + \cdots + w^{(m-1)k}| = m,$$

and it implies that

$$1 = |w^k| = |w^{2k}| = \cdots = |w^{(m-1)k}|.$$

Using Lemma 5.9, we obtain

$$1 = w^k = w^{2k} = \cdots = w^{(m-1)k},$$

which implies $w^k = 1$. However, since we know that $w^k \neq 1$, the assumption made leads to a contradiction. Therefore, we can conclude that $|v_k| < 1$. □

By utilizing the lemmas, we are able to express $F_r(z)$ and $G_r(z)$ in a finite form as stated in the subsequent propositions.

Proposition 5.11. *For $r \in V$ and $z \in \mathbb{R}$ such that $0 < z < 1$, the following equation holds:*

$$G_r(z) = 1 + \frac{z}{d(1-z)} + \frac{1}{d}\sum_{k=1}^{d-1}\frac{w^{-rk}v_k z}{1-v_k z}. \tag{5.5}$$

Proof. By Theorem 4.1, it follows that

$$\begin{aligned}
&G_r(z)\\
&= 1 + \sum_{n=1}^{\infty} \mathbb{P}(R_n = r) z^n\\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{d} + \frac{1}{d} \sum_{k=1}^{d-1} w^{-rk} v_k^n \right) z^n\\
&= 1 + \frac{1}{d} \sum_{n=1}^{\infty} z^n + \frac{1}{d} \sum_{k=1}^{d-1} w^{-rk} \sum_{n=1}^{\infty} (v_k z)^n.
\end{aligned}$$

By Lemma 5.10 and $0 < z < 1$, we can conclude that

$$G_r(z) = 1 + \frac{z}{d(1-z)} + \frac{1}{d} \sum_{k=1}^{d-1} \frac{w^{-rk} v_k z}{1 - v_k z}.$$

□

Proposition 5.12. *For $r \in V$ and $z \in \mathbb{R}$ such that $0 < z < 1$, the following equation holds:*

$$F_r(z) = \frac{z + (1-z) \sum_{k=1}^{d-1} \frac{w^{-rk} v_k z}{1 - v_k z}}{z + (1-z) \left(d + \sum_{k=1}^{d-1} \frac{v_k z}{1 - v_k z} \right)}. \tag{5.6}$$

Proof. By applying the Cauchy product theorem and Theorem 5.4, we can establish the equality:

$$\begin{aligned}
F_r(z) G_0(z) &= \left(\sum_{n=1}^{\infty} \mathbb{P}(N_r = n) z^n \right)\\
&\quad \times \left(1 + \sum_{n=1}^{\infty} \mathbb{P}(R_n = 0) z^n \right)\\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \mathbb{P}(N_r = k) \mathbb{P}(R_{n-k} = 0) \right) z^n\\
&= \sum_{n=1}^{\infty} \mathbb{P}(R_n = r) z^n\\
&= G_r(z) - 1.
\end{aligned}$$

It is evident that $G_0(z) \neq 0$, as $z > 0$ and $\mathbb{P}(R_n = 0) \geq 0$ for all $n \geq 1$. Therefore, $G_0(z)$ is a divisor, and we will have

$$\begin{aligned}
F_r(z) &= \frac{G_r(z) - 1}{G_0(z)}\\
&= \frac{\frac{z}{d(1-z)} + \frac{1}{d} \sum_{k=1}^{d-1} \frac{w^{-rk} v_k z}{1 - v_k z}}{1 + \frac{z}{d(1-z)} + \frac{1}{d} \sum_{k=1}^{d-1} \frac{v_k z}{1 - v_k z}};
\end{aligned}$$

Proposition 5.11

$$= \frac{z + (1-z) \sum_{k=1}^{d-1} \frac{w^{-rk} v_k z}{1 - v_k z}}{z + (1-z) \left(d + \sum_{k=1}^{d-1} \frac{v_k z}{1 - v_k z} \right)}.$$

□

Based on the proposed propositions, the expected values of the hitting time can be derived as stated in the following theorem.

