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New Existence and Stability Results of a Backward Impulsive FDEs

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ABSTRACT

This article focuses on studying a type of non-local and local backward problem involving impulsive fractional differential equations. The equations include a special type of derivative called the $\chi-$Caputo fractional derivative. The use of Krasnoselskii's fixed-point theorem and Banach's contraction principle helps us prove that there is only one solution and it definitely exists. In addition, we found some results about the stability of Hyers-Ulam and generalized Hyers-Ulam equations. Finally, some examples are given to demonstrate that the results are correct.

Keywords: $\chi-$Caputo fractional derative; FDEs; Generalized Hyers-Ulam stability; Local and non-local backward problem

1. Introduction

Usually, to describe, understand and analyze a specific phenomenon that surrounds our reality, which is set according to certain controls, we resort to a set of rules formulated in mathematical laws, operations and various procedures that can be deduced through the quantitative or qualitative study of the problem that we summarize in specific and various mathematical equations related to the nature of the problem or phenomenon studied, in order to be more specific, the type of model chosen, which is often represented by a set of differential equations (DEs), partial DEs, or systems of DEs within a set of conditions whether they are local or nonlocal initial conditions, final conditions etc. [1–5].

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Mathematical models of reality are the most important types of representation. Essentially, anything in the physical or biological world, whether natural or involving technology and human intervention, is subject to analysis by the models if it can be described in terms of mathematical expressions. Thus, optimization and control theory can be used to model industrial processes, traffic patterns, and sediment transport in streams and other situations; Information theory can be used to model message transmission, linguistic properties, and the like; Dimensional analysis and computer simulations can be used to model patterns of atmospheric circulation, stress distribution in engineering structures, growth and evolution of terrain, and a host of other processes in science and engineering.

The accuracy of our understanding and description of the original problem depends on several fundamental things, including the nature and style of the mathematical model chosen, as well as the discovery of new features about the phenomenon. In other words, the nature of the phenomenon dictates to us the chosen model, and not vice versa. Fractional calculus is about using integrals and derivatives of fractions instead of integer numbers. It is a good way to describe different kinds of operators that don't use the numbers [6–10].

Fractional calculus is non integer mathematical models allows us to describe the memory and hereditary properties of various phenomena and effects in natural and social sciences. For example, we should note the non-locality of power-law type, spatial dispersion of power type, fading memory, frequency dispersion of power type, intrinsic dissipation, the openness of systems (interaction with environment), fractional relaxation-oscillation, fractional viscoelasticity, fractional diffusion-waves, long-range interactions of power-law type, and many others see [11].

Among a several approaches to fractional derivative which are exist, for example, Caputo, Caputo-Fabrizio, Riemann-Liouville (RL), Hadamard, Caputo-Hadamard, Erdélyi-Kober, Caputo-Erdélyi-Kober. We refer the reader to monographs and papers [12–15]. There is a special kind of a kernel dependency. Therefore, in order to analyze fractional differential equations (FDEs) in a generic way, a fractional derivative with respect to another function, called $\chi-$Caputo derivative, was proposed [16]. This type of differentiation depends on a kernel , and for particular choices of χ function, we obtain some well known fractional derivatives like Caputo, Caputo-Hadamard or Caputo-Erdélyi-Kober fractional derivatives. This approach seems also suitable from the applications point of view [17–20]. $\chi-$Caputo derivative makes possible to control, in some sense, the process of modeling of the considered phenomenon by means of a proper choice of a "trial" function.

An explanation and a discussion about interrelations between the properties of the operator's kernels and the types of phenomena are given by [21] in the form of answer to a raised question in the title of their paper, we must know that, not all fractional derivatives and integrals can be used for modeling the processes with memory and hereditary properties. For example, the Kober and Erdélyi-Kober operators as well as the Caputo-Fabrizio integral and derivatives cannot be applied to describe phenomena with memory or spatial non-locality. These operators can be applied only to describe processes with continuously distributed scaling (dilation) and lag (delay), respectively [22, 23].

In recent years, there have been many results obtained on the existence, multiplicity and uniqueness of solutions of initial or boundary value problems (BVPs) for nonlinear FDEs is extensively studied using various tools of nonlinear analysis as fixed point (FP) theorems, degree theory and the method of upper and lower solutions, see [24–26]. The idea behind them is trying to find solutions of a given BVP by looking for FPs of a suitable functional defined on an appropriate function space. In the last 30 years, the FP theory has become a wonderful tool in studying the existence of solutions to DEs with integral structures, we refer the reader to the books due to [27,28].

In the other hand, the theory of impulsive DEs can be represented as an adequate mathematical model for describing many processes and phenomena in the real world, which are subjected during their development to short-term external influences. But, this duration is often negligible compared with the total duration of the studied phenomena and processes. Hence, it can be seen that these external effects are "instantaneous or not", i.e. those are in the form of impulses [29]. Thus, such processes tend to be more suitably modeled by impulsive DEs, which allow for discontinuities in the evolution of the state. Further, the DEs with impulsive effects often arise naturally, in chemical technology, population dynamics, physics, aeronautics, biotechnology, chemotherapy, optimal control, ecology, economics, engineering and describe the dynamics of processes in which sudden, discontinuous jumps occur [30–35].

In the literature, there have been many authors are interested in the solvability of fractional BVPs with impulses. Ahmad et al. studied the existence of solutions for impulsive anti-periodic BVP of fractional order [5]. Recently, Bonanno et al. in [36], and Rodríguez-López and Tersian in [37], considered the solvability of the same type of certain Dirichlet's BVP for FDEs with impulses, by using variational methods and a three critical points theorem. Lazreg et al. deal with some impulsive Caputo-Fabrizio FDEs in b-metric spaces by the using of $\alpha - \varphi$-Geraghty-type contraction [38].

However, as far as we know, there are few results on the existence and stability of solutions to backward impulsive differential equations. In this paper, By using some well known classical FP theorems, we establish the existence theory for the considered problems. Also we develop some results for Hyers-Ulam (U.H) and generalized (G.U.H) stabilities. Pertinent example is given to verify our results.

The main aim has two parts. We consider the following local backward problems for nonlinear impulsive FDE,

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\varkappa(q) = \phi\big(q, \varkappa(q)\big), \\ \triangle\varkappa\big|_{q=q_k} = \varrho_k\big(\varkappa(q_k^-)\big), \end{cases} \tag{1.1}$$

for $q \in \mathbb{J} = [\mathfrak{b}, \mathfrak{c}] \subset (0, \infty)$, $q \neq q_k$, $k \in \mathbb{K}_m = \{1, 2, \cdots, m\}$, with two cases local and nonlocal of initial value,

$$\varkappa(\mathfrak{c}) = \varkappa_{\mathfrak{c}}, \tag{1.2}$$

$$\varkappa(\mathfrak{c}) = \varkappa_{\mathfrak{c}} + \omega(\varkappa), \tag{1.3}$$

and we discuss Ulam stability the equation in the second case, where ${}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}$ is the χ−Caputo fractional derivative of order $0 < \upsilon < 1$, $\chi \in C^1(\mathbb{J})$ is an increasing function such that $\chi'(q) \neq 0$ for each $q \in \mathbb{J}$,

$$\mathfrak{b} = q_0 < q_1 < q_2 < \cdots < q_m < q_{m+1} = \mathfrak{c},$$

where $\triangle\varkappa|_{q=q_k}$ represent change of right and left hand limit of the discontinuity points at q_k, it is define as

$$\triangle\varkappa\big|_{q=q_k} = \varkappa(q_k^+) - \varkappa(q_k^-),$$

$\varkappa_{\mathfrak{c}} \in \mathbb{R}$, , ϱ_k, $k \in \mathbb{K}_m$, ϕ and ω in $C(\mathbb{J} \times \mathbb{R})$ and $C(\mathbb{J})$, respectively.

The content of the paper is organized as follows: In Sect. 2, some definitions, concepts and preparation results are given. In Sect. 3, we introduce a concept of a piecewise continuous solution for local backward Eq. (1.1) and Eq. (1.2) and give existence, uniqueness results of solutions, by using FP theorems. In Sect. 4, we extend a concept of a piecewise continuous solution for local backward Eq. (1.1) and Eq. (1.3) and give existence, uniqueness results of solutions to a nonlocal backward the problem via some FP theorems. In Sect. 5, we give four types of Ulam stability definitions ot generation of nonlocal FDEs: U.H, generalized U.H, Ulam-Hyers-Rassias (U.H.R) and generalized U.H.R stabilities. We present the four types of Ulam stability results for nonlocal impulsive FDE in Eq. (1.1) and Eq. (1.3) via the generalized Grönwall inequality through the fractional integral with respect to another function. Some examples are given in Sect. 6 to demonstrate the application of our main results.

2. Preliminaries

Let $\boldsymbol{E}_{\mathrm{B}}$ be a Banach space. We consider spaces $C(\mathbb{J})$,

$$PC(\mathbb{J}) = \Big\{\varkappa \in C(\mathbb{J} \setminus q_k) \,:\, \varkappa(q_k^-) = \varkappa(q_k), \\ \varkappa(q_k^+) \text{ exists}, k = 0, 1, \cdots, m\Big\},$$

together the norm

$$\|q\|_{PC} = \|\varkappa\|_C = \max_{q \in \mathrm{J}} |\varkappa(q)|,$$

$$C^+(\mathbb{J}) = \big\{\varkappa \in C(\mathbb{J}) \,:\, \varkappa(q) \geq 0, \forall q \in \mathbb{J}\big\},$$

$C^n(\mathbb{J})$ denote the spaces of n times continuously differentiable functions on $\mathbb{J}$ and $L^p(\mathfrak{b}, \mathfrak{c})$ denotes the space of Lebesgue integrable functions on $(\mathfrak{b}, \mathfrak{c})$.

Below we formulate some known results.

Theorem 2.1 (Ascoli-Arzelà). *Let K is compact and $W \subseteq C(K)$. Then W is compact if and only if W is closed, bounded, and equi-continuous.*

Theorem 2.2 (PC-type Ascoli-Arzelà). *Consider uniformly bounded subset $\mathfrak{W} \subset PC(\mathbb{J})$ such that,*
i) $\mathfrak{W}$ is equi-continuous in (q_k, q_{k+1}), $k \in \{0\} \cup \mathbb{K}_m$, where $q_0 = \mathfrak{b}$, $q_{m+1} = \mathfrak{c}$;
ii)

$$\mathfrak{W}(q) = \big\{\varkappa(q) \,:\, \varkappa \in \mathfrak{W},\ \mathbb{J}\backslash\{q_1, \cdots, q_m\}\big\},$$

and

$$\mathfrak{W}(q_k^+) = \Big\{\varkappa(q_k^+) \,:\, \varkappa \in \mathfrak{W}\Big\},$$
$$\mathfrak{W}(q_k^-) = \Big\{\varkappa(q_k^-) \,:\, \varkappa \in \mathfrak{W}\Big\},$$

are relatively compact subset of $\boldsymbol{E}_{\mathrm{B}}$.
Then $\mathfrak{W}$ is a relatively compact subset of $PC(\mathbb{J})$.

Theorem 2.3 (Schaefer). *Consider the convex subset $W \subseteq \boldsymbol{E}_{\mathrm{B}}$, which it contains zero and a completely continuous operator $O : W \to W$. If*

$$\Omega = \Big\{\varkappa \in U \,:\, \varkappa = \gamma O\varkappa,\ 0 \leq \gamma \leq 1\Big\},$$

is bounded, then O admits at least one FP in $\boldsymbol{E}_{\mathrm{B}}$.

Theorem 2.4 (Krasnoselskii). *Let $\emptyset \neq W \subseteq \boldsymbol{E}_{\mathrm{B}}$ be a closed convex and O_1, O_2 be two operators satisfying the conditions:*
i) $O_1\varkappa + O_2\acute{\varkappa} \in W$, for $\varkappa, \acute{\varkappa} \in W$;
ii) O_1 is a contraction mapping;
iii) O_2 is compact and continuous.
Then there exist $\mathring{\varkappa} \in W$ such that,

$$\mathring{\varkappa} = O_1\mathring{\varkappa} + O_2\mathring{\varkappa},$$

i.e, the operator $O_1 + O_2$ admits a FP on W.

The RL and $\chi-$RL fractional integral of order $\upsilon > 0$ are defined by,

$$\mathcal{I}_{\mathfrak{b}}^{\upsilon}\varkappa(q) = \int_{\mathfrak{b}}^{q} \frac{(q-\lambda)^{\upsilon-1}}{\Gamma(\upsilon)} q(\lambda)\,\mathrm{d}\lambda,$$

for $\varkappa \in PC(\mathbb{J})$;

$${}^{\chi}\mathcal{I}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q) = \int_{\mathfrak{b}}^{q} \frac{(\chi_{\lambda}(q))^{\upsilon-1}\chi'(\lambda)}{\Gamma(\upsilon)} \varkappa(q)\,\mathrm{d}\lambda,$$

for $\varkappa \in C(\mathbb{J})$, respectively, where

$$\chi_{\lambda}(q) := \chi(q) - \chi(\lambda),$$

and $\chi \in C^1(\mathbb{J})$ is an increasing function with $\chi'(q) \neq 0$ for all $q \in \mathbb{J}$ [9, 10, 16, 39]. The $\chi-$Caputo fractional derivative of the function $\varkappa \in C^n(\mathbb{J})$ of order $n-1 < \upsilon < n$ to the real function $\varkappa$ is defined as follows

$${}^{\chi}_{\mathrm{C}}\mathrm{D}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q) = \int_{\mathfrak{b}}^{q} \frac{(\chi_{\lambda}(q))^{\upsilon-1}\chi'(\lambda)}{\Gamma(n-\upsilon)} \varkappa_{\chi}^{[n]}(\lambda)\,\mathrm{d}\lambda,$$

$$\varkappa_{\chi}^{[n]}(q) = \left[\frac{1}{\chi'(q)}\frac{\mathrm{d}}{\mathrm{d}q}\right]^n \varkappa(q),$$

where $\chi \in C^n(\mathbb{J})$ with $\chi'(q) \neq 0$ for all $q \in \mathbb{J}$, $[\upsilon]$ is the largest integer less than or equal to υ [16, 39]. Throughout the paper, we use $n = [\upsilon]$ if υ is an integer and $n = [\upsilon] + 1$ otherwise.

In the following, we present some properties for left-sided integrals and derivatives. But, the same properties are also true for the right-sided ones.

