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Existence of Coincidence Point In C^*-Algebra Valued Metric Space

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ABSTRACT

In the present manuscript, we established some results on coincidence points for two pairs of compatible and weakly compatible mappings satisfying a generalized contraction with a pair of partially weakly increasing mapping in ordered C^*-algebra valued metric space. Moreover, integral type contractions in space provide a few implications for the main result. The result is an extension and generalization of several metric space results available.

Keywords: C^*-algebra valued metric space; Partially weakly increasing; Compatible mapping; Weakly compatible mapping.

1. Introduction and Preliminaries

One of the most important theorem of analysis was initially presented by Banach [1], which is used widely in various applications and fields of mathematics. It requires a contractive condition on a self-map that is simple to test in many situations, the structure of complete metric space. In essence, this concept provides a series of approximation solutions and valuable information about the fixed point rate of convergence. A key strategy for generalizing fixed point results in metric space is adding generalized metric space. Ma et al. made one such attempt in [2], giving the notion of C^*-algebra valued metric space and demonstrating some fixed point results. Several fixed point results have been achieved along with C^*-algebra valued b-metric space (more general) as a part of this line of investigations (see [3–21] and references cite therein)

In the present manuscript, we established some results on coincidence points for two pairs of compatible and weakly compatible mappings satisfying a generalized contraction with a pair of partially weakly increasing mapping in ordered C^*-algebra valued metric space. Moreover, integral type contractions in space provide a few implications for the main result. The result is an extension and generalization of

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several metric space results available.

Throughout this work, we denote the set of unital C^*-algebra with unity $I_{\mathbb{A}}$ by $\mathbb{A}$ and C^*-algebra valued metric space by C^*_{av}-metric space.

Definition 1.1 ([22]). A pair of self mappings (Γ_1, Γ_2) satisfying $\Gamma_1\varsigma \preceq \Gamma_2\Gamma_1\varsigma$ and $\Gamma_2\varsigma \preceq \Gamma_1\Gamma_2\varsigma$ on partially ordered set $\mho$, then the pair (Γ_1, Γ_2) is said to be weakly increasing.

Definition 1.2 ([23]). A pair of self mappings (Γ_1, Γ_2) satisfying $\Gamma_1\varsigma \preceq \Gamma_2\Gamma_1\varsigma$ on partially ordered set $\mho$, then the pair (Γ_1, Γ_2) is said to be partially weakly increasing.

Definition 1.3 ([24]). A pair of self mappings (Γ_1, Γ_2) with a self map Γ_3 satisfying

1. $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$;
2. $\Gamma_1\varsigma \preceq \Gamma_2\vartheta$ for all $\vartheta \in \Gamma_3^{-1}(\Gamma_1\varsigma)$ and $\Gamma_2\varsigma \preceq \Gamma_1\vartheta$ for all $\vartheta \in \Gamma_3^{-1}(\Gamma_2\varsigma)$;

then the pair (Γ_1, Γ_2) is said to be weakly increasing with respect to Γ_3 on a partially ordered set $(\mho, \preceq)$.

Definition 1.4 ([25]). A pair of self mappings (Γ_1, Γ_2) with a self map Γ_3 satisfying

1. $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$;
2. $\Gamma_1\varsigma \preceq \Gamma_2\vartheta$ for all $\vartheta \in \Gamma_3^{-1}(\Gamma_1\varsigma)$;

then the pair (Γ_1, Γ_2) is said to be partially weakly increasing (PWI) with respect to Γ_3 on a partially ordered set $(\mho, \preceq)$.

Definition 1.5 ([20]). Two self mappings Γ_1 and Γ_2 on C^*_{av}-metric space are said to be compatible if for an arbitrary sequence $\{\varsigma_n\} \subseteq \mho$ such that $\lim_{n\to\infty} \Gamma_1\varsigma_n = \lim_{n\to\infty} \Gamma_2\varsigma_n = t \in \mho$, then $d(\Gamma_2\Gamma_1\varsigma_n, \Gamma_1\Gamma_2\varsigma_n) \xrightarrow{\|.\|} \theta_{\mathbb{A}}$ as $n \to \infty$.

Definition 1.6 ([20]). If two self mappings Γ_1 and Γ_2 commute at all of their coincidence points, i.e. $\Gamma_1\Gamma_2\varsigma = \Gamma_2\Gamma_1\varsigma$ for all $\varsigma \in \{\varsigma \in \mho : \Gamma_1\varsigma = \Gamma_2\varsigma\}$ then Γ_1 and Γ_2 are called weakly compatible.

Definition 1.7 ([26]). Let $(\mho, \preceq)$ be a partially ordered set. The self-mappings Γ_3 is called monotone Γ_1-nondecreasing, if

$$\Gamma_1\varrho \preceq \Gamma_1\varsigma \quad \text{implies} \quad \Gamma_3\varrho \preceq \Gamma_3\varsigma,$$

for all $\varrho, \varsigma \in \mho$.

Definition 1.8 ([2]). Suppose $\mho$ is a nonempty set and a function $d : \mho \times \mho \to \mathbb{A}$ satisfying:

1. $d(\varsigma, \vartheta) \succeq \theta_{\mathbb{A}}$ and $d(\varsigma, \vartheta) = \theta_{\mathbb{A}}$ if and only if $\varsigma = \vartheta$;
2. $d(\varsigma, \vartheta) = d(\vartheta, \varsigma)$;
3. $d(\varsigma, \vartheta) \preceq d(\varsigma, \nu) + d(\nu, \vartheta)$;

for all $\varsigma, \vartheta, \nu \in \mho$. Then, d is called a C^*_{av}-metric and $(\mho, \mathbb{A}, d)$ is called a C^*_{av}-metric space.