Theorem 5.13 (Expected value of N_r). *For $r \in V$, the expected value of random variable N_r is given by:*

$$\mathbb{E}[N_r] = d + \sum_{k=1}^{d-1} \frac{(1 - w^{-rk}) v_k}{1 - v_k}. \tag{5.7}$$

Proof. We are aware that

$$\mathbb{E}[N_r] = \sum_{n=1}^{\infty} n \mathbb{P}(N_r = n) = F_r'(1).$$

The task at hand is to determine the value of $F_r'(1)$. Let us define

$$f(z) = \sum_{k=1}^{d-1} \frac{w^{-rk} v_k z}{1 - v_k z}$$

and

$$h(z) = d + \sum_{k=1}^{d-1} \frac{v_k z}{1 - v_k z}$$

for $0 < z \leq 1$. Consequently, we have

$$f(1) = \sum_{k=1}^{d-1} \frac{w^{-rk} v_k}{1 - v_k},$$

$$f'(1) = \sum_{k=1}^{d-1} \frac{w^{-rk} v_k}{(1 - v_k)^2},$$

$$h(1) = d + \sum_{k=1}^{d-1} \frac{v_k}{1 - v_k}, \text{ and}$$

$$h'(1) = \sum_{k=1}^{d-1} \frac{v_k}{(1 - v_k)^2}.$$

Proposition 5.12 allows us to express

$$F_r(z) = \frac{z + (1-z)f(z)}{z + (1-z)h(z)} \text{ for } 0 < z < 1.$$

By differentiating $F_r(z)$, we obtain

$$F_r'(z) = \Big(1 + (1-z)f'(z) - f(z) - F_r(z) \times (1 + (1-z)h'(z) - h(z))\Big) \times \Big(z + (1-z)h(z)\Big)^{-1}$$

for $0 < z < 1$. Based on the observation made in Remark 5.8, we are able to take the limit as $z \to 1^-$ on both sides of the aforementioned equations. Consequently, we obtain the result that $F_r(1) = 1$ and $F_r'(1) = d + \sum_{k=1}^{d-1} \frac{(1 - w^{-rk})v_k}{1 - v_k}$. □

To express $\mathbb{E}[N_r]$ in terms of a trigonometric sum, we can proceed as illustrated in the subsequent corollary.

Corollary 5.14. *For $r \in V$,*

$$\mathbb{E}[N_r] = d - 2\sum_{k=1}^{d-1} \sin\left(\frac{mk\pi}{d}\right)\sin\left(\frac{rk\pi}{d}\right) \left(m \sin\left(\frac{(m-r+1)k\pi}{d}\right)\sin\left(\frac{k\pi}{d}\right) + \sin\left(\frac{mk\pi}{d}\right)\sin\left(\frac{rk\pi}{d}\right)\right) \cdot \left(\sin^2\left(\frac{mk\pi}{d}\right) - 2m\cos\left(\frac{(m+1)k\pi}{d}\right)\sin\left(\frac{mk\pi}{d}\right)\sin\left(\frac{k\pi}{d}\right) + m^2\sin^2\left(\frac{k\pi}{d}\right)\right)^{-1}.$$

Proof. It is evident that $\mathbb{E}[N_r]$ must be a real number, since $N_0, N_1, ..., N_{d-1}$ are integers. According to Theorem 5.13, we have

$$\mathbb{E}[N_r] = \mathrm{Re}(\mathbb{E}[N_r]) = d + \sum_{k=1}^{d-1} \mathrm{Re}\left(\frac{(1 - w^{-rk})v_k}{1 - v_k}\right). \tag{5.8}$$

For each $k \in V - \{0\}$, we will find the real part of $\frac{(1 - w^{-rk})v_k}{1 - v_k}$. For simplicity, let us define, for all j,

$$S_j = \sin\left(\frac{jk\pi}{d}\right) \text{ and } C_j = \cos\left(\frac{jk\pi}{d}\right).$$

By applying the identity

$$e^{\theta i} - 1 = 2ie^{\frac{\theta}{2}i}\sin\frac{\theta}{2},$$

we can evaluate that

$$\frac{(1 - w^{-rk})v_k}{1 - v_k} = \frac{2iw^{\frac{(m-r+1)k}{2}} S_m S_r}{mS_1 - w^{\frac{(m+1)k}{2}} S_m}.$$