Lemma 2.5 ([16,39]). *Let $\upsilon_1, \upsilon_2 > 0$. Then*

$${}^{\chi}\mathcal{I}_{\mathfrak{b}^+}^{\upsilon_1}\left({}^{\chi}\mathcal{I}_{\mathfrak{b}^+}^{\upsilon_1}\varkappa(q)\right) = \mathcal{I}_{\mathfrak{b}^+}^{\upsilon_1+\upsilon_2}\varkappa(q),$$

a.e. $q \in \mathbb{J}$, for $\varkappa \in C(\mathbb{J})$ or the space of Lebesgue integrable functions on $(\mathfrak{b}, \mathfrak{c})$, $L^1(\mathbb{J})$. Also,

$${}^{\chi}_{\mathrm{C}}\mathrm{D}_{\mathfrak{b}^+}^{\upsilon}\left({}^{\chi}\mathcal{I}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q)\right) = \varkappa(q), \tag{2.1}$$

$\varkappa \in C(\mathbb{J})$, and for $q \in \mathbb{J}$,

$$\begin{aligned}&{}^{\chi}\mathcal{I}_{\mathfrak{b}^+}^{\upsilon}\left({}^{\chi}_{\mathrm{C}}\mathrm{D}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q)\right) \\ &= \varkappa(q) - \sum_{k=0}^{n-1} \frac{\varkappa_{\chi}^{[k]}(\mathfrak{b})}{k!}\left(\chi_{\mathfrak{b}}(q)\right)^k,\end{aligned} \tag{2.2}$$

for $\varkappa \in C^n(\mathbb{J})$, $n-1 < \upsilon < n$.

3. Position of local problem, results and discussion

In view of impulsive FDE Eq. (1.1) and Eq. (1.2), one can see that problem,

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q) = w(q), \\ \qquad \upsilon > 0,\ q \in \mathbb{J}, \\ \varkappa(\hat{q}) = \varkappa_{\hat{q}} \in \mathbb{R},\ \hat{q} \in \mathbb{J}, \end{cases} \tag{3.1}$$

has a unique solution $\varkappa \in C(\mathbb{J})$ which is defined by the integral structure,

$$\begin{aligned}\varkappa(q) &= \varkappa_{\hat{q}} \\ &- \int_{\mathfrak{b}}^{\hat{q}} \frac{\chi'(\lambda)\left(\chi_{\lambda}(\hat{q})\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda \\ &+ \int_{\mathfrak{b}}^{q} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda.\end{aligned} \tag{3.2}$$

Lemma 3.1. *Suppose $w \in C(\mathbb{J})$. A function $\varkappa \in C(\mathbb{J})$ is the solution of the given backward impulsive problem,*

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}_{\mathfrak{b}^+}^{\upsilon}\varkappa(q) = w(q), \\ \qquad q \in \mathbb{J},\ q \neq q_k, \\ \triangle\varkappa\big|_{q=q_k} = \varrho_k(\varkappa(q_k^-)), \\ \qquad k \in \mathbb{K}_m, \\ \varkappa(\mathfrak{c}) = \varkappa_{\mathfrak{c}}, \end{cases} \tag{3.3}$$

if and only if it satisfies,

$$\varkappa(q) = \tag{3.4}$$

$$\begin{cases} \varkappa_{\mathfrak{c}} - \sum_{p=1}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) w(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in [0, q_1], \\ \varkappa_{\mathfrak{c}} - \sum_{p=2}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) w(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in (q_1, q_2], \\ \varkappa_{\mathfrak{c}} - \sum_{p=3}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) w(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in (q_2, q_3], \\ \ldots, \\ \varkappa_{\mathfrak{c}} - \sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) w(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in (q_k, q_{k+1}), k \in \mathbb{K}_{m-1}, \\ \varkappa_{\mathfrak{c}} - \int_{\mathfrak{b}}^{\mathfrak{c}} w(\lambda) \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda, \\ \quad q \in (q_m, \mathfrak{c}], \end{cases}$$

where

$$\mathcal{K}(q,\lambda) = \begin{cases} \frac{1}{\Gamma(\upsilon)}\Big[\left(\chi_{\mathfrak{c}}(q)\right)^{\upsilon-1} \\ \quad - \left(\chi_{\lambda}(q)\right)^{\upsilon-1} \Big]\chi'(\lambda)], \\ \qquad \lambda \le q, \\ \frac{\left(\chi_{\mathfrak{c}}(q)\right)^{\upsilon-1}\chi'(\lambda)}{\Gamma(\upsilon)}, \\ \qquad q \le \lambda. \end{cases} \tag{3.5}$$

Proof. Assume $\varkappa$ satisfies Eq. (3.3). If $q \in [\mathfrak{b}, q_1]$, then we have

$${}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\varkappa(q) = w(q), \quad q \in [\mathfrak{b}, q_1).$$

By applying Eq. (2.2), we have

$$\varkappa(q) = d_0 + \int_{\mathfrak{b}}^{q} \frac{\chi'(\lambda)(\chi_{\lambda}(q))^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda. \tag{3.6}$$

Now applying impulsive condition $\varkappa(q_1^-)$, one can see that

$$\varkappa(q_1^-) = d_0 + \int_{\mathfrak{b}}^{q_1} \chi'(\lambda)\left(\chi_{\lambda}(q_1)\right)^{\upsilon-1} w(\lambda)\,\mathrm{d}\lambda. \tag{3.7}$$

Again for $q \in (q_1, q_2]$, then

$${}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon,\chi}_{\mathfrak{b}^+}\varkappa(q) = w(q), \quad q \in (q_1, q_2),$$

with $\triangle\varkappa\big|_{q=q_1} = \varrho_1(\varkappa(q_1^-))$. Lemma 3.2 implies that

$$\begin{aligned} \varkappa(q) &= \varkappa(q_1^+) \\ &\quad - \int_{\mathfrak{b}}^{q_1} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q_1)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda \\ &\quad + \int_{\mathfrak{b}}^{q} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda \\ &= \varkappa(q_1^-) + \varrho_1\left(\varkappa(q_1^-)\right) \\ &\quad - \int_{\mathfrak{b}}^{q_1} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q_1)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda \\ &\quad + \int_{\mathfrak{b}}^{q} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

Further using Eq. (3.7), we get

$$\varkappa(q) = d_0 + \varrho_1\left(\varkappa(q_1^-)\right) + \int_{\mathfrak{b}}^{q} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda.$$

Again if q in $[q_2, q_3]$, then

$${}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\varkappa(q) = w(q), \quad q \in (q_2, q_3),$$

with $\triangle\varkappa\big|_{q=q_2} = \varrho_1(\varkappa(q_2^-))$. Again applying Eq. (3.2) and using impulsive condition $\varkappa(q_2^-)$ in Eq. (3.8), we can obtain,

$$\begin{aligned} \varkappa(q) &= \varkappa(q_2^+) \\ &\quad - \int_{\mathfrak{b}}^{q_2} \frac{\chi'(\lambda)\left(\chi_{\lambda}(q_2)\right)^{\upsilon-1}}{\Gamma(\upsilon)} w(\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{3.8}$$

$$+\int_{\mathfrak{b}}^{q}\frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\lambda)\,\mathrm{d}\lambda$$
$$=\varkappa(q_2^-)+\varrho_2\left(\varkappa(q_2^-)\right)$$
$$-\int_{\mathfrak{b}}^{q_2}\frac{\chi'(\lambda)\left(\chi_{\lambda}(q_2)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\lambda)\,\mathrm{d}\lambda$$
$$+\int_{\mathfrak{b}}^{q}\frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\lambda)\,\mathrm{d}\lambda$$
$$=d_0+\varrho_1\left(\varkappa(q_1^-)\right)\varrho_2\left(\varkappa(q_2^-)\right)$$
$$+\int_{\mathfrak{b}}^{q}\frac{\chi'(\xi)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\lambda)\,\mathrm{d}\lambda.$$

Furthermore, by continuing this process, we obtain for $q \in [q_k, q_{k+1}]$ as,

$$\varkappa(q)=d_0+\sum_{p=1}^{k}\varrho_p\left(\varkappa(q_p^-)\right)+\int_{\mathfrak{b}}^{q}\frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\lambda)\,\mathrm{d}\lambda, \tag{3.9}$$

where $k \in \mathbb{K}_m$. From the boundary (final) condition of Eq. (3.3) we get

$$d_0=\varkappa_{\mathfrak{c}}-\sum_{p=1}^{m}\varrho_p\left(\varkappa(q_p^-)\right)-\int_{\mathfrak{b}}^{\mathfrak{c}}\frac{\chi'(\lambda)\left(\chi_{\lambda}(q)\right)^{\upsilon-1}}{\Gamma(\upsilon)}w(\xi)\,\mathrm{d}\lambda,$$

substituting the value of d_0 in Eq. (3.6), Eq. (3.8) and Eq. (3.9) lead us to prove the integral presentation Eq. (3.4). Conversely, if $\varkappa(q)$ satisfies Eq. (3.4), then we can prove that $\varkappa(q)$ is the solution of Eq. (3.3). This complete the proof. □

3.1 Some qualitative results

Lemma 3.2. *Assume that $\phi \in C(\mathbb{J}\times\mathbb{R})$. The function $\varkappa \in PC(\mathbb{J})$ is a solution of Eq. (1.1) and Eq. (1.2), if and only if $\varkappa$ is a solution of the integral equation*

$$\varkappa(q)=\begin{cases}\varkappa_{\mathfrak{c}}-\sum_{p=k+1}^{m}\varrho_p(\varkappa(q_p^-))-\int_{\mathfrak{b}}^{\mathfrak{c}}\mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda, & q\in[q_k,q_{k+1}),\ k\in\{0\}\cup\mathbb{K}_{m-1},\\ \varkappa_{\mathfrak{c}}-\int_{\mathfrak{b}}^{\mathfrak{c}}\mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda, & q\in(q_m,\mathfrak{c}],\end{cases} \tag{3.10}$$

where $\mathcal{K}(q,\lambda)$ is defined in Eq. (3.5).

First, before discuss some qualitative results, we need to consider the integral operator $O : PC(\mathbb{J}) \to PC(\mathbb{J})$ as

$$O\varkappa(q)=\begin{cases}\varkappa_{\mathfrak{c}}-\sum_{p=k+1}^{m}\varrho_p(\varkappa(q_p^-))-\int_{\mathfrak{b}}^{\mathfrak{c}}\mathcal{K}(q,\lambda)\phi\big(\lambda,\varkappa(\lambda)\big)\,\mathrm{d}\lambda, & q\in[q_k,q_{k+1}),\ k\in\{0\}\cup\mathbb{K}_{m-1},\\ \varkappa_{\mathfrak{c}}-\int_{\mathfrak{b}}^{\mathfrak{c}}\mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda, & q\in(q_m,\mathfrak{c}],\end{cases} \tag{3.11}$$

As we know, a FP of O is a solution of Eq. (1.1) and Eq. (1.2). Now, we discuss conditions which Eq. (1.1) and Eq. (1.2) has a unique solution. The following result is based on the Banach FP theorem.

Theorem 3.3. *Assume that the following assumptions hold:*
H1) *A $\Lambda > 0$ exists such that*

$$|\phi(q,\varkappa)-\phi(q,\acute{\varkappa})| \leqslant \Lambda\,|\varkappa-\acute{\varkappa}|,$$

for $q \in \mathbb{J}$, $\varkappa, \acute{\varkappa} \in \mathbb{R}$;
H2) *There exists a positive constants μ_k such that*

$$|\varrho_k(\varkappa)-\varrho_k(\acute{\varkappa})| \leqslant \mu_k\,|\varkappa-\acute{\varkappa}|,$$

for $\varkappa, \acute{\varkappa} \in \mathbb{R},\ k \in \mathbb{K}_m$. *If*

$$\sum_{p=1}^{m} \mu_p + \Lambda \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda < 1, \tag{3.12}$$

then Eq. (1.1) *and Eq.* (1.2) *has a unique solution in* $PC(\mathbb{J})$.

Proof. In view of Eq. (3.11), for each $q \in \mathbb{J}$ and any $\varkappa,\ \acute{\varkappa} \in C(\mathbb{J})$, we have,

$$\begin{aligned} \left|O\varkappa(q) - O\tilde{\varkappa}(q)\right| &\leqslant \sum_{p=1}^{m} \left|\varrho_p(\varkappa(q_p^-)) - \varrho_p(\acute{\varkappa}(q_p^-))\right| \\ &+ \int_{\mathfrak{b}}^{\mathfrak{c}} \left|\phi\left(\lambda, \varkappa(\lambda)\right) - \phi\left(\lambda, \acute{\varkappa}(\lambda)\right)\right| \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{3.13}$$

Hypothesis (H1), (H2) and Eq. (3.13), we get

$$\begin{aligned} \left|O\varkappa(q) - O\acute{\varkappa}(q)\right| &\leqslant \sum_{p=1}^{m} \mu_p \left|\varkappa(q_p^-) - \acute{\varkappa}(q_p^-)\right| \\ &+ \Lambda \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) \left|\varkappa(\lambda) - \acute{\varkappa}(\lambda)\right| \mathrm{d}\lambda, \end{aligned}$$

and so,

$$\begin{aligned} \max_{q\in\mathbb{J}} \left|O\varkappa(q) - O\acute{\varkappa}(q)\right| &\leqslant \sum_{p=1}^{m} \mu_p \max_{q\in\mathbb{J}} \left|\varkappa(q_p^-) - \acute{\varkappa}(q_p^-)\right| \\ &+ \Lambda \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) \left|\varkappa(\lambda) - \acute{\varkappa}(\lambda)\right| \mathrm{d}\lambda. \end{aligned}$$

Hence,

$$\begin{aligned} \left\|O\varkappa(q) - O\tilde{\varkappa}(q)\right\| &\leqslant \left(\sum_{p=1}^{m} \mu_p + \Lambda \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda\right) \\ &\times \left\|\varkappa - \acute{\varkappa}\right\| \leqslant \left\|\varkappa - \acute{\varkappa}\right\|. \end{aligned}$$

Eq. (3.12) implies that O is a contraction. Hence, O has a unique FP, indeed Eq. (1.1) and Eq. (1.2) has a unique solution. □

The following result is based via the Schaefer's FP theorem.