Definition 1.9 ([2]). Let $(\mho, \mathbb{A}, d)$ be a C^*_{av}-metric space, $\varsigma \in \mho$ and $\{\varsigma_n\}$ be a sequence in $\mho$. Then,

1. $\{\varsigma_n\}$ converges to ς with respect to $\mathbb{A}$, whenever for every $\epsilon > 0$ there is a natural number $\mathbb{N}$ such that $\|d(\varsigma_n, \varsigma)\| \leq \epsilon$ for all $n \geq N$;
2. $\{\varsigma_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$, whenever for every $\epsilon > 0$ there is a natural number $\mathbb{N}$ such that $\|d(\varsigma_n, \varsigma_m)\| \leq \epsilon$ for all $n, m \geq N$;
3. $(\mho, \mathbb{A}, d)$ is said to be complete with respect to $\mathbb{A}$, if every Cauchy sequence converges to a point $\varsigma \in \mho$ with respect to $\mathbb{A}$.

2. Main Result

In this section, we prove some coincidence point theorems in the framework of an ordered C^*_{av}-metric space.

Definition 2.1. Let $\mho$ be a nonempty set. Then, $(\mho, \mathbb{A}, d, \preceq)$ is called a partially ordered C^*_{av}-metric space if and only if d is a C^*_{av}-metric on a partially ordered set $(\mho, \preceq)$.

Theorem 2.2. *Consider four self mappings* Γ_1, Γ_2, Γ_3, *and* Γ_4 *on* $\mho$ *in ordered complete* C^*_{av}*-metric space* $(\mho, \mathbb{A}, d, \preceq)$ *satisfying:*

1. $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;
2. $\Gamma_3\vartheta$ *and* $\Gamma_4\varsigma$ *are comparable elements for all* $\varsigma, \vartheta \in \mho$;
3. *for some* $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + \beta + 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$, *we have*
$$\begin{aligned} d(\Gamma_1\varsigma, \Gamma_2\vartheta) &\preceq \alpha \frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} \\ &+\beta \frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)d(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d(\Gamma_4\varsigma, \Gamma_3\vartheta)} \\ &+\gamma(d(\Gamma_1\varsigma, \Gamma_4\varsigma) \\ &+d(\Gamma_2\vartheta, \Gamma_3\vartheta)) \\ &+\delta(d(\Gamma_1\varsigma, \Gamma_3\vartheta) \\ &+d(\Gamma_4\varsigma, \Gamma_2\vartheta)) \\ &+\lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta); \end{aligned} \tag{2.1}$$
4. *the pairs* (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are compatible and continuous;*
5. *the pairs* (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfying PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover, if* $\Gamma_3\varpi_{\mathbb{A}}$ *and* $\Gamma_4\varpi_{\mathbb{A}}$ *are comparable, then* $\varpi_{\mathbb{A}}$ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

Proof. Let $\varsigma_0 \in \mho$ be an arbitrary point. Since, $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$, one can find $\varsigma_1, \varsigma_2 \in \mho$ such that $\Gamma_1\varsigma_0 = \Gamma_3\varsigma_1$ and $\Gamma_2\varsigma_1 = \Gamma_4\varsigma_2$. Continuing this process, we construct a sequence $\{\varpi_n\}$ defined by

$$\begin{aligned} &\varpi_{2n+1} = \Gamma_1\varsigma_{2n} = \Gamma_3\varsigma_{2n+1} \\ &\text{and } \varpi_{2n+2} = \Gamma_2\varsigma_{2n+1} = \Gamma_4\varsigma_{2n+2}. \end{aligned}$$

Since, $\varsigma_1 \in \Lambda^{-1}_{\mathbb{A}}(\Gamma_1\varsigma_0)$, $\varsigma_2 \in \Gamma^{-1}_{\mathbb{A}}(\Gamma_2\varsigma_1)$ and the pairs (Γ_2, Γ_1) & (Γ_1, Γ_2) are partially weakly increasing with respect to Γ_3 and Γ_4 respectively. Therefore, we have

$$\begin{aligned} \Gamma_3\varsigma_1 = \Gamma_1\varsigma_0 &\preceq \Gamma_2\varsigma_1 = \Gamma_4\varsigma_2, \\ &\Rightarrow \varpi_1 \preceq \varpi_2; \\ \Gamma_2\varsigma_1 = \Gamma_4\varsigma_2 &\preceq \Gamma_1\varsigma_2 = \Gamma_3\varsigma_3, \\ &\Rightarrow \varpi_2 \preceq \varpi_3; \\ \Gamma_1\varsigma_2 = \Gamma_3\varsigma_3 &\preceq \Gamma_2\varsigma_3 = \Gamma_4\varsigma_4, \\ &\Rightarrow \varpi_3 \preceq \varpi_4. \end{aligned}$$