Furthermore, let

$$z_1 = 2iw^{\frac{(m-r+1)k}{2}} S_m S_r,$$

and

$$z_2 = mS_1 - w^{\frac{(m+1)k}{2}} S_m.$$

Consequently we have

$$\begin{aligned}
\operatorname{Re}(z_1) &= -2S_{m-r+1}S_mS_r,\\
\operatorname{Re}(z_2) &= mS_1 - C_{m+1}S_m,\\
\operatorname{Im}(z_1) &= 2C_{m-r+1}S_mS_r, \text{ and}\\
\operatorname{Im}(z_2) &= -S_{m+1}S_m.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\operatorname{Re}\left(\frac{(1-w^{-rk})v_k}{1-v_k}\right) = \operatorname{Re}\left(\frac{z_1}{z_2}\right)\\
&= \frac{\operatorname{Re}(z_1)\operatorname{Re}(z_2) + \operatorname{Im}(z_1)\operatorname{Im}(z_2)}{(\operatorname{Re}(z_2))^2 + (\operatorname{Im}(z_2))^2}\\
&= \Big(-2S_{m-r+1}S_mS_r(mS_1 - C_{m+1}S_m)\\
&\qquad - 2C_{m-r+1}S_{m+1}S_m^2S_r \Big)\\
&\quad \times \left((mS_1 - C_{m+1}S_m)^2 + S_{m+1}^2S_m^2\right)^{-1}\\
&= \Big(-2mS_{m-r+1}S_mS_rS_1 + 2S_m^2S_r\\
&\quad \times (S_{m-r+1}C_{m+1} - C_{m-r+1}S_{m+1}) \Big)\\
&\quad \times \left(m^2S_1^2 - 2mC_{m+1}S_mS_1 + S_m^2\right)^{-1}\\
&= -\frac{2S_mS_r(mS_{m-r+1}S_1 + S_mS_r)}{m^2S_1^2 - 2mC_{m+1}S_mS_1 + S_m^2}.
\end{aligned}$$

By substituting (5.9) back into (5.8), we can obtain the result. □

Finally, we proceed to derive the variance of the hitting time as follows:

Theorem 5.15 (Variance of N_r). *For $r \in V$, the variance of random variable N_r is :*

$$\begin{aligned}
\boldsymbol{Var}[N_r] &= 2\sum_{k=1}^{d-1} \frac{(1-w^{-rk})v_k}{(1-v_k)^2}\\
&+ \left(d + \sum_{k=1}^{d-1}\frac{(1-w^{-rk})v_k}{1-v_k}\right)\\
&\cdot \left(d - 1 + \sum_{k=1}^{d-1}\frac{(1+w^{-rk})v_k}{1-v_k}\right).
\end{aligned} \tag{5.9}$$

Proof. Using the definition of the expected value, we can deduce that $\mathbb{E}[N_r] = F_r'(1)$, and

$$\mathbb{E}[N_r^2] = \sum_{n=1}^{\infty} n^2\mathbb{P}(N_r = n) = F_r''(1) + F_r'(1).$$

Hence, we can determine that

$$\begin{aligned}
\mathbf{Var}[N_r] &= \mathbb{E}[N_r^2] - \mathbb{E}[N_r]^2\\
&= F_r''(1) + F_r'(1) - (F_r'(1))^2.
\end{aligned}$$

By defining $f(z)$ and $h(z)$ in the same manner as in the proof of Theorem 5.13, we establish the relationship

$$\begin{aligned}
F_r''(z) = &\Big((1-z)f''(z) - 2f'(z)\\
&- F_r(z)((1-z)h''(z) - 2h'(z))\\
&- 2F_r'(z)(1 + (1-z)h'(z) - h(z))\Big)\\
&\times \left(z + (1-z)h(z)\right)^{-1}.
\end{aligned}$$

Taking limit as $z \to 1^-$ allows us to obtain

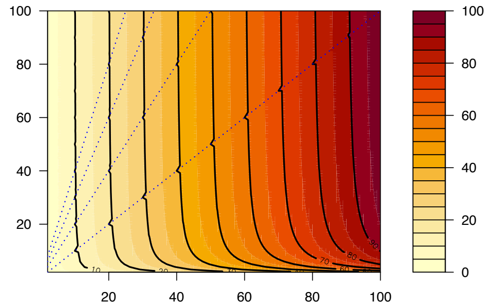
$$\begin{aligned}
F_r''(1) = &-2f'(1) + 2h'(1)\\
&- 2F_r'(1)(1 - h(1)).
\end{aligned}$$