Theorem 3.4. *Assume that* $\psi \in C(\mathbb{J} \times \mathbb{R})$ *and* $\varrho_k,\ kk \in \mathbb{K}_m$. *Then Eq.* (1.1) *and Eq.* (1.2) *has at least one solution in* $PC(\mathbb{J})$.

Proof. We will divide the proof in to many steps.

`Step 1:` Let $(\varkappa_n)$ be a sequence in $PC(\mathbb{J})$ such that $\varkappa_n \to \varkappa$ on $\mathbb{J}$. Then

$$\begin{aligned} \left|O\varkappa_n(q) - O\varkappa(q)\right| &\leqslant \sum_{p=1}^{m} \left|\varrho_p(\varkappa_n(q_p^-)) - \varrho_p(\varkappa(q_p^-))\right| \\ &+ \int_{\mathfrak{b}}^{\mathfrak{c}} \left|\phi\left(\lambda, \varkappa_n(\lambda)\right) - \phi\left(\lambda, \varkappa(\lambda)\right)\right| \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

Thus, $\|O\varkappa_n - O\varkappa\|_\infty \to 0$ as $n \to \infty$ and so, O is continuous.

`Step 2:` Let $\mathbb{S} \subset PC(\mathrm{J})$ be bounded, i.e there is $r > 0$ such that $\|\varkappa\| \leqslant r$ for all $\varkappa \in \mathbb{S}$, and let the functions $\psi : \mathbb{J} \times \mathbb{S} \to \mathbb{R}$ and $\varrho_k : \mathbb{S} \to \mathbb{R}$ are bounded by D_0 and D_k respectively i.e.

$$D_0 = \max_{(q,\varkappa)\in\mathbb{J}\times\mathbb{S}} |\phi(q,\varkappa)| + 1,$$

$D_k = \max_{q\in\mathbb{S}} |\varrho_k(\varkappa)|$. From Eq. (3.11), for any $\varkappa \in \mathbb{S}$ and for each $q \in \mathbb{J}$, we obtain,

$$|O\varkappa(q)| =$$

$$\begin{cases} \Big| \varkappa_{\mathfrak{c}} - \sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \phi(\lambda, \varkappa(\lambda)) \mathcal{K}(q,\lambda)\, \mathrm{d}\lambda \Big|, \\ \quad q \in [q_k, q_{k+1}), \\ \quad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big| \varkappa_{\mathfrak{c}} - \int_{\mathfrak{b}}^{\mathfrak{c}} \phi(\lambda, \varkappa(\lambda)) \mathcal{K}(q,\lambda)\, \mathrm{d}\lambda \Big|, \\ \quad q \in (q_m, \mathfrak{c}], \end{cases}$$

$$\leq |\varkappa_{\mathfrak{c}}| + \sum_{p=1}^{m} \left|\varrho_p(\varkappa(q_p^-))\right| + \int_{\mathfrak{b}}^{\mathfrak{c}} |\phi(\lambda, \varkappa(\lambda))| \mathcal{K}(q,\lambda)\, \mathrm{d}\lambda$$

$$\leq |\varkappa_{\mathfrak{c}}| + \sum_{p=1}^{m} D_p + D_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\, \mathrm{d}\lambda.$$

Thus,

$$\|O\varkappa(q)\| \leq |\varkappa_{\mathfrak{c}}| + \sum_{p=1}^{m} D_p + D_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\, \mathrm{d}\lambda.$$

Hence, $O(\mathbb{S})$ is uniformly bounded and this means that, $O\varkappa$ is maps bounded sets into the bounded sets in $PC(\mathbb{J})$.

`Step 3:` Now, we will prove that the operator O is bounded sets into the equi-continuous sets of $PC(\mathbb{J})$, for each $\varkappa \in \mathbb{S}$. Then For $\acute{q}_1, \acute{q}_2 \in \mathbb{J}$ with

$$\acute{q}_1 < \acute{q}_2, \quad q_k < \acute{q}_1 < \acute{q}_2 < q_{k+1},$$

$k = 0, 1, \cdots, m$, we have,

$$\left|O\varkappa(\acute{q}_2) - O\varkappa(\acute{q}_1)\right| = \tag{3.14}$$

$$\begin{cases} \Big| \int_{\mathfrak{b}}^{\mathfrak{c}} \Big(\mathcal{K}(\acute{q}_2,\lambda) \\ \quad -\mathcal{K}(\acute{q}_1,\lambda)\Big) \phi(\lambda, \varkappa(\lambda)\, \mathrm{d}\lambda \Big|, \\ \quad \acute{q}_1, \acute{q}_2 \in [q_k, q_{k+1}), \\ \quad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big| \int_{\mathfrak{b}}^{\mathfrak{c}} \Big(\mathcal{K}(\acute{q}_2,\lambda) \\ \quad -\mathcal{K}(\acute{q}_1,\lambda)\Big) \phi(\lambda, \varkappa(\lambda)\, \mathrm{d}\lambda \Big|, \\ \quad \acute{q}_1, \acute{q}_2 \in (q_m, \mathfrak{c}], \end{cases}$$

$$\leq \int_{\mathfrak{b}}^{\acute{q}_1} |(\mathcal{K}(\acute{q}_2,\lambda) - \mathcal{K}(\acute{q}_1,\lambda)| \, |\phi(\lambda, \varkappa(\lambda)|\, \mathrm{d}\lambda + \int_{\acute{q}_1}^{\acute{q}_2} |(\mathcal{K}(\acute{q}_2, \mathrm{d}\lambda) - \mathcal{K}(\acute{q}_1,\lambda)| \, |\phi(\lambda, \varkappa(\lambda))|\, \mathrm{d}\lambda,$$

$$\leq D_0 \Big(\int_{\mathfrak{b}}^{\acute{q}_1} |\mathcal{K}(\acute{q}_2,\lambda) - \mathcal{K}(\acute{q}_1,\lambda)|\, \mathrm{d}\lambda + \int_{\acute{q}_1}^{\acute{q}_2} |\mathcal{K}(\acute{q}_2,\lambda) - \mathcal{K}(\acute{q}_1,\lambda)|\, \mathrm{d}\lambda \Big),$$

$$= \frac{D_0}{\Gamma(\upsilon+1)} \left[2\left(\chi_{\acute{q}_1}(\acute{q}_2)\right)^{\upsilon} + \left(\chi_0(\acute{q}_1)\right)^{\upsilon} \right].$$

As $\acute{q}_1$ tends to $\acute{q}_2$, the right-hand side of the inequality Eq. (3.14) tends to zero and the convergence is independent of each $\varkappa \in \mathbb{S}$, which means that $O(\mathbb{S})$ is equi-continuous. Thus, the compactness of O follows by PC-type Ascoli-Arzelà's theorem.

`Step 4:` A priori bounds. Now it remains to show that the set,

$$\Omega(O) = \Big\{\varkappa \in PC(\mathbb{J}) \,:\, \varkappa = \gamma O(\varkappa)\Big\},$$

$0 < \gamma < 1$, is bounded. Let $\varkappa = \gamma O(\varkappa)$ and $0 < \gamma < 1$, $q \in \mathbb{J}$, then

$$|\varkappa(q)| = \lambda\, |O\varkappa(q)| =$$

$$\begin{cases} \gamma\Big|\varkappa_{\mathfrak{c}} - \sum_{p=k+11}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda\Big|, \\ \quad q \in [q_k, q_{k+1}), \\ \quad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \gamma\Big|\varkappa_{\mathfrak{c}} - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda\Big|, \\ \quad q \in (q_m, \mathfrak{c}], \end{cases}$$

$$\le |\varkappa_{\mathfrak{c}}| + \sum_{p=1}^{m} \left|\varrho_p(\varkappa(q_p^-))\right| + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,|\phi\,(\lambda,\varkappa(\lambda))|\,\mathrm{d}\lambda$$

$$\le |\varkappa_{\mathfrak{c}}| + \sum_{p=1}^{m} D_p + D_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda := D.$$

Hence, for every $q \in \mathbb{J}$, $\|O\varkappa(q)\| \le D$. This shows that the set $\Omega(O)$ is bounded. As a consequence of Schaefer's FP theorem, we deduce that O has a FP which is a solution of the Eq. (1.1) and Eq. (1.2). The proof is completed. □

4. Nonlocal backward impulsive FDEs

In this section, we generalize the results of the previous section to nonlocal impulsive FDEs Eq. (1.1) and Eq. (1.3). Let us put the following assumptions:

H3) There exists a positive constant θ_0 and a function $\Phi \in C(\mathbb{J}, \mathbb{R}^+)$ such that,

$$|\phi(q,\varkappa)| < \Phi(q)\,|\varkappa| + \theta_0, \quad q \in \mathbb{J},\ \varkappa \in \mathbb{R};$$

H4) The functions $\varrho_k \in C(\mathbb{R})$ and there exists positive constants θ_k^1 and θ_k^2 such that

$$|\varrho_k(\varkappa)| < {}_1\theta_k\,|\varkappa| + {}_2\theta_k,$$

$\varkappa \in \mathbb{R}$, $k \in \{0\} \cup \mathbb{K}_m$;

H5) There exists a positive constant C such that

$$\|\omega(\varkappa) - \omega(\acute{\varkappa})\| \leqslant C\,\|\varkappa - \acute{\varkappa}\|, \quad \varkappa, \acute{\varkappa} \in \mathbb{R};$$

H6) There exists positive constants $\tilde{D}$, K and r such that $|\omega(\varkappa)| \leqslant \tilde{D}\,|\varkappa| + K$ for any $\varkappa \in \mathbb{R}$.

Assume that $0 < \upsilon < 1$, (H1) is satisfied and $\omega \in C(\mathbb{R})$. Before going to discuss and prove some main results, we have to define the operator $\overline{O} : PC(\mathbb{J}) \to PC(\mathbb{J})$ which are involved to Eq. (1.1) and Eq. (1.3) as

$$\overline{O}\varkappa(q) = \tag{4.1}$$

$$\begin{cases} \varkappa_{\mathfrak{c}} + \omega(\varkappa(q)) - \sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in [q_k, q_{k+1}), \\ \quad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \varkappa_{\mathfrak{c}} + \omega(\varkappa(q)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda)\,\mathrm{d}\lambda, \\ \quad q \in (q_m, T]. \end{cases}$$

where $\mathcal{K}(q,\lambda)$ defined in Eq. (3.5).

Theorem 4.1. *Assume that, all conditions (H1), (H2), (H3) and (H5) are checked. If*

$$C + \sum_{p=1}^{m} \mu_p + \Lambda \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda < 1, \tag{4.2}$$

then Eq. (1.1) *has a unique solution in* $PC(\mathbb{J})$ *which can be shown by use of the same process of the proof of Theorem 3.3.*

Theorem 4.2. *If (H3), (H4) and (H6) are satisfied, and $C < 1$ and there exists a positive constant r such that,*

$$4r \ge \max\left\{\sum_{p=1}^{m} {}_2\theta_p \right. \tag{4.3}$$

$$+\theta_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda,\ |\varkappa_{\mathfrak{c}}| + K\Big\},$$

then Eq. (1.1) *and Eq.* (1.3) *has at least a solution in* $PC(\mathbb{J})$.

Proof. Let us put,

$$\max\Big\{\sum_{p=1}^{m} {}_1\theta_p + \max_{q\in\mathbb{J}} \int_0^{\mathfrak{c}} \mathcal{K}(q,\lambda)\Phi(\lambda)\,\mathrm{d}\lambda, \tilde{D}\Big\} \leq 4,$$

and define the operators, $\overline{O}_1$ and $\overline{O}_2$ on the compact set,

$$\mathbb{S}_r = \Big\{\varkappa \in PC(\mathbb{J}) \ :\ \|\varkappa\| \leqslant r\Big\},$$

by,

$$\overline{O}_1(\varkappa(q)) = \varkappa_{\mathfrak{c}} + \omega(\varkappa(q),$$

and

$$\overline{O}_2(\varkappa(q)) = \begin{cases} \sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda)\,\mathrm{d}\lambda, & q \in [q_k, q_{k+1}),\ k \in \{0\}\cup\mathbb{K}_{m-1}, \\ -\int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda, & q \in (q_m, \mathfrak{c}]. \end{cases}$$

For all $\varkappa(q) \in \mathbb{S}_r$, we can write

$$\begin{aligned} \left|\overline{O}_1(\varkappa(q))\right| &= |\varkappa_{\mathfrak{c}} + \omega(\varkappa(q)| \leq |\varkappa_{\mathfrak{c}}| + |\omega(\varkappa(q)| \\ &\leq |\varkappa_{\mathfrak{c}}| + \tilde{D}\,\|\varkappa\| + K \leq |\varkappa_{\mathfrak{c}}| + \tilde{D}r + K \\ &\leq \tfrac{r}{4} + \tfrac{r}{2} + \tfrac{r}{4} \leqslant r. \end{aligned}$$

Hence $\overline{O}_1(\mathbb{S}_r) \subset \mathbb{S}_r$. Let $\varkappa, \acute{\varkappa} \in PC(\mathbb{J})$, then

$$\begin{aligned} \left|\overline{O}_1(\varkappa(q)) - \overline{O}_1(\acute{\varkappa}(q))\right| &= |\omega(\varkappa(q)) - \omega(\acute{\varkappa}(q))| \\ &\leqslant C\,|\varkappa(\tau) - \acute{\varkappa}(q)|, \end{aligned}$$

as $C < 1$, then the operator $\overline{O}_1$ satisfies the contraction property. Now, since,

$$\left|\overline{O}_2(\varkappa(q))\right| = \begin{cases} \Big|-\sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda)\,\mathrm{d}\lambda\Big|, & q \in [q_k, q_{k+1}),\ k = k \in \{0\}\cup\mathbb{K}_{m-1}, \\ \Big|-\int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda)\,\mathrm{d}\lambda\Big|, & q \in (q_m, \mathfrak{c}], \end{cases}$$

$$\begin{aligned} &\leq \sum_{p=1}^{m} \left|\varrho_p\big(\varkappa(q_p^-)\big)\right| + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,|\phi(\lambda,\varkappa(\lambda)|\,\mathrm{d}\lambda \\ &\leq \sum_{p=1}^{m} {}_1\theta_p\,\|\varkappa\| + {}_2\theta_p + \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\Phi(\lambda)\,\mathrm{d}\lambda\,\|\varkappa\| + \theta_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda \\ &\leq r\Big[\sum_{p=1}^{m} {}_1\theta_p + \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\Phi(\lambda)\,\mathrm{d}\lambda\Big] + \sum_{p=1}^{m} {}_2\theta_p + \theta_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda \\ &\leq \tfrac{r}{4} + \tfrac{r}{4} = \tfrac{r}{2}, \end{aligned}$$

we can get

$$\begin{aligned} &\left|\overline{O}_1(\varkappa(q)) + \overline{O}_2(\varkappa(q))\right| \\ &\quad\leq \left|\overline{O}_1(\varkappa(q))\right| + \left|\overline{O}_2(\varkappa(q))\right| \end{aligned}$$

$$\leq \frac{r}{2} + \frac{r}{2} = r.$$

Then,

$$\overline{O}_1(\varkappa(q)) + \overline{O}_2(\varkappa(q)) \in \mathbb{S}_r, \quad \forall \varkappa, \acute{\varkappa} \in \mathbb{S}_r.$$

In view of (H1) and that $\mathcal{K}(q, \lambda)$ is continuous function on $\mathbb{J}^2$, it is clear that the operator $\overline{O}_2(\varkappa(q))$ is continuous and uniformly bounded on $\mathbb{S}_r$ by the inequality Eq. (4.3). Thanks to Theorem 3.4 $\overline{O}_2(\varkappa(q))$ is equicontinuity. Hence, by the PC-type Arzelà-Ascoli theorem, the operator $\overline{O}_2(\mathbb{S}_r)$ is relatively compact into $\mathbb{S}_r$ which implies that $\overline{O}_2$ is compact. Therefore, according to Krasnoselskii's theorem, there exists $\varkappa \in \mathbb{S}_r$ such that $\varkappa = \overline{O}(\varkappa)$, i.e. Eq. (1.1) and Eq. (1.3) has a least one solution. □

5. Stability results of nonlinear backward impulsive FDE

FDEs have been extensively analyzed from different angles. Amonge these, stability analysis in the Ulam's type concepts is consider as an important aspect that gained proper attention from many researchers [7, 16, 20, 39].