Repeating this process, we get $\varpi_{2n+1} \preceq \varpi_{2n+2}$ for all $n \geq 0$.
Define $d_n = d(\varpi_n, \varpi_{n+1})$. Suppose that $d_{2n} = \theta_{\mathbb{A}}$ i.e. $d(\varpi_{2n}, \varpi_{2n+1}) = \theta_{\mathbb{A}}$ for some n. Then, $\Gamma_1\varsigma_{2n} = \Gamma_3\varsigma_{2n+1} = \Gamma_2\varsigma_{2n-1} = \Gamma_4\varsigma_{2n}$. Thus, Γ_1 and Γ_4 have coincidence points. Hence the result. Now, suppose that $d_{2n} \succ \theta_{\mathbb{A}}$ for all $n \in \mathbb{N}$. Substituting $\varsigma = \varpi_{2n+1}$ and $\vartheta = \varpi_{2n+2}$, we have

$$\begin{aligned} &d(\varpi_{2n+1}, \varpi_{2n+2}) = \\ &d(\Gamma_1\varsigma_{2n}, \Gamma_2\varsigma_{2n+1}) \\ &\preceq \alpha d(\Gamma_1\varsigma_{2n}, \Gamma_4\varsigma_{2n}) \\ &(1 + d(\Gamma_2\varsigma_{2n+1}, \Gamma_3\varsigma_{2n+1}))/ \\ &1 + d(\Gamma_1\varsigma_{2n}, \Gamma_2\varsigma_{2n+1}) \\ &+\beta d(\Gamma_1\varsigma_{2n}, \Gamma_4\varsigma_{2n}) \\ &d(\Gamma_2\varsigma_{2n+1}, \Gamma_3\varsigma_{2n+1})/ \\ &d(\Gamma_4\varsigma_{2n}, \Gamma_3\varsigma_{2n+1}) \\ &+\gamma\big(d(\Gamma_1\varsigma_{2n}, \Gamma_4\varsigma_{2n}) \\ &+d(\Gamma_2\varsigma_{2n+1}, \Gamma_3\varsigma_{2n+1})\big) \end{aligned}$$

$$
\begin{aligned}
&+\delta\big(d(\Gamma_1\varsigma_{2n},\Gamma_3\varsigma_{2n+1})\\
&+d(\Gamma_4\varsigma_{2n},\Gamma_2\varsigma_{2n+1})\big)\\
&+\lambda d(\Gamma_4\varsigma_{2n},\Gamma_3\varsigma_{2n+1})\\
&=\alpha d(\varpi_{2n+1},\varpi_{2n})\\
&(1+d(\varpi_{2n+2},\varpi_{2n+1}))/\\
&1+d(\varpi_{2n+1},\varpi_{2n+2})\\
&+\beta d(\varpi_{2n+1},\varpi_{2n})\\
&d(\varpi_{2n+2},\varpi_{2n+1})/\\
&d(\varpi_{2n},\varpi_{2n+1})\\
&+\gamma\big(d(\varpi_{2n+1},\varpi_{2n})\\
&+d(\varpi_{2n+2},\varpi_{2n+1})\big)\\
&+\delta\big(d(\varpi_{2n+1},\varpi_{2n+1})\\
&+d(\varpi_{2n},\varpi_{2n+2})\big)\\
&+\lambda d(\varpi_{2n},\varpi_{2n+1})\\
&d_{2n+1}\preceq\alpha d_{2n}+\beta d_{2n+1}\\
&+\gamma(d_{2n}+d_{2n+1})\\
&+\delta(d_{2n}+d_{2n+1})+\lambda d_{2n}\\
&(I_{\mathbb{A}}-(\beta+\gamma+\delta)d_{2n+1}\\
&\preceq(\alpha+\gamma+\delta+\lambda)d_{2n}
\end{aligned}
$$

$$
\begin{aligned}
d_{2n+1} &\preceq \frac{(\alpha+\gamma+\delta+\lambda)}{(I_{\mathbb{A}}-(\beta+\gamma+\delta)}d_{2n}\\
d_{2n+1} &\preceq hd_{2n},
\end{aligned}
$$

where $h=\frac{(\alpha+\gamma+\delta+\lambda)}{(I_{\mathbb{A}}-(\beta+\gamma+\delta)}$ with $\|h\|\leq 1$. On the same argument, we can conclude that $d_{2n}\preceq hd_{2n-1}$, $d_{2n-1}\preceq hd_{2n-2}$ and so on. In general, we have $d_n\preceq hd_{n-1}$ for all $n\in\mathbb{N}$ i.e

$$
\begin{aligned}
d(\varpi_n,\varpi_{n+1}) &\preceq hd(\varpi_{n-1},\varpi_n)\\
&\preceq h^2d(\varpi_{n-2},\varpi_{n-1})\\
&\quad\dots\\
&\preceq h^nd(\varpi_0,\varpi_1).
\end{aligned}
$$

For any $p\in\mathbb{N}$, we get

$$
\begin{aligned}
&d(\varpi_{n+p},\varpi_n)\preceq d(\varpi_{n+p},\varpi_{n+p-1})\\
&+d(\varpi_{n+p-1},\varpi_{n+p-2})\\
&+\ldots+d(\varpi_{n+1},\varpi_n)\\
\preceq\ & h^{n+p-1}d(\varpi_0,\varpi_1)+h^{n+p-2}\\
&d(\varpi_0,\varpi_1)+\ldots+h^nd(\varpi_0,\varpi_1)\\
=\ & h^nd(\varpi_0,\varpi_1)(1+h+h^2+\ldots+h^{p-1})\\
=\ & h^nd(\varpi_0,\varpi_1)\frac{1-h^p}{1-h}.
\end{aligned}
$$