Therefore,

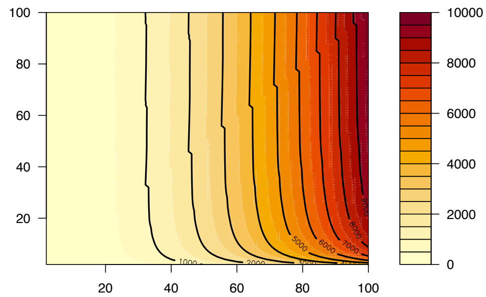
$$\begin{aligned}
\mathbf{Var}[N_r] &= F_r''(1) + F_r'(1) - (F_r'(1))^2\\
&= -2f'(1) + 2h'(1) - 2F_r'(1)(1 - h(1))\\
&\quad + F_r'(1) - (F_r'(1))^2\\
&= 2(h'(1) - f'(1)) + F_r'(1)(2h(1)\\
&\quad - 1 - F_r'(1)).
\end{aligned}$$

□

We have omitted the illustration of the proof for the second moment of the hitting time in terms of a trigonometric sum, as it involves a complicated form. Fig. 2 presents heatmap and contour plots illustrating the expected hitting time and variance. The plots explore the range of values for m and d from 2 to 100, with $r = 1$ chosen for simplicity due to the symmetric nature of the cycle graph's vertices. The blue lines on the plot indicate points where m and d are divisible. We also plotted values of r other than $r = 1$ and observed that the patterns did not differ significantly from the case of $r = 1$. The simulation results can be reproduced using the code available in our GitHub repository. Consequently, we have chosen not to display the plots for the other r values. The results demonstrate a general increasing trend in both the expected value and variance as m and d increase. Notably, the impact of increasing m on the expected value and variance is more pronounced compared to the minimal effect of increaseing d, as evidenced by the vertical color pattern. Additionally, subtle variations in the contours are observed at the points where m and d are divisible. Specifically, when d is held constant, the expected hitting time at these points slightly surpasses the neighboring values.



(a) Expected value



(b) Variance

Fig. 2. Heatmap and contour plot of the expected variance of hitting time, displaying d values on the vertical axis and value of m on the horizontal axis ranging from 2 to 100, where $r = 1$.

6. Experiment on Board Game with Circular Walk

Board games have been extensively studied in mathematics education due to their effectiveness in teaching various mathematical concepts, including probability and linear algebra [15, 17–19]. Notably, they offer valuable opportunities for simulating real-world problems, as demonstrated by the Hunter vs. Rabbit game [8].

In this particular study, our focus is on analyzing the random walk on a board game with a circular walk. The game board consists of a square grid with $d = 12$ square s, each labeled from 0 to 11. The objective is to investigate the behavior of players as they engage in the game. A key rule of the game involves rolling an m-sided die, where m can have values of 4, 6, 8, 10, 12, or 20, that are the commonly used physical dice shapes. The number rolled on the die determines the number of spaces a player advances on the board. We chose a board size of $d = 12$ in order to investigate the behavior of PMFs under different conditions. Specifically, our objective is to analyze scenarios where d is less than, equal to, and greater than the number of sides on the die (m). By examining these cases, we can gain insights into how the probability distributions evolve and vary based on the relationship between d and m, in which $d < m$, $d = m$, and $d > m$.

Through an analysis of this random walk process, our objective is to gain insights into the dynamics and characteristics of the random walk. To achieve this, we will address the following key questions:

1. Does the random walker visit each square on the board with equal probability in the long run?

2. How does the probability of the walker's first visit to each square change as the number of walking steps increases, and what are the different behaviors exhibited by different squares?

3. How many steps does it take for the walker to reach a square for the first time, and does this duration depend on the value of the maximum forward movement m?

4. How do the probability distribution and hitting time evolve and vary based on the relationship between the number of vertices or states d and the maximum forward movement m?

The PMF of R_n, the hitting time, as well as its expected value and variance, were calculated for different values of m and n, displayed in Figs. 3, 4, and 5. Specifically, we considered $d = 12$ and $m \in \{4, 6, 8, 10, 12, 20\}$, and $n \in \{1, 2, 4, 8, 16, 32\}$. We have also provided the R code for reproducing the simulation on GitHub repository: https://github.com/QuantFILab/Forward-Jump-Random-Walk and https://github.com/QuantFILab/Forward-Jump-Random-Walk.