Based on the fundamental definition of Ulam's type concepts for some nonlinear backward impulsive FDEs and generalized Grönwall inequality through the fractional integral with respect to another function, the notion was later modified to more general types, and their results were successfully applied to various problems [17, 18].

In this section, we will adopt a number of sufficient conditions to review the H.U type stability results to the considered Eq. (1.1)-Eq. (1.3).

Let $\sigma > 0$, $\varepsilon > 0$ and $\varphi \in PC(\mathbb{J})$ is non decreasing. We consider the following inequalities (Ulam's type stability concepts for Eq. (1.1) and Eq. (1.3),

$$\begin{cases} \left| {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\tilde{\varkappa}(q) - \phi(q, \tilde{\varkappa}(q)) \right| \leq \varepsilon, \\ \qquad q \in \mathbb{J}, q \neq q_k, \\ \left| \triangle\tilde{\varkappa}(q_k) - \varrho_k(\tilde{\varkappa}(q_k^-)) \right| \leq \varepsilon, \\ \qquad k \in \mathbb{K}_m. \end{cases} \tag{5.1}$$

$$\begin{cases} \left| {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\tilde{\varkappa}(q) - \phi(q, \tilde{\varkappa}(q)) \right| \leq \varphi(q), \\ \quad q \in \mathbb{J},\ q \neq q_k, \\ \left| \triangle\tilde{\varkappa}(q_k) - \varrho_k(\tilde{\varkappa}(q_k^-)) \right| \leq \sigma, \\ \qquad k \in \mathbb{K}_m. \end{cases} \tag{5.2}$$

$$\begin{cases} \left| {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\tilde{\varkappa}(q) - \phi(q, \tilde{\varkappa}(q)) \right| \leq \varepsilon\varphi(q), \\ \qquad q \in \mathbb{J},\ q \neq q_k, \\ \left| \triangle\tilde{\varkappa}(q_k) - \varrho_k(\tilde{\varkappa}(q_k^-)) \right| \leq \varepsilon\sigma, \\ \qquad k \in \mathbb{K}_m. \end{cases} \tag{5.3}$$

Definition 5.1. The said nonlocal backward impulsive FDE Eq. (1.1) and Eq. (1.3), for $q \in \mathbb{J}$ is,

D1: H.U stable if there exists a real number $c_{\phi,m} > 0$ such that for any $\varepsilon > 0$, and for each solution $\tilde{\varkappa} \in PC(\mathbb{J})$ of the given systems of inequalities Eq. (5.1), there exists a solution $\varkappa \in PC(\mathbb{J})$ of Eq. (1.1) and Eq. (1.3) with $|\tilde{\varkappa}(q) - \varkappa(q)| \leq c_{\phi,m}\varepsilon$;

D2: Generalized U.H (G.U.H) stable if there exists non-decreasing function $\alpha_{\phi,m} \in PC(\mathbb{J})$ with $\alpha_{\phi,m}(0) = 0$ such that for each solution $\tilde{\varkappa} \in PC(\mathbb{J})$ of the given systems of inequalities Eq. (5.1), there exists a solution $\varkappa \in PC(\mathbb{J})$ of Eq. (1.1) and Eq. (1.3) with

$$|\tilde{\varkappa}(q) - \varkappa(q)| \leq \alpha_{\phi,m}(\varepsilon);$$

D3: U.H.R stable with respect to (φ, α) if there exists a real number $c_{\phi,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and each solution $\tilde{\varkappa} \in PC(\mathbb{J})$ of the given systems of inequalities Eq. (5.3), there exists a solution $\varkappa \in PC(\mathbb{J})$ of Eq. (1.1) and Eq. (1.3) with,

$$|\tilde{\varkappa}(q) - \varkappa(q)| \leq c_{\phi,m,\varphi}\varepsilon\big[\varphi(q) + \sigma\big];$$

D4: Generalized Ulam-Hyers-Rassias (G.U.H.R) stable with respect to (φ, σ) if there exists a real number $c_{\phi,m,\varphi} > 0$ such that for each solution $\tilde{\varkappa} \in PC(\mathbb{J})$ of the given systems of inequalities Eq. (5.2), there exists a solution $\varkappa \in PC(\mathbb{J})$ of Eq. (1.1) and Eq. (1.3) with,

$$|\tilde{\varkappa}(q) - \varkappa(q)| \leq c_{\phi,m,\varphi}\big[\varphi(q) + \sigma\big].$$

Remark 5.2. *It is clear that:*
(i) Definition D1 implies that Definition D2;
(ii) Definition D3 implies that Definition D4;
(iii) Definition D3 for $\varphi(q) = \beta = 1$ implies that Definition D1.

Remark 5.3. *A function $\tilde{\varkappa} \in PC(\mathbb{J})$ is a solution of the given systems of inequalities Eq.* (5.1) *if and only if there exist a function $\tilde{\varkappa}_\circ \in PC(\mathbb{J})$ and a sequence $\tilde{\varkappa}_k$, $k \in \mathbb{K}_m$ (which depend on $\tilde{\varkappa}$) such that,*

i) $|\tilde{\varkappa}_\circ(q)| \leqslant \varepsilon$, $q \in \mathbb{J}$ *and* $|\tilde{\varkappa}_k| \leqslant \varepsilon$, *with* $k \in \mathbb{K}_m$;

ii) ${}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\tilde{\varkappa}(q) = \phi(q, \tilde{\varkappa}(q)) + \tilde{\varkappa}_\circ(q)$, $q \in \mathbb{J}$, $q \neq q_k$;

iii) $\triangle\varkappa(q_k) = \varrho_k(\varkappa(q_k^-)) + \tilde{\varkappa}_k$.

One can have similar remarks for the given systems of inequalities Eq. (5.2) and Eq. (5.3). So, the Ulam stabilities of nonlocal backward impulsive FDEs are some special types of data dependence of the solutions of impulsive FDEs.

Lemma 5.4. *The solution of below mention problem,*

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\tilde{\varkappa}(q) = \phi\big(q, \tilde{\varkappa}(q)\big) + \tilde{\varkappa}_\circ(q), \\ \qquad q \in \mathbb{J},\ q \neq q_k; \\ \triangle\tilde{\varkappa}(q_k) = \varrho_k\big(\tilde{\varkappa}(q_k^-)\big) + \tilde{\varkappa}_k, \\ \qquad k \in \mathbb{K}_m; \\ \tilde{\varkappa}(\mathfrak{c}) = \varkappa_{\mathfrak{c}} + \omega(\tilde{\varkappa}(q)). \end{cases} \tag{5.4}$$

is

$$\tilde{\varkappa}(q) = \begin{cases} \varkappa_{\mathfrak{c}} + \omega(\tilde{\varkappa}(q)) \\ \quad -\Big(\sum_{p=k+11}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) + \tilde{\varkappa}_p\Big) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} [\phi(\lambda, \tilde{\varkappa}(\lambda)) \\ \quad + \tilde{\varkappa}_\circ(\lambda)]\mathcal{K}(q,\lambda)\,\mathrm{d}\lambda, \\ \qquad q \in [q_k, q_{k+1}), \\ \qquad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \varkappa_{\mathfrak{c}} + \omega(\tilde{\varkappa}(q)) \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\Big[\phi(\lambda, \tilde{\varkappa}(\lambda)) \\ \quad + \tilde{\varkappa}_\circ(\lambda)\Big]\mathrm{d}\lambda, \\ \qquad q \in (q_m, \mathfrak{c}], \end{cases} \tag{5.5}$$

where $\mathcal{K}(q,\lambda)$ defined in Eq. (3.5) *for all $q, \lambda \in \mathbb{J}$.*

Proof. The solution of Eq. (5.4) can be easily obtained through using Remark 5.3. □

Proposition 5.5. *If the function $\tilde{\varkappa} \in PC(\mathbb{J})$ is a solution of the given systems of inequalities Eq.* (5.1)*, then it is a solution of the following systems of inequalities*

$$\begin{cases} \Big|\tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) \\ \quad + \sum_{p=k+1}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda\Big| \\ \leq \Big[m + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda\Big]\varepsilon, \\ \qquad q \in [q_k, q_{k+1}), \\ \qquad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big|\tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda\Big| \\ \leq \varepsilon \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda \\ \qquad q \in (q_m, \mathfrak{c}], \end{cases} \tag{5.6}$$

where $\mathcal{K}(q,\lambda)$ is defined by Eq. (3.5).

Proof. The proof of this proposition can be easily obtained through using Eq. (5.4) in Lemma 5.4 and Remark 5.3. □

We have similar remarks for the solutions of the given systems of inequalities Eq. (5.2) and Eq. (5.3).

Lemma 5.6 ([16,39] Generalized Grönwall inequality through the fractional integral with respect to another function)**.** *Let $\varkappa_1, \varkappa_2$ be two integrable functions and $\hbar$ continuous, with domain $[\mathfrak{b},\mathfrak{c}]$. Let $\beta \in C^1(\mathbb{J})$ an increasing function such that $\chi'(q) \neq 0$, for each $q \in \mathbb{J}$. Assume that*
i) $\varkappa_1$ and $\varkappa_2$ are nonnegative;
ii) $\hbar$ in nonnegative and nondecreasing.
If for each $q \in \mathbb{J}$,

$$\varkappa_1(q) \le \varkappa_2(q) + \hbar(q)\int_{\mathfrak{b}}^{q} \chi'(\lambda)\left(\chi_\lambda(q)\right)^{\upsilon-1}\varkappa_1(\lambda)\,\mathrm{d}\lambda,$$

then

$$\varkappa_1(q) \le \varkappa_2(q) + \int_{\mathfrak{b}}^{q}\sum_{j=0}^{\infty}\frac{[\hbar(q)\Gamma(\upsilon)]^j}{\Gamma(j\upsilon)} \times \chi'(\lambda)\left(\chi_\lambda(q)\right)^{j\upsilon-1}\varkappa_2(\lambda)\,\mathrm{d}\lambda.$$

Remark 5.7. *Under the hypothesis of Lemma 5.6, let $\hbar(q) = b$, for each $q \in \mathbb{J}$. Then for each $q \in \mathbb{J}$, we have,*

$$\varkappa_1(q) \le \varkappa_2(q) + \int_{\mathfrak{b}}^{q}\sum_{j=0}^{\infty}\frac{[b\Gamma(\upsilon)]^j}{\Gamma(j\upsilon)} \times \chi'(\lambda)\left(\chi_\lambda(q)\right)^{j\upsilon-1}\varkappa_2(\lambda)\,\mathrm{d}\lambda.$$

Remark 5.8. *Under the hypothesis of Lemma 5.6, let $\varkappa_2$ be a nondecreasing function for each $q \in [\mathfrak{b},\mathfrak{c}]$. Then, we have*

$$\varkappa(q) \le \varkappa_2(q)\mathcal{E}_\upsilon\Big(\hbar(q)\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^{\upsilon}\Big),$$

$\forall\,(q,\lambda) \in \mathbb{J}\times[\mathfrak{b},q]$. *where*

$$\mathcal{E}_\upsilon(q) = \sum_{j=0}^{\infty}\frac{1}{\Gamma(j\upsilon+1)}q^j,$$

with $\upsilon > 0$, is the Mittag-Leffler function.

Remark 5.9. *Let $\varkappa_1, \varkappa_2$ be two integrable functions, and $\hbar_1$, $\hbar_2$ are two nonnegative and continuous on functions $\mathbb{J}$. Let $\chi \in C^1(\mathbb{J})$ an increasing function such that $\chi'(q) \neq 0$, $\forall q \in \mathbb{J}$. Assume that,*
i) $\varkappa_1$ and $\varkappa_2$ are nonnegative;
ii) $\varkappa_2$ and $\hbar_1$, $\hbar_2$ are nondecreasing.
If for all $q \in \mathbb{J}$,

$$\varkappa_1(q) \le \varkappa_2(q)\hbar_1(q)\int_{\mathfrak{b}}^{q}\chi'(\lambda) \times\left(\chi_\lambda(q)\right)^{\upsilon-1}\varkappa_1(\lambda)\,\mathrm{d}\lambda + \hbar_2(q)\int_{\mathfrak{b}}^{\mathfrak{c}}\chi'(\lambda)\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon-1}\varkappa_1(\lambda)\,\mathrm{d}\lambda,$$

then

$$\varkappa_1(q) \le \varkappa_2(q)\mathcal{E}_\upsilon\Big(\hbar_1(q)\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^{\upsilon}\Big) \times\mathcal{E}_\upsilon\Big(\hbar_2(\mathfrak{c})\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon}\Big),$$

$\forall(q,\lambda) \in \mathbb{J}\times[\mathfrak{b},q]$.