Taking the limit as $n\to\infty$ with the norm on both sides, we get

$$
\begin{aligned}
\|d(\varpi_{n+p},\varpi_n)\| &\leq \|h\|^n\|\beta\|\left\|\frac{1-h^p}{1-h}\right\|\\
&\to 0 \text{ as } n\to\infty,
\end{aligned}
$$

where $\|\beta\|=d(\varpi_0,\varpi_1)$. Hence, $\{\varpi_n\}$ is a Cauchy sequence in $(\mho,\mathbb{A},d)$.
Now, we prove that Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point.
Since, $\{\varpi_n\}$ is a Cauchy sequence in a complete C^*_{av}-metric space. Therefore, there exists $\varpi\in\mho$ such that

$$
\begin{aligned}
&\lim_{n\to\infty}\|d(\varpi_{2n+1},\varpi)\|=\\
&\lim_{n\to\infty}\|d(\Gamma_3\varsigma_{2n+1},\varpi)\|\\
&=\lim_{n\to\infty}\|d(\Gamma_1\varsigma_{2n},\varpi)\|=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\|d(\varpi_{2n},\varpi)\|=\\
&\lim_{n\to\infty}\|d(\Gamma_4\varsigma_{2n},\varpi)\|\\
&=\lim_{n\to\infty}\|d(\Gamma_2\varsigma_{2n-1},\varpi)\|=0.
\end{aligned}
$$

Hence,

$$\Gamma_4\varsigma_{2n}\to\varpi \quad \text{and } \Gamma_1\varsigma_{2n}\to\varpi \quad \text{as } n\to\infty.$$

Since, the pair (Γ_1,Γ_4) is compatible, therefore

$$\lim_{n\to\infty}\|d(\Gamma_4\Gamma_1\varsigma_{2n},\Gamma_1\Gamma_4\varsigma_{2n})\|=0.$$

Moreover, from $\lim_{n\to\infty}\|d(\Gamma_1\varsigma_{2n},\varpi)\|=0$, $\lim_{n\to\infty}\|d(\Gamma_4\varsigma_{2n},\varpi)\|=0$ and the continuity of Γ_1 and Γ_4, we get

$$\lim_{n\to\infty}\|d(\Gamma_4\Gamma_1\varsigma_{2n},\Gamma_4\varpi)\|=0$$

$$= \lim_{n\to\infty} \|d(\Gamma_1\Gamma_4\varsigma_{2n}, \Gamma_1\varpi)\|.$$

Now,

$$\begin{aligned}
&\|d(\Gamma_4\varpi, \Gamma_1\varpi)\| \\
\leq\ & \|d(\Gamma_4\varpi, \Gamma_4\Gamma_1\varsigma_{2n})\| \\
&+\|d(\Gamma_4\Gamma_1\varsigma_{2n}, \Gamma_1\varpi)\| \\
\leq\ & \|d(\Gamma_4\varpi, \Gamma_4\Gamma_1\varsigma_{2n})\| \\
&+\|d(\Gamma_4\Gamma_1\varsigma_{2n}, \Gamma_1\Gamma_4\varsigma_{2n})\| \\
&+\|d(\Gamma_1\Gamma_4\varsigma_{2n}, \Gamma_1\varpi)\|. \qquad (2.2)
\end{aligned}$$

Taking limit as $n \to \infty$ in Eq. (2.2), we get

$$\|d(\Gamma_4\varpi, \Gamma_1\varpi)\| \leq 0.$$

Thus, $\Gamma_1\varpi = \Gamma_4\varpi$ i.e. ϖ is a coincidence point of the pair (Γ_1, Γ_4).
Similarly, it can be proved that $\Gamma_2\varpi = \Gamma_3\varpi$ i.e. ϖ is a coincidence point of the pair (Γ_2, Γ_3).

We have to prove that ϖ is a common coincidence point of the mappings. Let $\Gamma_3\varpi$ and $\Gamma_4\varpi$ are comparable. By Eq. (2.1), we have

$$\begin{aligned}
& d(\Gamma_1\varpi, \Gamma_2\varpi) \preceq \\
& \alpha\frac{d(\Gamma_1\varpi, \Gamma_4\varpi)(1 + d(\Gamma_2\varpi, \Gamma_3\varpi))}{1 + d(\Gamma_1\varpi, \Gamma_2\varpi)} \\
& +\beta\frac{d(\Gamma_1\varpi, \Gamma_4\varpi)d(\Gamma_2\varpi, \Gamma_3\varpi)}{d(\Gamma_4\varpi, \Gamma_3\varpi)} \\
& +\gamma\big(d(\Gamma_1\varpi, \Gamma_4\varpi) + d(\Gamma_2\varpi, \Gamma_3\varpi)\big) \\
& +\delta\big(d(\Gamma_1\varpi, \Gamma_3\varpi) + d(\Gamma_4\varpi, \Gamma_2\varpi)\big) \\
& +\lambda d(\Gamma_4\varpi, \Gamma_3\varpi) \\
=\ & \delta\big(d(\Gamma_1\varpi, \Gamma_2\varpi) + d(\Gamma_1\varpi, \Gamma_2\varpi)\big) \\
& +\lambda d(\Gamma_1\varpi, \Gamma_2\varpi) \\
=\ & (2\delta + \lambda)d(\Gamma_1\varpi, \Gamma_2\varpi),
\end{aligned}$$

which is a contradiction. Hence, $\Gamma_1\varpi = \Gamma_2\varpi = \Gamma_4\varpi = \Gamma_3\varpi$. This completes the proof. □

We obtain the following consequences from Theorem 2.2 on taking zero value to $\alpha, \beta, \gamma, \delta$ and λ as exceptional cases.

