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Some New Results on Fixed Points for ϖ-Distances in Complex-Valued Metric Spaces

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ABSTRACT

In this paper, we introduce the notion of a ϖ-distance in complete complex-valued metric spaces and prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces. Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results.

Keywords: ϖ-distance; Ω-distance; c-distance; Complex-valued metric spaces; Fixed point; Generalized c-distance; wt-distance; w-distance

1. Introduction

The Banach's contraction mapping principle is widely recognized as the source of a metric fixed point theory. The existence and uniqueness of a fixed point of operators or mappings has been a subject of a great interest since the work of a Banach in 1922 [1]. The concept of nonexpansive mappings has also been widely studied in the following works [2, 3]. This principle has been applied in different spaces by mathematicians, for example D-metric spaces, quasimetric spaces, quasi b-metric spaces, b-metric-like spaces, Dislocated quasi-b-metric spaces, and G-metric spaces (see [4-8]) have already been obtained. A new space called the complex valued metric space which is more general than well-know metric spaces has been introduced by Azam et al. [9]. Naturally, this new idea

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can be utilized to define complex valued normed spaces and complex valued inner product spaces which, in turn, offer a lot of scope for further investigations. Many authors have studied a fixed point theory in complex valued metric space (see [10-16]).

On the other hand, in 1996, Takahashi et al.[17] introduced the notion of a w-distance on a metric space and proved a nonconvex minimization theorem which generalizes Caristi's fixed-point results and the ϵ-variational principle. After that in 2011, Cho et al. [18] introduced the concept of a c-distance in a cone metric space. For more details about c-distance (see [19, 20] and the references contained therein). The concept of a Ω-distance of G-metric spaces and constructed some fixed point theorems in G-metric spaces by using the notion of a Ω-distance introduced by Saadati et al. [21]. Moreover, the results of Saadati et al. was made clear to shoot up by Shatanawi and Pitea [22]. Recently, a new concept of a wt-distance on b-metric spaces, which is a b-metric version of the w-distance of Takahashi et al. [17] was introduced by Saadati et al. [23] and proved some fixed point results in a partially ordered b-metric space.

Moreover, Mohanta [24] has generalized the results of Saadati et al. [23]. The concept of a generalized c-distance on a cone b-metric space was introduced by Xu et al. [25] which is a generalization of c-distance of Cho et al. [18] and proved some fixed and common fixed point results in ordered cone b-metric spaces using this distance . For more details about generalized c-distance see [26].

The above concept, we introduce the notion of a ϖ-distance in complete complex-valued metric spaces and prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces. Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results.

2. Preliminaries

Throughout this paper, we will write $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ and $\mathbb{N} = \{1, 2, 3, ...\}$. Let $\mathbb{C}$ be the set of complex numbers and $\sigma_1, \sigma_2 \in \mathbb{C}$, we define a partial order $\prec$ and $\precsim$ on $\mathbb{C}$ as follows:

(i) $\sigma_1 \prec \sigma_2$ if and only if $\mathrm{Re}(\sigma_1) < \mathrm{Re}(\sigma_2)$ and $\mathrm{Im}(\sigma_1) < \mathrm{Im}(\sigma_2)$

(ii) $\sigma_1 \precsim \sigma_2$ if and only if $\mathrm{Re}(\sigma_1) \leq \mathrm{Re}(\sigma_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

Now, we recall some property of a complex valued metric space.

Definition 2.1 ([9]). Let X be a nonempty set. Suppose that the mapping $\Gamma : X \times X \to \mathbb{C}$ satisfies the following conditions:

(Γ_1) $0 \precsim \Gamma(\zeta, \eta)$, for all $\zeta, \eta \in X$;

$(\Gamma_2)$$\Gamma(\zeta, \eta) = 0$ if and only if $\zeta = \eta$ for all $\zeta, \eta \in X$;

(Γ_3) $\Gamma(\zeta, \eta) = \Gamma(\eta, \zeta)$ for all $\zeta, \eta \in X$;

(Γ_4) $\Gamma(\zeta, \eta) \precsim \Gamma(\zeta, \sigma) + \Gamma(\sigma, \eta)$, for all $\zeta, \eta, \sigma \in X$.

Then Γ is called a *complex valued metric* on X and (X, Γ) is called a *complex valued metric space*.