Lemma 5.10. *Let $\varkappa_1 \in PC(\mathbb{J})$ satisfy the following inequality*

$$|\varkappa_1(q)| \le \varkappa_2(\tau) + b_1\int_{\mathfrak{b}}^{q}\chi'(\lambda)\left(\chi_\lambda(q)\right)^{\upsilon-1}|\varkappa_1(\lambda)|\,\mathrm{d}\lambda + \sum_{p=1}^{k}\mu_p\left|\varkappa_1(q_p^-)\right| + b_2\int_{\mathfrak{b}}^{q}\chi'(\lambda)\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon-1}|\varkappa_1(\lambda)|\,\mathrm{d}\xi,$$

for $q \in [q_k, q_{k+1})$, where $k \in \mathbb{K}_m$, $\varkappa_2$ is nonnegative continuous and nondecreasing on $\mathbb{J}$, and $b_1, b_2, \mu_p \ge 0$ are constants. Then

$$|\varkappa_1(q)| \le \varkappa_2(q)\Big[1+ \tag{5.7}$$

$$\mu\mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg]^k \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big),$$

$\forall(q,\lambda) \in (\mathbb{J} \setminus q_k) \times [\mathfrak{b}, q]$, *where*

$$\mu = \max\Big\{\mu_1, \mu_2, \ldots, \mu_m\Big\}.$$

Proof. Indeed, from Remark 5.9, we derive

$$|\varkappa_1(\tau)| \le \varkappa_2(q) \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big), \tag{5.8}$$

$\forall(q,\lambda) \in [\mathfrak{b}, q_1] \times [\mathfrak{b}, q]$, and

$$|\varkappa_1(q)| \le \left[\varkappa_2(q) + \sum_{p=1}^{k}\mu_p\left|\varkappa_1(\tau_p^-)\right|\right] \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big), \tag{5.9}$$

$\forall(q,\lambda) \in (\mathbb{J} \setminus q_k) \times [\mathfrak{b}, q]$. By Eq. (5.8), inequality Eq. (5.7) holds for $k = 0$. By induction suppose Eq. (5.7) holds for $k = p < m$. Then by Eq. (5.9) and since $\varkappa_2(a)$ and $\mathcal{E}_\upsilon(q)$ are nondecreasing,

$$\forall(q,\lambda) \in \left(\mathbb{J} \setminus \left\{q_{p+1}\right\}\right) \times [\mathfrak{b}, q],$$

we derive

$$\begin{aligned}
|\varkappa_1(q)| &\le \left(\varkappa_2(q) + \sum_{i=1}^{p+1}\mu_i\left|\varkappa_1(q_i^-)\right|\right) \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big), \\
&\le \Bigg[\varkappa_2(q) + \sum_{i=1}^{p}\mu_p\varkappa_2(q_i^-)\Bigg[1+ \mu\,\mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q_i^-)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg]^{i-1} \\
&\quad \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q_i^-)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg] \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big) \\
&\le \Bigg[\varkappa_2(q) + \mu\sum_{i=1}^{p}\varkappa_2(q)\Bigg[1+ \mu\,\mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg]^{i-1} \\
&\quad \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg] \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big) \\
&= \varkappa_2(q)\Bigg[1+ \mu\,\mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Bigg]^{p+1} \\
&\quad \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon)\left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon)\left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big).
\end{aligned}$$

This finishes the proof. □

Remark 5.11. *Let* $\varkappa_1 \in PC(\mathrm{J})$ *satisfy the following inequality*

$$|\varkappa_1(q)| \le \varkappa_2(q) + b_1\int_{\mathfrak{b}}^{q}\chi'(\lambda)\left(\chi_\lambda(q)\right)^{\upsilon-1}|\varkappa_1(\lambda)|\,\mathrm{d}\lambda$$

$$+ \sum_{p=k+1}^{m} \mu_p \left|\varkappa_1(q_p^-)\right|$$
$$+ b_2 \int_{\mathfrak{b}}^{q} \chi'(\lambda) \left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon-1} |\varkappa_1(\lambda)| \, \mathrm{d}\lambda,$$

for $q \in [q_k, q_{k+1})$, *where* $k \in \{0\} \cup \mathbb{K}_{m-1}$, $\varkappa_2$ *is nonnegative continuous and nondecreasing on* $\mathbb{J}$, *and* $b_1, b_2, \mu_p \geq 0$ *are constants. Then,*

$$|\varkappa_1(q)| \leq \varkappa_2(q) \Big[1+ \mu \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon) \left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon) \left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big)\Big]^{m-k} \times \mathcal{E}_\upsilon\Big(b_1\Gamma(\upsilon) \left(\chi_\lambda(q)\right)^\upsilon\Big) \times \mathcal{E}_\upsilon\Big(b_2\Gamma(\upsilon) \left(\chi_\lambda(\mathfrak{c})\right)^\upsilon\Big), \tag{5.10}$$

for all $(q,\lambda) \in (\mathbb{J} \setminus q_k) \times [\mathfrak{b}, q]$, *where,*

$$\mu = \max\left\{\mu_1, \mu_2, \ldots, \mu_m\right\}.$$

Now, we give the main results, generalized U.H.R. stable results, in this section.

Theorem 5.12. *If the assumption (H1) and (H5) hold. Suppose there exists* $\eta_\varphi > 0$ *such that,*

$$\int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\varphi(\lambda) \, \mathrm{d}\lambda \leq \eta_\varphi \varphi(q), \quad \forall q \in \mathbb{J}.$$

Then Eq. (1.1) *and Eq.* (1.3) *is G.U.H.R stable with respect to* (χ, φ).

Proof. Let $\tilde{\varkappa} \in PC(\mathbb{J})$ be a solution of the given systems of inequalities Eq. (5.1). Denote by $\varkappa \in PC(\mathbb{J})$ the unique solution of the backward impulsive FDE Eq. (1.1) and Eq. (1.3). By the given systems of differential inequalities Eq. (5.1) (see Eq. (5.6) in Proposition 5.5), for each $q \in \mathbb{J}$, we have,

$$\begin{cases} \Big|\tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) + \sum\limits_{p=k+1}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \tilde{\varkappa}(\lambda) \, \mathrm{d}\lambda\Big| \\ \leq m\chi + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\varphi(\lambda) \, \mathrm{d}\lambda, \quad q \in [q_k, q_{k+1}), \; k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big|\tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \tilde{\varkappa}(\lambda) \, \mathrm{d}\xi\Big| \\ \leq \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\varphi(\lambda) \, \mathrm{d}\lambda, \quad q \in (q_m, \mathfrak{c}]. \end{cases}$$

Hence, for each $q \in \mathbb{J}$, we can write

$$\left|\tilde{\varkappa}(q) - \varkappa(q)\right| = \begin{cases} \Big|\tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\varkappa(\tau)) + \sum\limits_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \varkappa(\lambda)) \, \mathrm{d}\lambda\Big|, \\ \quad q \in [q_k, q_{k+1}), \; k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big| - \varkappa_{\mathfrak{c}} - \omega(\varkappa(q)) + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda, \varkappa(\lambda)) \, \mathrm{d}\lambda\Big|, \\ \quad q \in (q_m, \mathfrak{c}], \end{cases}$$

$$
= \begin{cases} \Big| \tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\varkappa(q)) \\ \quad + \sum_{p=k+1}^{m} \varrho_p(\varkappa(q_p^-)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda \\ \quad + \omega(\tilde{\varkappa}(q)) - \omega(\tilde{\varkappa}(q)) \\ \quad + \sum_{p=k+1}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) \\ \quad - \sum_{p=k+1}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda))\,\mathrm{d}\lambda \Big|, \\ \qquad q \in [q_k, q_{k+1}), \\ \qquad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big| - \varkappa_{\mathfrak{c}} - \omega(\varkappa(q)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\varkappa(\lambda))\,\mathrm{d}\lambda \\ \quad + \omega(\tilde{\varkappa}(q)) - \omega(\tilde{\varkappa}(q)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda \\ \quad - \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda \Big|, \\ \qquad q \in (q_m, \mathfrak{c}], \end{cases}
$$

$$
\leq \begin{cases} \Big| \tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) \\ \quad + \sum_{p=k+1}^{m} \varrho_p(\tilde{\varkappa}(q_p^-)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda \Big| \\ \quad + |\omega(\tilde{\varkappa}(q)) - \omega(\varkappa(q))| \\ \quad + \sum_{p=k+1}^{m} \left|\varrho_p(\tilde{\varkappa}(q_p^-)) - \varrho_p(\varkappa(q_p^-))\right| \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) \Big|\phi(\lambda,\tilde{\varkappa}(\lambda) \\ \quad -\phi(\lambda,\varkappa(\lambda)\Big|\,\mathrm{d}\lambda, \\ \qquad q \in [q_k, q_{k+1}), \\ \qquad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \Big| \tilde{\varkappa}(q) - \varkappa_{\mathfrak{c}} - \omega(\tilde{\varkappa}(q)) \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\phi(\lambda,\tilde{\varkappa}(\lambda)\,\mathrm{d}\lambda \Big| \\ \quad + |\omega(\tilde{\varkappa}(q)) - \omega(\varkappa(q))| \\ \quad + \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda) |\phi(\lambda,\tilde{\varkappa}(\lambda) \\ \quad -\phi(\lambda,\varkappa(\lambda)|\,\mathrm{d}\lambda \Big|, \\ \qquad q \in (q_m, \mathfrak{c}], \end{cases}
$$

$$
\leq \begin{cases} \left(m + q_{\varphi}\right)(\chi + \varphi(q)) \\ \quad + C\,|\tilde{\varkappa}(q) - \varkappa(q)| \\ \quad + \sum_{p=1}^{m} \mu_p \left|\tilde{\varkappa}(q_p^-) - \varkappa(q_p^-)\right| \\ \quad + \gamma \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,|\tilde{\varkappa}(\lambda) - \varkappa(\lambda)|\,\mathrm{d}\lambda, \\ \qquad q \in [q_k, q_{k+1}), \\ \qquad k \in \{0\} \cup \mathbb{K}_{m-1}, \\ \left(m + \eta_{\varphi}\right)(\chi + \varphi(q)) \\ \quad + C\,|\tilde{\varkappa}(q) - \varkappa(q)| \\ \quad + \gamma \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,|\tilde{\varkappa}(\lambda) - \varkappa(\lambda)|\,\mathrm{d}\lambda, \\ q \in (q_m, \mathfrak{c}]. \end{cases}
$$

Hence for each $q \in \mathbb{J}$, it follows

$$
\left|\tilde{\varkappa}(q) - \varkappa(q)\right| \leq \tfrac{1}{1-C} \tag{5.11}
$$

$$\times \begin{cases} (m+\eta_{\varphi})\,(\chi+\varphi(q)) \\ \quad +\sum_{p=1}^{m} \mu_p \left|\tilde{\varkappa}(q_p^-) - \varkappa(\tau_p^-)\right| \\ \quad +\frac{\gamma}{\Gamma(\upsilon)} \int_{\mathfrak{b}}^{q} \chi'(\lambda) \\ \quad \times \left(\chi_\lambda(q)\right)^{\upsilon-1} \left|\tilde{\varkappa}(\lambda) - \varkappa(\lambda)\right| \,\mathrm{d}\lambda \\ \quad +\frac{\gamma}{\Gamma(\upsilon)} \int_{\mathfrak{b}}^{\mathfrak{c}} \chi'(\lambda) \left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon-1} \\ \quad \times \left|\tilde{\varkappa}(\lambda) - \varkappa(\lambda)\right| \,\mathrm{d}\lambda, \end{cases} \tag{5.12}$$

for $q \in [q_k, q_{k+1})$ with $k \in \{0\} \cup \mathbb{K}_m$. Using the systems of inequalities Eq. (5.11) and Eq. (5.7) in Lemma 5.10, we get

$$\begin{aligned} &\left|\tilde{\varkappa}(q) - \varkappa(q)\right| \\ &\le \begin{cases} \frac{(m+\eta_\varphi)(\chi+\varphi(q))}{1-C}\Big[1+ \\ \quad \mu\,\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(q)\right)^{\upsilon}\Big) \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon}\Big)\Big]^{m-k} \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(q)\right)^{\upsilon}\Big) \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon}\Big), \\ \quad \forall (q,\lambda) \in (\mathbb{J} \setminus q_k) \times [\mathfrak{b}, q], \\ \qquad k \in \{0\} \cup \mathbb{K}_m, \end{cases} \\ &\le (\sigma + \varphi(q))\; c_{\phi,m,\varphi}, \end{aligned}$$

$\forall\, q \in \mathbb{J}$, where,

$$c_{\phi,m,\varphi} = \max\begin{Bmatrix} \frac{(m+\eta_\varphi)(\chi+\varphi(q))}{1-C} \\ \quad \times\Big[1+\mu\,\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(q)\right)^{\upsilon}\Big) \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon}\Big)\Big]^{m-k} \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(q)\right)^{\upsilon}\Big) \\ \quad \times\mathcal{E}_\upsilon\Big(\gamma\left(\chi_\lambda(\mathfrak{c})\right)^{\upsilon}\Big), \\ \quad k \in \{0\} \cup \mathbb{K}_m, \end{Bmatrix},$$

$$\forall (q,\lambda) \in (\mathbb{J} \setminus q_k) \times [\mathfrak{b}, q],$$

and $\mu = \max\{\mu_1, \mu_2, \ldots, \mu_m\}$. Thus, Eq. (1.1) and Eq. (1.3) is G.U.H.R. stable with respect to (χ, φ). The proof is completed. □

Remark 5.13. *By similar process we can extend the above results to the case of Eq.* (1.1) *and Eq.* (1.3).

6. Illustrative applications

In this section, we will show the correctness of the proved theorems with various examples. First, we introduce the suitable algorithm which we use to illustrate our method. Algorithm 1 shows the iterative procedure to estimate numerical values of all parameters for nonlinear impulsive FDE Eq. (1.1).

Algorithm 1 Iterative procedure to estimate Eq. (1.1).