Corollary 2.3. *Consider four self mappings Γ_1, Γ_2, Γ_3, and Γ_4 on $\mho$ in ordered complete C^*_{av}-metric space $(\mho, \mathbb{A}, d, \preceq)$ satisfying:*

1. *$\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;*
2. *for all $\varsigma, \vartheta \in \mho$, $\Gamma_3\vartheta$ and $\Gamma_4\varsigma$ are comparable elements ;*
3. *for some*

(*a*) *$\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha+2(\gamma+\delta) + \lambda \preceq I_{\mathbb{A}}$ with*

$$\begin{aligned}
& d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \\
& \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} \\
& +\gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) \\
& +\delta\big(d(\Gamma_1\varsigma, \Gamma_3\vartheta) + d(\Gamma_4\varsigma, \Gamma_2\vartheta)\big) \\
& +\lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),
\end{aligned}$$

or

(*b*) *$\alpha, \gamma, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha+2\gamma+\lambda \preceq I_{\mathbb{A}}$ with*

$$\begin{aligned}
& d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \\
& \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} \\
& +\gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) \\
& +\lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),
\end{aligned}$$

or

(*c*) *$\alpha, \delta, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha+2\delta+\lambda \preceq I_{\mathbb{A}}$ with*

$$\begin{aligned}
& d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \\
& \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)}
\end{aligned}$$

$$+\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big)$$
$$+\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta),$$

or

(*d*) $\gamma,\delta,\lambda\in[0,1)$ *with* $\theta_{\mathbb{A}}\preceq 2(\gamma+\delta)+\lambda\preceq I_{\mathbb{A}}$ *with*

$$d(\Gamma_1\varsigma,\Gamma_2\vartheta)\preceq \gamma\big(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)\big) +\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big) +\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta),$$

or

(*e*) $\beta,\gamma,\delta,\lambda\in[0,1)$ *with* $\theta_{\mathbb{A}}\preceq \beta+2(\gamma+\delta)+\lambda\preceq I_{\mathbb{A}}$ *with*

$$d(\Gamma_1\varsigma,\Gamma_2\vartheta)\preceq \beta\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} +\gamma\big(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)\big) +\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big) +\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta),$$

or

(*f*) $\beta,\gamma,\lambda\in[0,1)$ *with* $\theta_{\mathbb{A}}\preceq \beta+2\gamma+\lambda\preceq I_{\mathbb{A}}$ *with*

$$d(\Gamma_1\varsigma,\Gamma_2\vartheta)\preceq \beta\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} +\gamma\big(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)\big) +\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta),$$

or

(*g*) $\beta,\delta,\lambda\in[0,1)$ *with* $\theta_{\mathbb{A}}\preceq \beta+2\delta+\lambda\preceq I_{\mathbb{A}}$ *with*

$$d(\Gamma_1\varsigma,\Gamma_2\vartheta)\preceq \beta\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} +\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big) +\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta),$$

or

(*h*) $\alpha,\beta,\lambda\in[0,1)$ *with* $\theta_{\mathbb{A}}\preceq \alpha+\beta+\lambda\preceq I_{\mathbb{A}}$ *with*

$$d(\Gamma_1\varsigma,\Gamma_2\vartheta)\preceq \alpha\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)} +\beta\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} +\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta).$$

4. *the pairs* (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *are compatible and continuous;*

5. *pairs* (Γ_1,Γ_2) *and* (Γ_2,Γ_1) *satisfying PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover if* $\Gamma_3\varpi$ *and* $\Gamma_4\varpi$ *are comparable, then* ϖ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

We replace the pairs of compatible mappings with the pairs of weakly compatible mappings and omit the condition of continuity of the mappings in the following theorem.

Theorem 2.4. *Consider four self mappings* Γ_1, Γ_2, Γ_3, *and* Γ_4 *on* $\mho$ *in regular partially ordered complete* C^*_{av}*-metric space* $(\mho,\mathbb{A},d,\preceq)$ *satisfying:*

1. $\Gamma_1(\mho)\subseteq\Gamma_3(\mho)$ *and* $\Gamma_2(\mho)\subseteq\Gamma_4(\mho)$;

2. *for all* $\varsigma,\vartheta\in\mho$, $\Gamma_3\vartheta$ *and* $\Gamma_4\varsigma$ *are comparable elements ;*

3. *for some* $\alpha, \beta, \gamma, \delta, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + \beta + 2(\gamma+\delta) + \lambda \preceq I_{\mathbb{A}}$

$$\begin{aligned} d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \\ &\alpha\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \\ &+\beta\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \\ &+\gamma\big(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)\big) \\ &+\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big) \\ &+\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta); \end{aligned} \tag{2.3}$$

4. *the pairs* (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *are weakly compatible;*

5. *pairs* (Γ_1,Γ_2) *and* (Γ_2,Γ_1) *satisfying PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover, if* $\Gamma_3\varpi$ *and* $\Gamma_4\varpi$ *are comparable then* ϖ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