Example 2.2 ([9]). Defined $\Gamma : X \times X \to \mathbb{C}$ as follows:

$$\Gamma(\sigma_1, \sigma_2) = \begin{cases} \frac{2}{3}|\zeta_1 - \zeta_2| + \frac{i}{2}|\zeta_1 - \zeta_2| \\ if \sigma_1, \sigma_2 \in X_1, \\ \frac{1}{2}|\eta_1 - \eta_2| + \frac{i}{3}|\eta_1 - \eta_2| \\ if \sigma_1, \sigma_2 \in X_2, \\ (\frac{2}{3}\zeta_1 + \frac{1}{2}\eta_2) + i(\frac{1}{2}\zeta_1 + \frac{1}{3}\eta_2) \\ if \sigma_1 \in X_1,\ \sigma_2 \in X_2, \\ (\frac{1}{2}\eta_1 + \frac{2}{3}\zeta_2) + i(\frac{1}{3}\eta_1 + \frac{1}{2}\zeta_2) \\ if \sigma_1 \in X_2,\ \sigma_2 \in X_1, \end{cases}$$

Let $X_1 = \{\sigma \in \mathbb{C} : 0 \leq Re(\sigma) \leq 1, Im(\sigma) = 0\}$, $X_2 = \{\sigma \in \mathbb{C} : 0 \leq Im(\sigma) \leq 1, Re(\sigma) = 0\}$ and let $X = X_1 \cup X_2$. Let $\sigma_1 = \zeta_1 + i\eta_1$, $\sigma_2 = \zeta_2 + i\eta_2 \in X$. Then (X, Γ) is a complete complex valued metric space.

Example 2.3 ([9])**.** Let $X = C([1,3], \mathbb{R})$, $a > 0$ and for every $\zeta, \eta \in X$ let $M_{\zeta\eta} = \max_{t\in[1,3]} |\zeta(t) - \eta(t)|$, $\Gamma(\zeta, \eta) = M_{\zeta\eta}\sqrt{1+a^2}e^{i\tan^{-1}a}$. Then (X, Γ) is a complex valued metric space.

Definition 2.4 ([9])**.** Let (X, Γ) be a complex valued metric space.

(i) A point $\zeta \in X$ is called *interior point* of a set $B \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $N(\zeta, r) := \{\eta \in X : \Gamma(\zeta, \eta) \prec r\} \subseteq B$.

(ii) A point $\zeta \in X$ is called *limit point* of a set $B \subseteq X$ whenever for every $0 \prec r \in \mathbb{C}$ such that $N(\zeta, \eta) \cap (X - B) \neq \emptyset$.

(iii) A subset $B \subseteq X$ is called *open* whenever each element of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called *closed* whenever each limit point of B belongs to B.

(v) The family $F = \{N(\zeta, r) : \zeta \in X, 0 \prec r\}$ is a sub-basis for a topology on X. we denote this complex topology τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.5 ([9])**.** Let (X, Γ) be a complex valued metric space and $\{\zeta_n\}$ be a sequence in X and $\zeta \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N, \Gamma(\zeta_n, \zeta) \prec c$, then $\{\zeta_n\}$ is said to be *convergent*, $\{\zeta_n\}$ converges to ζ and ζ is limit point of $\{\zeta_n\}$. We denote this by $\zeta_n \to \zeta$ as $n \to \infty$ or $\lim_{n\to\infty} \zeta_n = \zeta$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N, \Gamma(\zeta_n, \zeta_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{\zeta_n\}$ is said to be *Cauchy sequence.*

(iii) If every Cauchy sequence in X is convergent, then (X, Γ) is said to be a *complete complex valued metric space.*

Lemma 2.6 ([9])**.** Let (X, Γ) be a complex valued metric space and let $\{\zeta_n\}$ be a sequence in X. Then $\{\zeta_n\}$ converges to ζ if and only if $|\Gamma(\zeta_n, \zeta)| \to 0$ as $n \to \infty$.

Lemma 2.7 ([9])**.** Let (X, Γ) be a complex valued metric space and let $\{\zeta_n\}$ be a sequence in X. Then $\{\zeta_n\}$ is a Cauchy sequence if and only if $|\Gamma(\zeta_n, \zeta_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Definition 2.8. Let $\mathbb{C}$ be the set of complex number and $M \subseteq \mathbb{C}$.

(i) f is continuous at $\sigma_0 \in \mathbb{C}$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$, if $|\sigma - \sigma_0| < \delta$, then $|f(\sigma) - f(\sigma_0)| < \epsilon$ for all $\sigma \in \mathbb{C}$.

(ii) f is continuous on M if and only if f is continuous at σ_0 for all $\sigma_0 \in M$.