Require: Input υ, χ, $\mathfrak{b}$, $\mathfrak{c}$, $\varkappa_{\mathfrak{c}}$, q, ϕ, k, ϱ_k, Λ, μ_k.

```
column ← 1;
for s = 1 to y_υ do
    n ← 1.
    svar ← 𝔟.
    sum1 ← sum(μ_k).
    while svar <= 𝔠 do
        pm(n, column) ← n;
        pm(n, column+1) ← svar;
        pm(n, column + 2 ) ← sum1 + μ_p + Λ max_{q∈𝕁} ∫_𝔟^𝔠 𝒦(q, λ) dλ;
        n ← n + 1;
        svar ← svar + 𝔠/10;
    end while
    column ← column+3;
end for
```

Ensure: Eq. (3.12).

Example 6.1. We consider nonlinear local impulsive FDE

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}^{\upsilon}_{\mathfrak{b}^+}\varkappa(q) = \frac{q}{15+\exp(q)+|\varkappa(q)|}, \\ \qquad q \in \mathbb{J} = [0,\pi],\ q \neq q_k, \\ \triangle\varkappa\big|_{q=q_k} = \frac{1}{15k+|\varkappa(q_k^-)|}, \\ \qquad k \in \mathbb{K}_5,\ q_k = \frac{\pi}{5}k, \\ \varkappa(\pi) = 1. \end{cases} \tag{6.1}$$

for $\upsilon = \frac{1}{2}, \frac{2}{3}, \frac{6}{7} \in (0,1)$. Clear $\chi(q) = \sqrt{q}$, $\mathfrak{b} = 0$, $\mathfrak{c} = \pi$, $\varkappa_{\mathfrak{c}} = 1$, $q_1 = \frac{\pi}{5}$, $q_2 = \frac{2\pi}{5}$, $q_3 = \frac{3\pi}{5}$, $q_4 = \frac{4\pi}{5}$, and $q_5 = \pi$. We set

$$\phi(q,\varkappa) = \frac{q}{15+\exp(q)+|\varkappa|},$$

and $\varrho_k(\varkappa) = \frac{1}{15k+|\varkappa|}$. By using this values, we obtain,

$$\begin{aligned} &|\phi(q,\varkappa) - \phi(q,\acute{\varkappa})| \\ &\quad = \left|\frac{q}{15+\exp(q)+|\varkappa|} - \left(\frac{q}{15+\exp(q)+|\acute{\varkappa}|}\right)\right| \\ &\quad = \frac{q\,\big||\varkappa|-|\acute{\varkappa}|\big|}{(15+\exp(q)+|\varkappa|)(15+\exp(q)+|\acute{\varkappa}|)} \\ &\quad \leqslant \frac{\pi|\varkappa-\acute{\varkappa}|}{(15+1)^2} = \frac{\pi}{256}\,|\varkappa-\acute{\varkappa}|, \end{aligned}$$

$$\begin{aligned} &|\varrho_k(\varkappa) - \varrho_k(\acute{\varkappa})| \\ &\quad = \left|\frac{1}{15k+|\varkappa|} - \frac{1}{15k+|\acute{\varkappa}|}\right| \\ &\quad \leqslant \frac{\big||\varkappa|-|\acute{\varkappa}|\big|}{(15k)^2} \leqslant \frac{|\varkappa-\acute{\varkappa}|}{15^2} = \frac{1}{225}\,|\varkappa-\acute{\varkappa}|, \end{aligned}$$

Then the conditions (H1) and (H2) holds with $\Lambda = \frac{\pi}{256}$, $\mu_k = \frac{1}{225}$, with $k \in \mathbb{K}_5$. Since

$$\sum_{k=1}^{5}\mu_k + \max_{q\in\mathbb{J}}\int_0^{\pi} K(q,\lambda)\,\mathrm{d}\lambda \tag{6.2}$$

$$\begin{aligned} &= m\mu_k + \frac{\Lambda(m+1)}{\Gamma(\upsilon_i+1)}\chi(\pi)^{\upsilon_i} \\ &= \frac{5}{225} + \frac{6\pi}{256\Gamma(\upsilon_i+1)}\cdot\chi(\pi)^{\upsilon_i} \\ &\simeq \left\{\begin{array}{ll} 0.1328, & \upsilon = 1/2, \\ 0.1416, & \upsilon = 2/3, \\ 0.1491, & \upsilon = 6/7, \end{array}\right\} < 1. \end{aligned}$$

By using Algorithm 4 we obtain all parameters which are shown in Table 1. Further in Fig. 1, we have plotted the results for the nonlinear local impulsive FDE Eq. (6.1) and three cases of υ. The condition Eq. (3.12) is satisfied. Thus by Theorem 3.3, the nonlinear local impulsive FDE Eq. (6.1) has a unique solution. Now, let $\widetilde{\varkappa} \in PC(\mathbb{J})$ be a

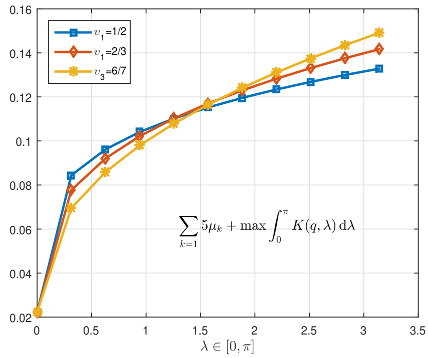


Fig. 1. Representation of Eq. (6.2) for three cases of υ in Example 6.1.

solution of the inequality Eq. (5.1). Denote by $\varkappa$ the unique solution to the backward impulsive FDE Eq. (6.1), then we obtain,

$$\begin{aligned} &\Bigg|\varkappa(q) - 1 + \sum_{p=0}^{k-1}\varrho_{5-p}\big(\varkappa(q_{5-p}^-)\big) \\ &\quad + \frac{1}{\Gamma(\upsilon_i)}\sum_{p=0}^{k}\int_{q_{5-p}}^{q_{5-p+1}}\left(\sqrt{q_{5-p+1}} - \sqrt{\lambda}\right)^{-\upsilon_i} \\ &\quad \times \left(\frac{\pi}{2\sqrt{\lambda}(15+e^{\lambda}+|\widetilde{\varkappa}(q)|)}\right)\mathrm{d}\lambda \\ &\quad - \frac{1}{\Gamma(\upsilon_i)}\sum_{p=0}^{k}\int_{q_{5-p}}^{q_{5-p+1}}\left(\sqrt{q} - \sqrt{\lambda}\right)^{-\upsilon_i} \\ &\quad \times \frac{\frac{\pi}{2\sqrt{\lambda}}}{15+e^{\lambda}+|\widetilde{\varkappa}(q)|}\Bigg|\,\mathrm{d}\lambda \\ &\leqslant \sum_{p=1}^{5}|\widetilde{\varkappa}_p| \\ &\quad + \frac{1}{\Gamma(\upsilon_i)}\sum_{p=1}^{5}\int_0^{\pi}\left(\sqrt{\pi} - \sqrt{\lambda}\right)^{-\upsilon_i}\varepsilon\frac{1}{2\sqrt{\lambda}}\,\mathrm{d}\lambda \\ &\quad - \frac{1}{\Gamma(\upsilon_i)}\sum_{p=1}^{5}\int_0^{\pi}\left(\sqrt{q} - \sqrt{\lambda}\right)^{-\upsilon_i}\varepsilon\frac{1}{2\sqrt{\lambda}}\,\mathrm{d}\lambda \end{aligned}$$

$$\leqslant \left(5+\frac{10}{\Gamma(\upsilon_i)}\sqrt[4]{\pi}-\frac{2}{\Gamma(\upsilon_i)}\sqrt[4]{q}\right)\varepsilon$$

$$\simeq \begin{cases} 11.0090\varepsilon, & \upsilon = 1/2, \\ 12.8654\varepsilon, & \upsilon = 2/3, \\ 14.6319\varepsilon, & \upsilon = 6/7, \end{cases}$$

Hence for each $q \in (q_{5-k}, q_{5-k+1})$ and $k \in \mathbb{K}_5$, we can write

$$\|\widetilde{\varkappa}-\varkappa\|_{PC}$$

$$\leqslant \frac{\left(5+\frac{10\sqrt[4]{\pi}}{\Gamma(\upsilon_i)}-\frac{2\sqrt[4]{q}}{\Gamma(\upsilon_i)}\right)\varepsilon}{1-\frac{1}{45}-\frac{5\pi\sqrt[4]{\pi}}{256\Gamma(\upsilon_i+1)}-\frac{\pi}{256\Gamma(\upsilon_i+1)}\sqrt[4]{q}}$$

$$\simeq \left\{\begin{array}{ll} 14.1274\varepsilon, & \upsilon = 1/2, \\ 16.7158\varepsilon, & \upsilon = 2/3, \\ 19.1119\varepsilon, & \upsilon = 6/7, \end{array}\right\} = c_{\phi,m}\varepsilon.$$

Fig. 2 shows the curve of $c_{\phi,m}$ the non-

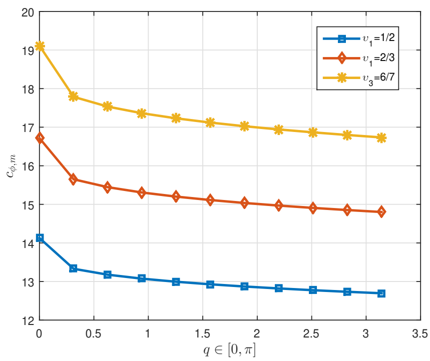


Fig. 2. Representation of $c_{\phi,m}$ for three cases of υ in Example 6.1.

linear impulsive FDE Eq. (6.1) with three cases of υ_i. This implies that Eq. (6.1) is generalized U.H stable.

Example 6.2. We consider nonlinear local impulsive FDE Eq. (6.1) in Example 6.1 with $\upsilon = \frac{2}{3}$ and four cases of $\chi_i(q)$,

$$\chi_1(q) = 2^q, \tag{6.3}$$
$$\chi_2(q) = q,$$
$$\chi_3(q) = \ln(q+0.01),$$
$$\chi_4(q) = \sqrt{q},$$

as,

$$\begin{cases} {}^{\chi}_{\mathrm{C}}\mathrm{D}_{0^+}^{2/3}\varkappa(q) = \frac{q}{15+\exp(q)+|\varkappa(q)|}, \\ \quad q \in \mathbb{J} = [0,\pi],\ q \neq q_k \\ \Delta\varkappa\big|_{q=q_k} = \frac{1}{15k+|\varkappa(q_k^-)|},\ \varkappa(\pi) = 1, \\ \quad k \in \mathbb{K}_5,\ q_k = \frac{\pi}{5}k. \end{cases} \tag{6.4}$$

Since

$$\sum_{k=1}^{5}\mu_k + \max_{q\in\mathbb{J}}\int_0^{\pi} K(q,\lambda)\,\mathrm{d}\lambda \tag{6.5}$$

$$= m\mu_k + \frac{\Lambda(m+1)}{\Gamma(\frac{2}{3}+1)}\chi_i(\pi)^{\upsilon}$$

$$= \frac{5}{225} + \frac{6\pi}{256\Gamma\left(\frac{5}{3}\right)}\cdot\chi_i(\pi)^{2/3}$$

$$\simeq \left\{\begin{array}{ll} 0.3705, & \chi_1(q) = 2^q, \\ 0.1972, & \chi_2(q) = q, \\ 0.1254, & \chi_3(q) = \ln(q+1.01), \\ 0.1417, & \chi_4(q) = \sqrt{q}, \end{array}\right\}$$

$$< 1.$$

One can use Algorithm 5 for computing all variables in this example which are shown in Table 2. we have plotted the results for the nonlinear local impulsive FDE Eq. (6.4) and four cases of χ in Fig. 3. The condition Eq. (3.12) is satisfied. Thus by Theorem 3.3, the nonlinear local impulsive FDE Eq. (6.4) has a unique solution. Now, let $\widetilde{\varkappa} \in PC(\mathbb{J})$ be a solution of the inequality Eq. (5.1). Denote by $\varkappa$ the unique solution to the backward impulsive FDE Eq. (6.4), then we obtain

$$\Bigg|\varkappa(q) - 1 + \sum_{p=0}^{k-1}\varrho_{5-p}(\varkappa(q_{5-p}^-))$$

$$+ \frac{1}{\Gamma(\frac{2}{3})}\sum_{p=0}^{k}\int_{q_{5-p}}^{q_{5-p+1}}(\chi_i(q_{5-p+1})$$

$$- \chi_i(\lambda))^{-1/3}\left(\frac{\pi\chi_i'(\lambda)}{15+e^{\lambda}+|\widetilde{\varkappa}(q)|}\right)\mathrm{d}\lambda$$

Table 1. Numerical results of Eq. (6.2) and $c_{\phi,m}$ in Example 6.1 for three cases of υ.