Proof. On the similar lines of Theorem (2.2), $\{\varpi_n\}$ is a Cauchy sequence and there exists $\varpi \in \mho$ such that

$$\lim_{n\to\infty} d(\varpi_n,\varpi) = \theta_{\mathbb{A}}.$$

Since, $\Gamma_3(\mho)$ is complete and $\{\varpi_{2n+1}\} \subset \Gamma_3(\mho)$. Therefore, $\varpi \in \Gamma_3(\mho)$. Hence, there exists $u \in \mho$ such that $\varpi = \Gamma_3 u$ and

$$\lim_{n\to\infty} d(\varpi_{2n+1},\Gamma_3 u) = \lim_{n\to\infty} d(\Gamma_3\varsigma_{2n+1},\Gamma_3 u) = \theta_{\mathbb{A}}.$$

Similarly, there exists $v \in \mho$ such that $\varpi = \Gamma_3 u = \Gamma_4 v$ and

$$\lim_{n\to\infty} d(\varpi_{2n},\Gamma_4 v) = \lim_{n\to\infty} d(\Gamma_4\varsigma_{2n},\Gamma_4 v) = \theta_{\mathbb{A}}.$$

Now, we shall show that v is a coincidence point of the pair (Γ_1,Γ_4).
Since, $\Gamma_3\varsigma_{2n+1} \to \varpi = Hv$ as $n \to \infty$ and regularity of $\mho$, $\Gamma_3\varsigma_{2n+1} \preceq \Gamma_4 v$. Substituting $\varsigma = v$ and $\vartheta = \varsigma_{2n+1}$ in the contractive condition, we have

$$\begin{aligned} d(\Gamma_1 v, \Gamma_2\varsigma_{2n+1}) \preceq \\ &\alpha\frac{d(\Gamma_1 v,\Gamma_4 v)(1+d(\Gamma_2\varsigma_{2n+1},\Gamma_3\varsigma_{2n+1}))}{1+d(\Gamma_1 v,\Gamma_2\varsigma_{2n+1})} \\ &+\beta\frac{d(\Gamma_1 v,\Gamma_4 v)d(\Gamma_2\varsigma_{2n+1},\Gamma_3\varsigma_{2n+1})}{d(\Gamma_4 v,\Gamma_3\varsigma_{2n+1})} \\ &+\gamma\big(d(\Gamma_1 v,\Gamma_4 v)+d(\Gamma_2\varsigma_{2n+1},\Gamma_3\varsigma_{2n+1})\big) \\ &+\delta\big(d(\Gamma_1 v,\Gamma_3\varsigma_{2n+1})+d(\Gamma_4 v,\Gamma_2\varsigma_{2n+1})\big) \\ &+\lambda d(\Gamma_4 v,\Gamma_3\varsigma_{2n+1}), \end{aligned} \tag{2.4}$$

Taking limit as $n \to \infty$ in Eq. (2.4) implies, $d(\Gamma_1 v,\varpi) = \theta_{\mathbb{A}}$. Hence, $\Gamma_1 v = \varpi = \Gamma_4 v$.

Since, the pair (Γ_1,Γ_4) are weakly compatible, therefore $\Gamma_1\varpi = \Gamma_1\Gamma_4 v = \Gamma_4\Gamma_1 v = \Gamma_4\varpi$. Thus, ϖ is a coincidence point of the pair (Γ_1,Γ_4).

Similarly, it can be shown that ϖ is a coincidence point of the pair (Γ_2,Γ_3). The rest of the proof can be done using similar arguments as in Theorem (2.2). □

Corollary 2.5. *Consider four self mappings* Γ_1, Γ_2, Γ_3, *and* Γ_4 *on* $\mho$ *in regular partially ordered complete* C^*_{av}*-metric space* $(\mho,\mathbb{A},d,\preceq)$ *satisfying:*

1. $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;

2. *for all* $\varsigma,\vartheta \in \mho$, $\Gamma_3\vartheta$ *and* $\Gamma_4\varsigma$ *are comparable elements*

   (a)

$$\begin{aligned} d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \\ &\alpha\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \\ &+\gamma\big(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)\big) \\ &+\delta\big(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta)\big) \\ &+\lambda d(\Gamma_4\varsigma,\Gamma_3\vartheta), \end{aligned}$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(b)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} + \gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\alpha, \gamma, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(c)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} + \delta\big(d(\Gamma_1\varsigma, \Gamma_3\vartheta) + d(\Gamma_4\varsigma, \Gamma_2\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\alpha, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(d)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) + \delta\big(d(\Gamma_1\varsigma, \Gamma_3\vartheta) + d(\Gamma_4\varsigma, \Gamma_2\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\gamma, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(e)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \beta\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)d(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d(\Gamma_4\varsigma, \Gamma_3\vartheta)} + \gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) + \delta\big(d(\Gamma_1\varsigma, \Gamma_3\vartheta) + d(\Gamma_4\varsigma, \Gamma_2\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\beta, \gamma, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(f)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \beta\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)d(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d(\Gamma_4\varsigma, \Gamma_3\vartheta)} + \gamma\big(d(\Gamma_1\varsigma, \Gamma_4\varsigma) + d(\Gamma_2\vartheta, \Gamma_3\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\beta, \gamma, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(g)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \beta\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)d(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d(\Gamma_4\varsigma, \Gamma_3\vartheta)} + \delta\big(d(\Gamma_1\varsigma, \Gamma_3\vartheta) + d(\Gamma_4\varsigma, \Gamma_2\vartheta)\big) + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\beta, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(h)

$$d(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq \alpha\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d(\Gamma_1\varsigma, \Gamma_2\vartheta)} + \beta\frac{d(\Gamma_1\varsigma, \Gamma_4\varsigma)d(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d(\Gamma_4\varsigma, \Gamma_3\vartheta)} + \lambda d(\Gamma_4\varsigma, \Gamma_3\vartheta),$$

for some $\alpha, \beta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + \beta + \lambda \preceq I_{\mathbb{A}}$;