In Fig. 3, the PMFs of the arriving state are displayed in a grid format. The columns represent the dice faces, while each row corresponds to a specific number of walking steps. In the first walking step, the probabilities of visiting each state are equal when $d \leq m$. However, if d exceeds m, certain square s may have zero chance of being visited due to the maximum length of the jump. On the other hand, when $m = 20$, some square s have a higher chance of being visited due to the circular nature of the walk, allowing for revisiting of certain states.

For a few early walking steps, such as $n = 2$ or $n = 3$, the distribution exhibits a bell-shaped pattern with a single peak. However, in the long term, the walker will visit each square equally. In other words, the results demonstrate that as the values of m or n increase, the distribution of R_n tends to approximate a uniform distribution. This observation is consistent with Eq. (4.3), where the denominator m^n becomes large, resulting in each R_n having a probability that approaches $1/m$. The convergence rate to the uniform distribution is faster with increasing m, where m dominates d, and it becomes instantaneous when $d = m$.

Fig. 4 presents the hitting times, N_r, in a grid format. Each column in the grid represents a dice face with values of $m = 2$, 6, 8, 10, 12, and 20, while the rows correspond to the square numbers, denoted as $n = 1, 2, 3, \ldots, 17$. It is evident that the probabilities of first visiting the squares starting from the reference square, $X_1 = 0$, vary for each square, except in the case of uniform walking when $m = d = 12$. Notably, the probabilities of first visiting decline exponentially as the number of walking steps increases. Additionally, in the case where $m < d$, the distribution exhibits a right-hand shift due to the symmetry of the cycle graph and the possibility of shifting the ref-

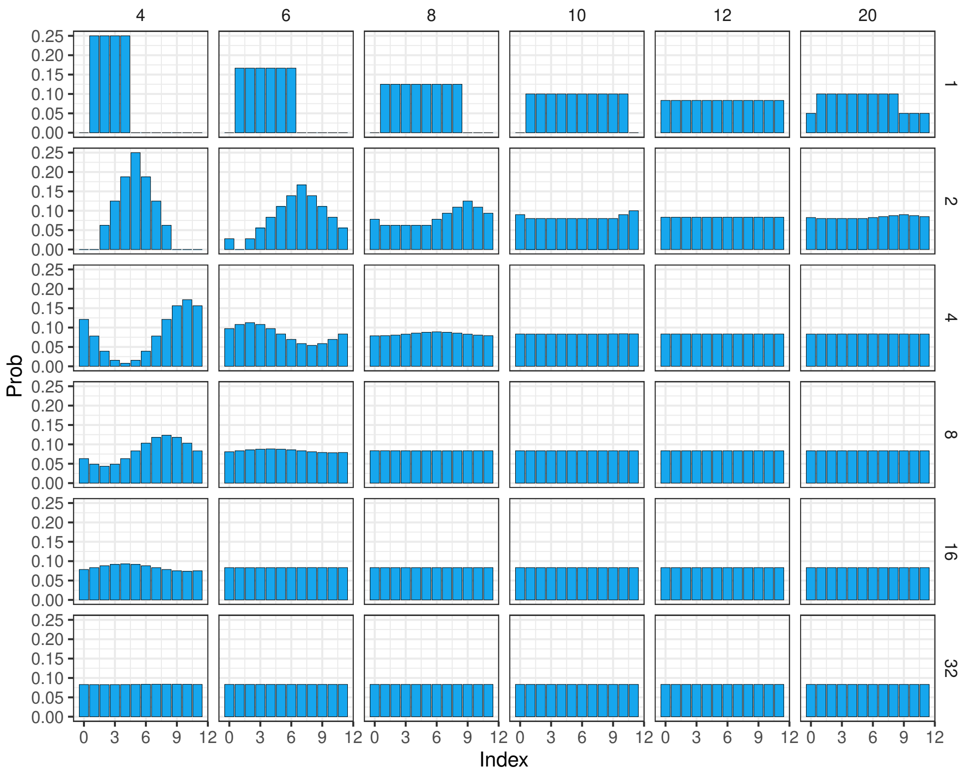


Fig. 3. Probability mass functions of the arriving state, each column in the grid represents a dice face, with values of $m = 2, 6, 8, 10, 12$, and 20. The rows in the grid correspond to the number of walking steps, denoted as $n = 1, 2, 4, 8, 16, 32$.

erence vertex, as discussed in Remark 3.3. The distributions where $m < d$ appear to be multimodal since they require multiple steps of walking to reach the target square. Furthermore, the distributions are smooth in the early steps when $m \geq d$ since the walker has a chance to jump to any square on the board.