	$\boldsymbol{\upsilon_1 = \frac{1}{2}}$		$\boldsymbol{\upsilon_2 = \frac{2}{3}}$		$\boldsymbol{\upsilon_3 = \frac{6}{7}}$	
$\lambda,\ q$	Eq. (6.2)	$c_{\phi,m}$	Eq. (6.2)	$c_{\phi,m}$	Eq. (6.2)	$c_{\phi,m}$
0.0000	0.0222	14.1274	0.0222	16.7158	0.0222	19.1119
0.3142	0.0844	13.3296	0.0777	15.6491	0.0695	17.7865
0.6283	0.0962	13.1765	0.0921	15.4445	0.0859	17.5324
0.9425	0.1041	13.0734	0.1022	15.3068	0.0980	17.3615
1.2566	0.1102	12.9935	0.1102	15.2000	0.1079	17.2289
1.5708	0.1152	12.9273	0.1170	15.1115	0.1165	17.1191
1.8850	0.1196	12.8703	0.1230	15.0353	0.1242	17.0246
2.1991	0.1234	12.8199	0.1283	14.9680	0.1311	16.9411
2.5133	0.1268	12.7747	0.1331	14.9075	0.1375	16.8660
2.8274	0.1300	12.7334	0.1376	14.8524	0.1435	16.7976
3.1416	0.1328	12.6954	0.1417	14.8016	0.1491	16.7346

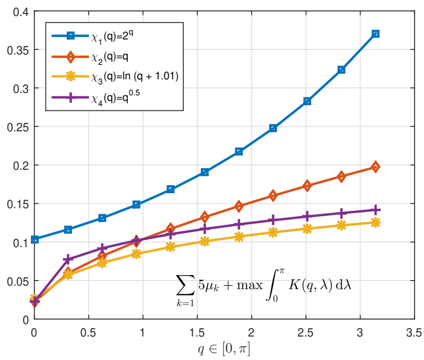


Fig. 3. Representation of Eq. (6.5) for four cases of χ in Example 6.2.

$$- \frac{1}{\Gamma(\frac{2}{3})} \sum_{p=0}^{k} \int_{q_{5-p}}^{q_{5-p+1}} (\chi_i(q) - \chi_i(\lambda))^{-1/3} \frac{\pi \chi_i'(\lambda)}{15+e^{\lambda}+|\widetilde{\varkappa}(q)|} \Bigg| \mathrm{d}\lambda$$

$$\leqslant \sum_{p=1}^{5} |\widetilde{\varkappa}_p| + \frac{1}{\Gamma(\frac{2}{3})} \sum_{p=1}^{5} \int_0^{\pi} (\chi_i(\pi) - \chi_i(\lambda))^{-1/3} \varepsilon\, \chi_i'(\lambda)\, \mathrm{d}\lambda - \frac{1}{\Gamma(\frac{2}{3})} \sum_{p=1}^{5} \int_0^{\pi} (\chi_i(q) - \chi_i(\lambda))^{-1/3} \varepsilon\, \chi_i'(\lambda)\, \mathrm{d}\lambda$$

$$\leqslant \left(5 + \frac{10}{\Gamma(\frac{2}{3})} (\chi_i(\pi))^{2/3} - \frac{2}{\Gamma(\frac{2}{3})} (\chi_i(q))^{2/3}\right) \varepsilon$$

$$\simeq \begin{cases} 5.0000, & \chi_1(q) = 2^q, \\ 17.8701, & \chi_2(q) = q, \\ 5.0000, & \chi_3(q) = \ln(q+1.01), \\ 13.7875, & \chi_4(q) = \sqrt{q}, \end{cases}$$

Hence, for $q \in (q_{5-k}, q_{5-k+1})$ and $k \in \mathbb{K}_5$,

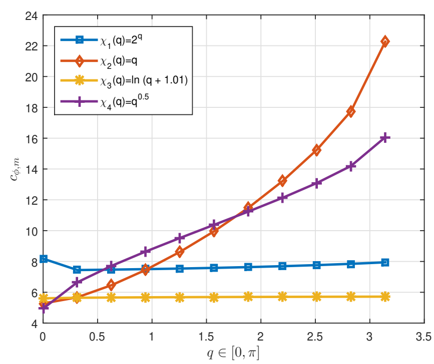


Fig. 4. Representation of $c_{\phi,m}$ for three cases of υ in Example 6.2.

we can write

$$\|\widetilde{\varkappa} - \varkappa\|_{PC}$$

$$\leqslant \frac{\left(5+\frac{10\sqrt[4]{\pi}}{\Gamma(\upsilon_i)}-\frac{2\sqrt[4]{q}}{\Gamma(\upsilon_i)}\right)\varepsilon}{1-\frac{1}{45}-\frac{5\pi(\chi(\pi))^{2/3}}{256\Gamma\left(\frac{5}{3}\right)}-\frac{\pi}{256\Gamma\left(\frac{5}{3}\right)}(\chi(\pi))^{2/3}}$$
$$\simeq \left\{\begin{array}{ll} 7.9432, & \chi_1(q)=2^q, \\ 22.2592, & \chi_2(q)=q, \\ 5.7171, & \chi_3(q)=\ln(q+1.01), \\ 16.0634, & \chi_4(q)=\pi^q, \end{array}\right\}$$
$$= c_{\phi,m}\varepsilon.$$

Fig. 4 shows the curve of $c_{\phi,m}$ the nonlinear impulsive FDE Eq. (6.4) with four cases of $\chi_i(q)$. It means that Eq. (6.4) is generalized U.H stable.

Example 6.3. In this example, base on Eq. (1.1) and Eq. (1.3), we consider the following backward impulsive FDE

$$\left\{\begin{array}{l} {}^{\chi}_{C}\mathrm{D}^{1/2}_{\mathfrak{b}^+}\varkappa(q) = \frac{\exp(-q)\cos(|\varkappa(q)|)}{50+q}, \\ \quad q \in \mathbb{J} = [0,1], q \neq q_k, \\ \triangle\varkappa|_{q=q_k} = \frac{1}{100k}\left[\sin\left(\left|\varkappa(q_k^-)\right|\right)\right. \\ \quad \left.+\left|\varkappa(q_k^-)\right|\right], \\ \quad k \in \mathbb{K}_5,\ q_k = \frac{1}{5}k, \\ \varkappa(1) = \frac{1}{2} + \frac{1}{40}\cos(|\varkappa|), \end{array}\right. \tag{6.6}$$

with for cases of

$$\begin{aligned} &\chi_1(q) = 2^q, \\ &\chi_2(q) = q, \\ &\chi_3(q) = \ln(q+1), \\ &\chi_4(q) = \sqrt{q}. \end{aligned} \tag{6.7}$$

We define

$$\begin{aligned} \phi(q,\varkappa) &= \frac{1}{50+q}\exp(-q)\cos(|\varkappa|), \\ \varrho_k(\varkappa) &= \frac{1}{100k}\left[\sin(|x|)+|x|\right], \\ \omega(\varkappa) &= \frac{1}{40}\cos(|\varkappa|). \end{aligned}$$

Then, for $\varkappa, \acute{\varkappa} \in PC(\mathbb{J})$ we have

$$\begin{aligned} |\phi(q,\varkappa)-\phi(q,\acute{\varkappa})| &= \left|\frac{\exp(-q)\cos(|\varkappa|)}{50+q} - \frac{\exp(-q)\cos(|\acute{\varkappa}|)}{50+q}\right| \\ &\leqslant \frac{1}{50}\left|\cos(|\varkappa|)-\cos(|\acute{\varkappa}|)\right| \\ &\leqslant \frac{1}{50}\left||\varkappa|-|\acute{\varkappa}|\right| \leqslant \frac{1}{50}\left|\varkappa-\acute{\varkappa}\right|, \end{aligned}$$

and

$$\begin{aligned} |\varrho_k(\varkappa)-\varrho_k(\acute{\varkappa})| &\leqslant \left|\frac{\sin(|\varkappa|)+|\varkappa|}{100k} - \frac{\sin(|\acute{\varkappa}|)+|\acute{\varkappa}|}{100k}\right| \\ &\leqslant \frac{1}{100}\Big[\left|\sin(|\varkappa|)-\sin(|\acute{\varkappa}|)\right| \\ &\quad + \left||\varkappa|-|\acute{\varkappa}|\right|\Big] \leqslant \frac{1}{50}\left|\varkappa-\acute{\varkappa}\right|, \end{aligned}$$

$$\begin{aligned} |\omega(\varkappa)-\omega(\acute{\varkappa})| &\leqslant \frac{1}{40}\left|\cos(|\varkappa|)-\cos(|\acute{\varkappa}|)\right| \\ &\leqslant \frac{1}{20}\left|\varkappa-\acute{\varkappa}\right|. \end{aligned}$$

Hence, the conditions (H1), (H2) and (H5) holds with $\Lambda = \frac{1}{50}$, $\mu_k = \frac{1}{50}$, with $k \in \mathbb{K}_5$, $C = \frac{1}{20}$. Thus, by employing Eq. (4.2), and using Algorithm 6, we get

$$C + \sum_{p=1}^{5}\mu_p + \max_{q\in\mathbb{J}}\int_0^1 \mathcal{K}(q,\lambda)\,\mathrm{d}\lambda \tag{6.8}$$
$$\begin{aligned} &= \frac{1}{20} + \frac{1}{10} + \frac{\Lambda(m+1)}{\Gamma(\upsilon+1)}\chi(\mathfrak{c})^{\upsilon} \\ &= \frac{3}{20} + \frac{\frac{6}{50}}{\Gamma\left(\frac{3}{2}\right)}\left(\chi_i(\mathfrak{c})\right)^{1/2} \\ &\simeq \left\{\begin{array}{ll} 0.3414, & \chi_1(q)=2^q, \\ 0.2854, & \chi_2(q)=q, \\ 0.2627, & \chi_3(q)=\ln(q+1), \\ 0.2854, & \chi_4(q)=\sqrt{q}, \end{array}\right\} \\ &< 1. \end{aligned} \tag{6.9}$$

These results are shown in Table 3 and in Fig.s 5a and 5b, we have plotted the results for the nonlinear local impulsive FDE Eq. (6.4) with four cases of χ. The condition Eq. (4.2) is satisfied. Thus by Theorem 4.1, Eq. (6.6) has a unique solution. Now, let $\widetilde{\varkappa} \in PC(\mathbb{J})$ be a solution of the inequality Eq. (5.2). Denote by $\varkappa$ the unique solution to the backward impulsive

Table 2. Numerical results of Eq. (6.5) and $c_{\phi,m}$ in Example 6.2 with $\upsilon = \frac{2}{3}$ and for four cases of χ_i.

	$\chi_1(q) = 2^q$		$\chi_2(q) = q$		$\chi_3(q) = \ln(q+1.01)$		$\chi_4(q) = \sqrt{q}$	
$\lambda,\ q$	Eq. (6.5)	$c_{\phi,m}$	Eq. (6.5)	$c_{\phi,m}$	Eq. (6.5)	$c_{\phi,m}$	Eq. (6.5)	$c_{\phi,m}$
0.0000	0.1038	8.1804	0.0222	5.2756	0.0260	5.6108	0.0222	4.9426
0.3142	0.1165	7.4427	0.0599	5.6733	0.0572	5.6437	0.0777	6.6498
0.6283	0.1313	7.4700	0.0821	6.4608	0.0732	5.6607	0.0921	7.7144
0.9425	0.1483	7.5018	0.1006	7.4531	0.0846	5.6730	0.1022	8.6471
1.2566	0.1680	7.5390	0.1172	8.6222	0.0936	5.6826	0.1102	9.5235
1.5708	0.1908	7.5824	0.1324	9.9618	0.1009	5.6906	0.1170	10.3791
1.8850	0.2171	7.6332	0.1467	11.4829	0.1072	5.6973	0.1230	11.2394
2.1991	0.2476	7.6928	0.1602	13.2186	0.1126	5.7032	0.1283	12.1304
2.5133	0.2828	7.7629	0.1730	15.2442	0.1173	5.7083	0.1331	13.0899
2.8274	0.3235	7.8455	0.1853	17.7540	0.1216	5.7129	0.1376	14.1997
3.1416	0.3705	7.9433	0.1972	22.2592	0.1254	5.7171	0.1417	16.0634

Table 3. Numerical results of Eq. (6.8) and η_φ in Example 6.3 with $\upsilon = \frac{1}{2}$ and for four cases of χ_i.

	$\chi_1(q) = 2^q$		$\chi_2(q) = q$		$\chi_3(q) = \ln(q+1)$		$\chi_4(q) = \sqrt{q}$	
q	Eq. (6.8)	η_φ	Eq. (6.8)	η_φ	Eq. (6.8)	η_φ	Eq. (6.8)	η_φ
0.0000	0.2854	1.1284	0.1500	0.0000	0.1500	0.0000	0.1500	0.0000
0.1000	0.2902	1.1682	0.1928	0.3568	0.1918	0.3484	0.2261	0.6345
0.2000	0.2951	1.2094	0.2106	0.5046	0.2078	0.4818	0.2406	0.7546
0.3000	0.3002	1.2520	0.2242	0.6180	0.2194	0.5780	0.2502	0.8351
0.4000	0.3055	1.2962	0.2356	0.7137	0.2285	0.6545	0.2577	0.8974
0.5000	0.3110	1.3419	0.2457	0.7979	0.2362	0.7185	0.2639	0.9489
0.6000	0.3167	1.3892	0.2549	0.8740	0.2428	0.7736	0.2692	0.9931
0.7000	0.3226	1.4382	0.2633	0.9441	0.2486	0.8220	0.2739	1.0321
0.8000	0.3287	1.4889	0.2711	1.0093	0.2538	0.8651	0.2781	1.0672
0.9000	0.3350	1.5414	0.2785	1.0705	0.2585	0.9040	0.2819	1.0990
1.0000	0.3415	1.5958	0.2854	1.1284	0.2627	0.9394	0.2854	1.1284

FDE Eqs. (1.1)-(1.3), then we have

$$
\begin{aligned}
&\int_0^1 \mathcal{K}(q,\lambda)\varphi(\lambda)\,\mathrm{d}\lambda \\
&\leqslant \max_{0\leqslant q,\lambda\leqslant 1} \mathcal{K}(q,\lambda)\int_0^1 \varphi(\lambda)\,\mathrm{d}\lambda \\
&\leqslant \frac{(\chi_i(\mathfrak{c}))^{1/2}}{\Gamma(\frac{3}{2})}\varphi(q) \\
&\simeq \left\{\begin{array}{ll} 1.5957\varphi(q), & \chi_1(q) = 2^q, \\ 1.1283\varphi(q), & \chi_2(q) = q, \\ 0.9394\varphi(q), & \chi_3(q) = \ln(q+1), \\ 1.1283\varphi(q), & \chi_4(q) = \sqrt{q}, \end{array}\right\} \\
&= \eta_\varphi\varphi(q),
\end{aligned}
$$

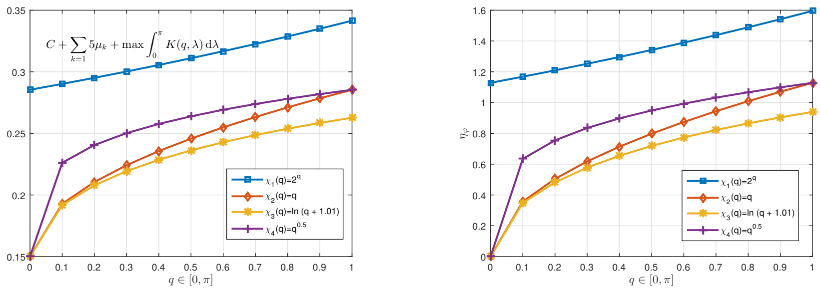
here $\eta_\varphi = \frac{1}{\Gamma(\frac{2}{3})}(\ln(2))^{1/2} > 0$. The condition of Theorem 5.12 is satisfied. Thus, Eq. (6.6) is G.U.H.R stable with respect to η_φ.