3. *the pairs* (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are weakly compatible;*
4. *the pairs* (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *are PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover, if* $\Gamma_3\varpi$ *and* $\Gamma_4\varpi$ *are comparable then* ϖ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

Some integral consequences of the main result for self mappings are as follows: Let ξ denote the set of all functions $\phi : \mathbb{A}^+ \to \mathbb{A}^+$ satisfying the following hypothesis:

1. each ϕ is Lebesgue integrable function on every compact subset of $\mathbb{A}^+$ and
2. for $\alpha > \theta_{\mathbb{A}}$, $\int_{\theta_{\mathbb{A}}}^{\alpha} \phi(t) \succ \theta_{\mathbb{A}}$ for $t \in \mathbb{A}^+$.

Corollary 2.6. *Consider four self mappings Γ_1, Γ_2, Γ_3, and Γ_4 on $\mho$ in regular partially ordered complete C^*_{av}-metric space $(\mho, \mathbb{A}, d, \preceq)$ satisfying:*

1. *$\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;*
2. *for all $\varsigma, \vartheta \in \mho$, $\Gamma_3\vartheta$ and $\Gamma_4\varsigma$ are comparable elements*

(a)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha + 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(b)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \gamma, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(c)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \delta, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq \alpha + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(d)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\gamma, \delta, \lambda \in [0, 1)$ with $\theta_{\mathbb{A}} \preceq 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(e)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}$$

$$\phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta,\gamma,\delta,\lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2(\gamma+\delta) + \lambda \preceq I_{\mathbb{A}}$.

(f)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta)} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta,\gamma,\lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(g)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta,\delta,\lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(h)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha,\beta,\lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + \beta + \lambda \preceq I_{\mathbb{A}}$;

3. *the pairs* (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *are compatible and continuous;*
4. *the pairs* (Γ_1,Γ_2) *and* (Γ_2,Γ_1) *are PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1,Γ_4) *and* (Γ_2,Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover if* $\Gamma_3\varpi$ *and* $\Gamma_4\varpi$ *are comparable, then* ϖ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

Corollary 2.7. *Consider four self mappings* Γ_1, Γ_2, Γ_3, *and* Γ_4 *on* $\mho$ *in regular partially ordered complete* C^*_{av}*-metric space* $(\mho,\mathbb{A},d,\preceq)$ *satisfying:*

1. $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;
2. *for all* $\varsigma,\vartheta \in \mho$, $\Gamma_4\varsigma$ *and* $\Gamma_3\vartheta$ *are comparable elements*

(a)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha,\gamma,\delta,\lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2(\gamma+\delta) + \lambda \preceq I_{\mathbb{A}}$.

(b)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \gamma, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(c)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \delta, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(d)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\gamma, \delta, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(e)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta, \gamma, \delta, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2(\gamma + \delta) + \lambda \preceq I_{\mathbb{A}}$.

(f)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \gamma \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_4\varsigma)+d(\Gamma_2\vartheta,\Gamma_3\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta, \gamma, \lambda \in [0,1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\gamma + \lambda \preceq I_{\mathbb{A}}$.

(g)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \delta \int_{\theta_{\mathbb{A}}}^{(d(\Gamma_1\varsigma,\Gamma_3\vartheta)+d(\Gamma_4\varsigma,\Gamma_2\vartheta))} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\beta, \delta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \beta + 2\delta + \lambda \preceq I_{\mathbb{A}}$.

(h)

$$\int_{\theta_{\mathbb{A}}}^{d(\Gamma_1\varsigma,\Gamma_2\vartheta)} \phi(t) \preceq \alpha \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)(1+d(\Gamma_2\vartheta,\Gamma_3\vartheta))}{1+d(\Gamma_1\varsigma,\Gamma_2\vartheta)}} \phi(t) + \beta \int_{\theta_{\mathbb{A}}}^{\frac{d(\Gamma_1\varsigma,\Gamma_4\varsigma)d(\Gamma_2\vartheta,\Gamma_3\vartheta)}{d(\Gamma_4\varsigma,\Gamma_3\vartheta)}} \phi(t) + \lambda \int_{\theta_{\mathbb{A}}}^{d(\Gamma_4\varsigma,\Gamma_3\vartheta)} \phi(t),$$

for some $\alpha, \beta, \lambda \in [0, 1)$ *with* $\theta_{\mathbb{A}} \preceq \alpha + \beta + \lambda \preceq I_{\mathbb{A}}$*;*

3. *the pairs* (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are weakly compatible;*

4. *the pairs* (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *are PWI with respect to* Γ_3 *and* Γ_4.

Then, the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *have a coincidence point* ϖ *in* $\mho$. *Moreover, if* $\Gamma_3\varpi$ *and* $\Gamma_4\varpi$ *are comparable then* ϖ *is a coincidence point of* Γ_1, Γ_2, Γ_3 *and* Γ_4.

3. Conclusion

We replace $\Gamma_1\mho = \Gamma_3\mho = I_{\mathbb{A}}\mho$ and the condition of partially weakly increasing of the pair by Γ_2 is monotone Γ_4-nondecreasing in the Theorems (2.2)-(2.4), one can also obtain the required results which is present by the Kalyani et al. [26] and some corollary by Sumit Chandok [28].