In Fig. 5, the expected value and standard deviation of the hitting time are displayed in a grid format. Each column represents a dice face with values of $m = 2, 6, 8, 10, 12$, and 20. The expected values of the hitting time for the board size $d = 12$ range from 7.75 to 12 walking steps. The expected values differ more significantly as the difference between m and d increases, particularly in the case of $m < d$. Specifically, the board with a 12-sided dice has the longest expected walking step at 12 steps, while the fastest is with a 4-sided dice at 7.75 steps. Moreover, the results demonstrate that a larger possible jump of the walker leads to a longer average time to visit a certain square for the first time, except in the case when $d = m = 12$. Similar to the standard deviation, a larger possible jump of the walker leads to increased variation in the average time taken to first visit a certain square.

In conclusion, our analysis of the random walk on a circular board game has yielded findings. We addressed four key questions and obtained the following in-

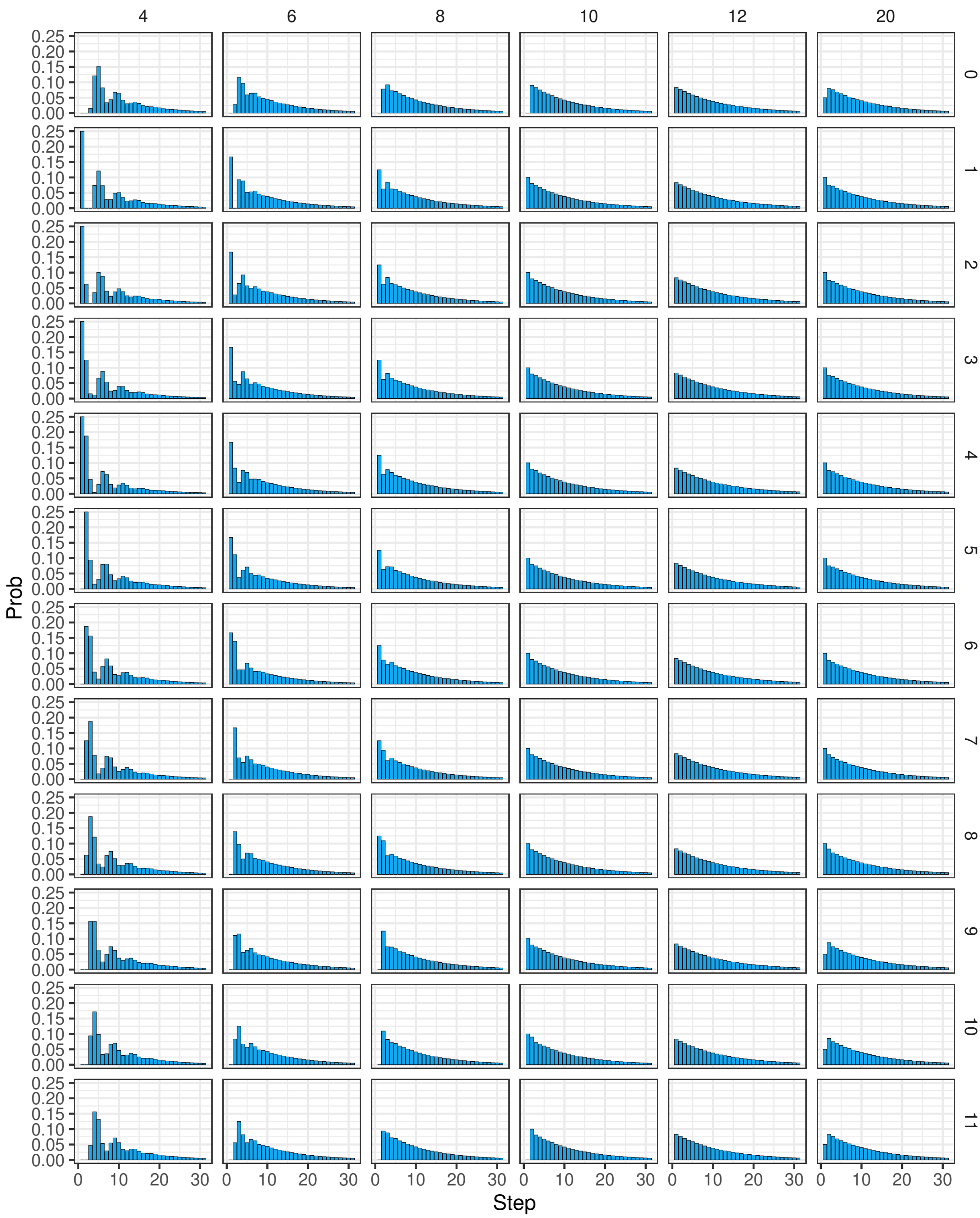


Fig. 4. Hitting times, each column in the grid represents a dice face, with values of m = 2, 6, 8, 10, 12, and 20. The rows in the grid correspond to the square 's number, denoted as $n = 1, 2, 3, \ldots, 17$.

sights: Firstly, all walkers eventually visit each square uniformly in the long run, with faster convergence for higher values of m or n. Secondly, the number of steps needed for a walker to reach a square for the first time depends on m. The expected hitting

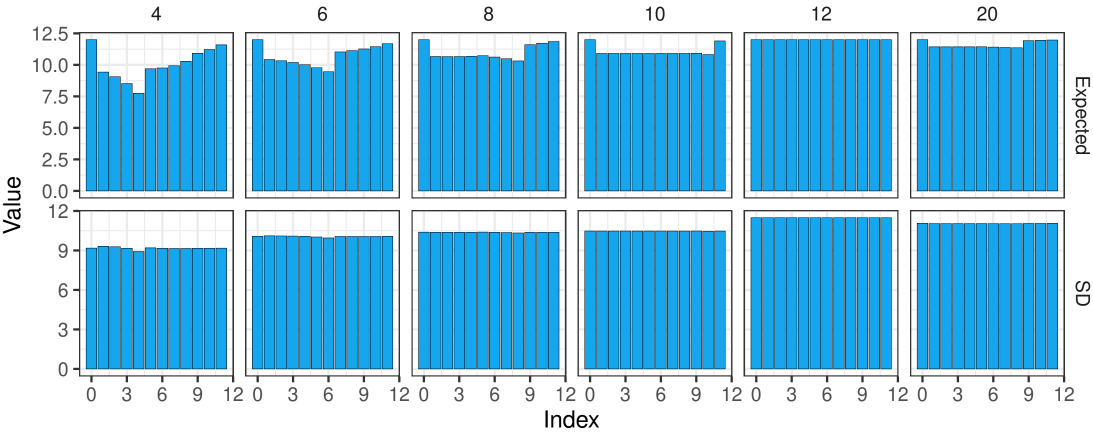


Fig. 5. Expected value and standard deviation of the hitting time, each column in the grid represents a dice face, with values of m = 2, 6, 8, 10, 12, and 20.

time varies significantly as the gap between m and d increases. The probability distribution and hitting time evolve and vary based on the relationship between d and m. When $m < d$, the distribution exhibits a right-hand shift due to graph symmetry and shifting the reference vertex, resulting in multimodal distributions requiring multiple steps. Conversely, when $m \geq d$, the distributions are smooth initially as the walker can jump to any square on the board with equal probability.

7. Conclusion

In conclusion, this study has made contributions to the analysis of the forward jump random walk on a cycle graph and its application in a board game context. By deriving analytic formulas for the PMF of the arriving state, the hitting time, and its expected value and variance, we have provided insights into the dynamics of random walks on cycle graphs. The use of a combinatorial method for formula derivation offers an alternative approach to existing eigenvector-based methods and expands the toolbox of techniques for analyzing random walks.

The findings of the analysis reveal the dependence of the random walk's behavior on key parameters, including the maximum jump length (m), the number of vertices or states (d), and the number of walking steps (n). The observation that the PMF of the arriving state converges to a uniform distribution as m or n increases implies that, in the long run, the random walk will equally visit all states. This insight enhances our understanding of the equilibrium properties of random walks on cycle graphs. Moreover, the sensitivity of the expected value and variance of the hitting time to changes in m, along with the minimal effect of changes in d, highlights the impact of jump length on average hitting times. These findings provide practical implications for predicting and analyzing hitting times in various scenarios, particularly those involving larger jumps. By observing the behavior of the random walk within a game context, we illustrate how the formulas can be applied to analyze movement and exploration patterns. This application opens up possibilities for further investigations into random walks in different game setups or even in broader fields such as simulation modeling.

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