Example 6.4. Consider nonlocal impulsive FDE

$$
\begin{cases}
{}^{\chi}_{\mathrm{C}}\mathrm{D}^{1/2}_{\mathfrak{b}^+}\varkappa(q) = \frac{q\sin(\varkappa)+\sqrt{q}}{1+q}, \\
\qquad q \in \mathbb{J} = [0,1],\ q \neq q_k, \\
\Delta\varkappa|_{q=q_k} = \frac{\exp(-|\varkappa(q_k^-)|)+|\varkappa(q_k^-)|}{20k}, \\
\qquad k \in \mathbb{K}_5,\ q_k = \frac{1}{5}k, \\
\varkappa(1) = 1+ \\
\qquad \frac{1}{10}\left(\arctan(|\varkappa|)+1\right),
\end{cases} \tag{6.10}
$$

with $\chi(q) = \sqrt{1+|q|}$ for three cases $\upsilon = \frac{1}{2}, \frac{3}{4}, \frac{11}{12} \in (0,1)$. Clearly, $\varkappa_{\mathfrak{c}} = 1$,

$$
\begin{aligned}
\phi(q,\varkappa) &= \frac{\sqrt{q}}{1+q}\left(\sqrt{q}\sin(\varkappa)+1\right), \\
\varrho_k(\varkappa) &= \frac{\exp(-|\varkappa|)+|\varkappa|}{20k}, \\
\omega(\varkappa) &= \frac{\arctan(|\varkappa|)+1}{10}.
\end{aligned}
$$



(a) **(b)**

Fig. 5. Representation of Eq. (6.8) for four cases of χ in Example 6.3.

Also, for $\varkappa, \acute{\varkappa} \in \mathbb{R}$, we have

$$\begin{aligned}&|\phi(q,\varkappa) - \phi(q,\acute{\varkappa})|\\ &\quad = \left|\frac{q\sin(\varkappa)+\sqrt{q}}{1+q} - \frac{q\sin(\acute{\varkappa})+\sqrt{q}}{1+q}\right|\\ &\quad \leqslant \Lambda\,|\varkappa - \acute{\varkappa}|,\end{aligned}$$

where $\Lambda = 1$. Thus,

$$|\phi(q,\varkappa)| \leqslant q\,|\varkappa| + 1,$$

$$|\varrho_k(\varkappa)| \leqslant \frac{|\varkappa|+1}{20},$$

$$|\omega(\varkappa)| \leq \frac{1}{10}\,|\varkappa| + \frac{1}{10},$$

and for $\varkappa, \acute{\varkappa}$, we have

$$\begin{aligned}&|\omega(\varkappa) - \omega(\acute{\varkappa})| =\\ &\quad \left|\frac{\arctan(|\varkappa|)+1}{10} - \frac{\arctan(|\acute{\varkappa}|)+1}{10}\right|\\ &\quad \leqslant \frac{1}{10}\big|\,|\varkappa| - |\acute{\varkappa}|\,\big|.\end{aligned}$$

Thus, assumption (H3)-(H6) hold by considering $\Phi(q) = q$, $\theta_0 = 1$, ${}_1\theta_k = {}_2\theta_k = \frac{1}{20}$, for $k \in \mathbb{K}_5$, $C = \frac{1}{10}$, and $\tilde{D} = K = \frac{1}{10}$. Thus, thanks to Algorithm 7, Eq. (4.3) implies that

$$\max\left\{\sum_{p=1}^{m} {}_2\theta_p \right. \tag{6.11}$$

$$\left. + \theta_0 \max_{q\in\mathbb{J}} \int_{\mathfrak{b}}^{\mathfrak{c}} \mathcal{K}(q,\lambda)\,d\lambda, |\varkappa_{\mathfrak{c}}| + K\right\}$$

$$\leq \max\left\{\frac{1}{4} + \frac{\Lambda(m+1)}{\Gamma(\upsilon_i+1)}\left(\sqrt{1+|q|}\right)^{\upsilon_i}, \frac{11}{10}\right\}$$

$$= \max\left\{\frac{1}{4} + \frac{6}{\Gamma(\upsilon_i+1)}\left(\sqrt{1+|q|}\right)^{\upsilon_i}, 1.1\right\}$$

$$\simeq \max\left\{\left\{\begin{array}{ll} 8.3012, & \upsilon_1 = 1/2,\\ 8.7162, & \upsilon_2 = 3/4,\\ 8.7698, & \upsilon_3 = 11/12, \end{array}\right\}, 1.1\right\}$$

$$= \left\{\begin{array}{ll} 8.3012, & \upsilon_1 = 1/2,\\ 8.7162, & \upsilon_2 = 3/4,\\ 8.7698, & \upsilon_3 = 11/12, \end{array}\right\}$$

$$\leq 4r, \tag{6.12}$$

here $r = 2.5$ for all cases υ_i. in Hence, Eq. (4.3) in Theorem 4.2 hold. This implies that Eq. (6.10) has at least a solution in $PC(\mathbb{J})$. In the other hand,

$$\max\left\{\sum_{p=1}^{5} {}_1\theta_p \right. \tag{6.13}$$

$$\left. + \max_{q\in\mathbb{J}} \int_0^1 \mathcal{K}(q,\lambda)\,\Phi(\lambda)\,d\lambda,\ \widetilde{D}\right\}$$

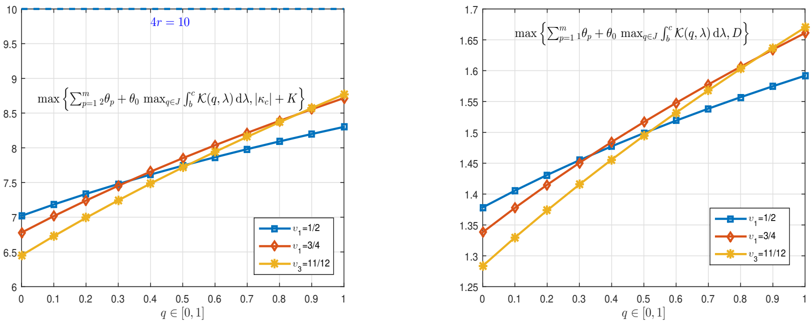
$$= \max\left\{\frac{1}{4} + \frac{\left(\sqrt{1+|q|}\right)^{\upsilon_i}}{\Gamma(\upsilon_i+1)}, \frac{1}{10}\right\}$$

$$\simeq \max\left\{\left\{\begin{array}{ll} 1.5918, & \upsilon_1 = 1/2,\\ 1.6610, & \upsilon_2 = 3/4,\\ 1.6699, & \upsilon_3 = 11/12, \end{array}\right\}, \frac{1}{10}\right\}$$

$$= \left\{\begin{array}{ll} 1.5918, & \upsilon_1 = 1/2,\\ 1.6610, & \upsilon_2 = 3/4,\\ 1.6699, & \upsilon_3 = 11/12, \end{array}\right\} \tag{6.14}$$

$$< 4.$$

Figs. 6- 6b show the curve of Eq. (6.11)



(a) **(b)**

Fig. 6. Representation of Eqs. (6.11)-(6.13) for three cases of υ in Example 6.4.

and Eq. (6.13) for three cases υ for nonlinear local impulsive FDE Eq. (6.10). The condition of Theorem 4.2 is satisfied. Thus, Eq. (6.10) has at least a solution in $PC(\mathbb{J})$.

7. Conclusion

Usually, the DEs are given under the initial conditions in a forward manner, that is, starting at $q = \mathfrak{b}$. But for other classes of problems in which the initial state set is unknown the procedure may be more convenient if one considers backward initial conditions, i.e., at $q = \mathfrak{c}$. This approach plays a vital role in many physical areas. A typical example of such a problem is the backward heat problem, also known as the terminal value problem. For application in stochastic DEs, see, for example [40]. However, as far as we know, there are few results on the existence and stability of solutions to backward impulsive differential equations. In this paper, By using some well known classical FP theorems, we establish the existence theory for the considered problems. Also we develop some results for H.U stability and generalized stability. Pertinent examples are given to verify our results. Comparing the data obtained from the selection of several types of local backward problems for nonlinear impulsive FDE is very significant.

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Appendices

Table 4. Algorithm 1: MATLAB lines to calculate all parameters in Example 6.1 for three cases of υ.

```
syms v e;
upsilon=[0.5 2/3 6/7];
[xupsilon yupsilon]=size(upsilon);
chi=sqrt(v);
mathfrakb=0; mathfrakc=pi;
varkappa_mathfrakc=1;
q=[pi/5 2*pi/5 3*pi/5 4*pi/5 pi];
phi=v/(15+exp(v)+abs(e));
k=[1 2 3 4 5];
[xk yk]=size(k);
varrho_k=1./(k+abs(e));
Lambda=pi/256;
upmu_k=[1/225 1/225 1/225 1/225 1/225];
[xupmu_k yupmu_k]=size(upmu_k);
a=mathfrakb; T=pi;
column=1;
for s=1:yupsilon
    n=1;
    svar=a;
    sum1=sum(upmu_k);
    while svar<=T
        pm(n,column)=n;
        pm(n,column+1)=svar;
        pm(n,column+2)=sum1 + ...
          Lambda * ( yupmu_k + 1 ) *…
          eval( subs( chi, {v}, {svar} ) )^...
          ( upsilon(s) ) / gamma( upsilon(s) + 1 );
        n=n+1;
        svar=svar+pi/10;
    end;
    column=column+3;
end;
```

Table 5. Algorithm 2: MATLAB lines to calculate all parameters in Example 6.2 for four cases of $\chi_i(q)$.

```
syms v e;
upsilon=2/3;
chi=[2^v v log(v+1.01) sqrt(v)];
[xchi ychi]=size(chi);
mathfrakb=0; mathfrakc=pi;
varkappa_mathfrakc=1;
q=[pi/5 2*pi/5 3*pi/5 4*pi/5 pi];
phi=v/(15+exp(v)+abs(e));
k=[1 2 3 4 5];
[xk yk]=size(k);
varrho_k=1./(k+abs(e));
Lambda=pi/256;
upmu_k=[1/225 1/225 1/225 1/225 1/225];
[xupmu_k yupmu_k]=size(upmu_k);
a=mathfrakb; T=pi;
column=1;
for s=1:ychi
    n=1;
    svar=a;
    sum1=sum(upmu_k);
    while svar<=T
        pm(n,column)=n;
        pm(n,column+1)=svar;
        pm(n,column+2)=sum1
          + Lambda * (yupmu_k+1)
          * eval( subs( chi(s), {v},
          {svar} ) )^(upsilon) /
          gamma(upsilon+1);
        A=5 + yupmu_k * ( int( (
          eval( subs( chi(s), {v},
          {mathfrakc}) ) - chi(s) )^(
          upsilon - 1 ) * diff(chi(s) ),
          'v', a, svar ) ) - ( int ( ( eval(
          subs( chi(s), {v}, {svar} ) )
          - chi(s) )^(upsilon - 1 ) *
          diff(chi(s)), 'v',a, mathfrakc));
        pm(n,column+3)=abs(A);
        pm(n,column+4)=abs(A)
          / ( 1 - 1/45 - 5 * mathfrakc *
          ( eval( subs( chi(s), {v},
          {mathfrakc} ) ) ) ^( upsilon)
          / ( 256 * gamma( upsilon+1) )
          - mathfrakc * ( eval( subs(
          chi(s), {v}, {svar} ) ) )^(
          upsilon ) / ( 256 *
          gamma( upsilon + 1 ) ) );
        n=n+1;
        svar=svar+pi/10;
    end;
    column=column+5;
end;
```

Table 6. Algorithm 3: MATLAB lines to calculate all parameters in Example 6.3 for four cases of $\chi_i(q)$.

```
syms v e;
upsilon=1/2;
chi=[2^v v log(v+1) sqrt(v)];
[xchi ychi]=size(chi);
mathfrakb=0; mathfrakc=1;
varkappa_mathfrakc=1;
q=[1/5 2/5 3/5 4/5 1];
phi=exp(-v)/(15+exp(v)+abs(e));
k=[1 2 3 4 5];
[xk yk]=size(k);
varrho_k=1./(k+abs(e));
omega=cos(abs(v))/40;
Lambda=1/50;
upmu_k=[1/50 1/50 1/50 1/50 1/50];
[xupmu_k yupmu_k]=size(upmu_k);
C=1/20;
a=mathfrakb; T=mathfrakc;
column=1;
for s=1:ychi
    n=1;
    svar=a;
    sum1=sum(upmu_k);
    while svar<=T
        pm(n,column)=n;
        pm(n,column+1)=svar;
        pm(n,column+2)=C + sum1
          + Lambda * ( yupmu_k + 1 )
          * eval( subs( chi(s), {v},
          {svar}) )^( upsilon ) /
          gamma(upsilon + 1);
        A= ( eval( subs( chi(s), {v},
          {svar})) )^( upsilon) /
          gamma( upsilon +1);
        pm(n,column+3)=A;
        n=n+1;
        svar=svar+1/10;
    end;
    column=column+4;
end;
```

Table 7. Algorithm 4: MATLAB lines to calculate all parameters in Example 6.4 for three cases υ.

```
syms v e;
upsilon=[1/2 3/4 11/12];
[xupsilon yupsilon]=size(upsilon);
chi=sqrt(1+abs(v));
mathfrakb=0; mathfrakc=1;
varkappa_mathfrakc=1;
q=[1/5 2/5 3/5 4/5 1];
phi=exp(-v)/(15+exp(v)+abs(e));
k=[1 2 3 4 5];
[xk yk]=size(k);
varrho_k=1./(k+abs(e));
omega=cos(abs(v))/40;
Lambda=1;
upmu_k=[1/50 1/50 1/50 1/50 1/50];
[xupmu_k yupmu_k]=size(upmu_k);
C=1/20;
a=mathfrakb; T=mathfrakc;
column=1;
for  s=1:yupsilon
    n=1;
    svar=a;
    while svar<=T
        pm(n,column)=n;
        pm(n,column+1)=svar;
        pm(n,column+2)=1/4 +
          Lambda * ( yupmu_k + 1 )
          * eval( subs(chi, {v}, {svar} ) )^
          ( upsilon(s) ) / gamma(
          upsilon(s) + 1 );
        pm(n,column+3)=1/4 +
          eval( subs( chi, {v}, {svar} ) )^
          ( upsilon(s) ) / gamma(
          upsilon(s) + 1 );
        n=n+1;
        svar=svar+1/10;
    end;
    column=column+4;
end;
```