References

[1] Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 1922;3:133-81.

[2] Ma, Z, Jiang, L, Sun, H. C^*-Algebra valued metric spaces and related fixed point theorems. Fixed Point Theory Appl. 2014;206.

[3] Ansari, A. H. Note on $\phi - \psi$-contractive type mappings and related fixed point. In The 2nd Regional Conference on Mathematics and Applications PNU. 2014:377–80.

[4] Rhoades, B. E. Some theorems on weakly contractive maps. Nonlinear Anal. 2001;47:2683–93.

[5] Kumar, D, Rishi, D, Park, C, Lee, J. R. On fixed point in C^*-algebra valued metric spaces using C_*-class function. Int. J. Nonlinear Anal. Appl. 2021;12:1157-61.

[6] Skof, F. Teorema di punti fisso per applicazioni negli spazi metrici. Atti Aooad Soi Torino. 1977;111:323-9.

[7] Ya Alber, I, Guerre-Delabriere, S. Principle of weakly contractive maps in Hilbert spaces. In: Gohberg I., Lyubich Y. (eds) New Results in Operator Theory and Its Applications, Oper. Theory: Adv. Appl. 1997;98.

[8] Khan, M S, Swalesh, M, Sessa, S. Fixed points theorems by altering distances between the points. Bull. Aust. Math. Soc. 1984;30:1–9.

[9] Chandok, S, Kumar, D, Park, C. C^*-algebra valued partial metric space and fixed point theorems. Proc. Indian Acad. Sci. 2019;129:37.

[10] Murthy, P P, Tas, K, Patel, U. D. Common fixed point theorems for generalized (ϕ, ψ)-weak contraction condition in complete metric spaces. J. Inequal Appl. 2015;139.

[11] Mani, G, Gnanaprakasam, A J, Subbarayan, P, Chinnachamy, S, George, R, Mitrović, Z. D. Applications to nonlinear fractional differential equations via common fixed point on C^*-algebra-valued bipolar metric spaces. Fractal Fract. 2023;7:534.

[12] Zhang, Q, Song, Y. Fixed point theory for generalized ϕ-weak contractions. Appl Math Lett. 2009;22:75–8.

[13] Janardhanan, G, Mani, G, Michael, E A R, Thabet, S T M, Kedim, I. Solution of a nonlinear fractional-order initial value problem via a C^*-algebra-valued R-metric space. Fixed Point Theory Algorithms Sci. Eng. 2024;7.

[14] Maheswari, J U, Anbarasan, A, Gunaseelan, M, Parvaneh, V, Bonab, S. H. Solving an integral equation via C^*-algebra-valued partial b-metrics. Fixed Point Theory Algorithms Sci. Eng. 2022;18.

[15] Mani, G, Gnanaprakasam, A J, Khumalo, P B, Gaba, Y. U. Common coupled fixed point theorems on C^*-algebra-valued partial metric spaces. J. Funct. Spaces. 2022:8 pages.

[16] Dhariwal, R, Kumar, D. Conditionally compatible mappings, non-compatible mappings, faintly compatible mappings and some results on common fixed point in C^*-algebra-valued metric spaces. Sahand Commun. Math. Anal. 2025;22:109-24.

[17] Dhariwal, R, Kumar, D. Generalized expansive mapping, equiexpansive mapping in C^*-algebra valued metric space and some fixed points results. Indian J. Pure Appl. Math. 2024. https://doi.org/10.1007/s13226-024-00664-7

[18] Mani, G, Gnanaprakasam, A J, Ege, O, Aloqaily, A, Mlaiki, N. Fixed point results in C^* -algebra-valued partial b-metric spaces with related application. Mathematics. 2023;11:1158.

[19] Mani, G, Joseph, G A, Isı́k, H, Jarad, F. Fixed point results in C^*-algebra-valued bipolar metric spaces with an application. AIMS Math. 2023;8:7695-713.

[20] Xin, Q, Jiang, L, Ma, Z. Common fixed point theorems in C^*-algebra valued metric spaces. JNSA. 2016;9:4617–27.

[21] Malhotra, A, Kumar, D, Park, C. C^*-algebra valued R-metric space and fixed point theorems. AIMS Math. 2022;7:6550-64.

[22] Altun, I, Simsek, H. Some fixed point theorems on ordered metric spaces and application. Fixed Point Theory Appl. 2010 Article ID 621492.

[23] Abbas, M, Nazir, T, Radenović, S. Common fixed points of four maps in partially ordered metric spaces. Appl. Math. Lett. 2011;24:1520–6.

[24] Nashine, H K, Samet, B. Fixed point results for mappings satisfying $\psi-\phi$-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal. 2011;74:2201–9.

[25] Esmaily, J, Vaezpour, S M, Rhoades, B. E. Coincidence point theorem for generalized weakly contractions in ordered metric spaces. Appl. Math. Comput. 2012;219:1536–48.

[26] Rao, N S, Kalyani, K, Khatri, K. Contractive mapping theorems in Partially ordered metric spaces. Cubo. 2020;22:203-14.

[27] Mustafa, Z, Jaradat, M M M, Ansari, A H, Popovı́c B Z, Jaradat H M. C-class functions with new approach on coincidence point results for generalized $(\psi-\phi)$-weakly contractions in ordered b-metric spaces. SpringerPlus 2016;5.

[28] Chandok, S. Some common fixed point results for rational type contraction mappings in partially ordered metric spaces. Math. Bohem. 2013;138:407-13.