Remark 2.9 ([14])**.** We obtained that following statements hold:

(i) If $\sigma_1 \precsim \sigma_2$ and $\sigma_2 \precsim \sigma_3$, then $\sigma_1 \precsim \sigma_3$.

(ii) If $\sigma \in \mathbb{C}$, $a, b \in \mathbb{R}$ and $a \leq b$, then $a\sigma \precsim b\sigma$.

(iii) If $0 \precsim \sigma_1 \precsim \sigma_2$, then $|\sigma_1| \leq |\sigma_2|$.

3. ϖ-distance

In this section, we introduce the notion of ϖ-distance in complete complex-valued metric spaces and prove some lemma in such a space.

Let us recall that a complex-valued function f defined on a complex-valued metric space X is said to be lower semi-continuous at a point σ in X if eirher $\liminf_{\zeta_n\to\sigma} f(\zeta_n) = \infty$ or $f(\sigma) \leq$

$\liminf_{\zeta_n \to \sigma} f(\zeta_n)$, whenever $\zeta_n \in X$ for each $n \in \mathbb{N}$ and $\zeta_n \to \sigma$

Definition 3.1. Let (X, Γ) be a complex valued metric space. Then a function $p : X \times X \to \mathbb{C}$ is called a ϖ-distance on X if the following are satisfied:

(1) $p(\zeta, \eta) \succeq 0$ for all $\zeta, \eta \in X$;

(2) $p(\zeta, \eta) \preceq p(\zeta, \sigma) + p(\sigma, \eta)$ for all $\zeta, \eta, \sigma \in X$;

(3) for all $\zeta \in X$, $p(\zeta, \cdot) : X \to \mathbb{C}$ is lower semicontinuous i.e., if $\zeta \in X$, $\eta_n \to \eta \in X$ then $p(\zeta, \eta) \preceq \liminf_{n\to\infty} p(\zeta, \eta_n)$;

(4) for all $\epsilon \succ 0$, there exists $\delta \succ 0$, such that $p(\zeta, \sigma) \preceq \delta$ and $p(\sigma, \eta) \preceq \delta$ imply $p(\zeta, \eta) \preceq \epsilon$, where $\epsilon, \delta \in \mathbb{C}$.

Now, we provide examples of ϖ-distance in complex-valued metric space.

Example 3.2. Let (X, Γ) be a complex valued metric space. Then $p = \Gamma$ is a ϖ-distance on X

Proof. (1), (2) and (3) are clearly. To show (4). Let $\epsilon \succ 0$. Setting $\delta = \frac{\epsilon}{2}$. Then, we have $\Gamma(\zeta, \eta) \preceq \Gamma(\zeta, \sigma) + \Gamma(\sigma, \eta) = p(\zeta, \sigma) + p(\sigma, \eta) \preceq \epsilon$, where $p(\zeta, \sigma) \preceq \delta$ and $p(\sigma, \eta) \preceq \delta$. □

Example 3.3. Let (X, Γ) be a complex valued metric space. Then a function $p : X \times X \to \mathbb{C}$ defined by $p(\zeta, \eta) = c$ for every $\zeta, \eta \in X$, p is a ϖ-distance on X, where $c \in \mathbb{C}$ and $c \succ 0$.

Proof. (1), (2) and (3) clear. To show (4), let $\epsilon \succ 0$. Setting $\delta = \frac{c}{2}$. Then, $p(\zeta, \sigma) \preceq \delta$ and $p(\sigma, \eta) \preceq \delta$ imply $\Gamma(\zeta, \eta) \preceq \epsilon$. □

Example 3.4. Let X be a normed linear space with $\|\cdot\|$ and let $\Gamma(\zeta, \eta) = \|\zeta - \eta\| + i\|\zeta - \eta\|$, for all $\zeta, \eta \in X$. Then a function $p : X \times X \to \mathbb{C}$ defined by $p(\zeta, \eta) = (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i$ for every $\zeta, \eta \in X$, p is a ϖ-distance on X

Proof. Let $\zeta, \eta, \sigma \in X$. Then,

$$\begin{aligned} p(\zeta, \eta) =& (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i \\ \preceq& p(\zeta, \sigma) + p(\sigma, \eta). \end{aligned}$$

(1), (2) and (3) clear. To show (4), let $\epsilon \succ 0$. Setting $\delta = \frac{\epsilon}{2}$. Then,

$$\begin{aligned} \Gamma(\zeta, \eta) =& \|\zeta - \eta\| + i\|\zeta - \eta\| \\ \preceq& (\|\zeta\| + \|\eta\|) + (\|\zeta\| + \|\eta\|)i \\ \preceq& p(\zeta, \sigma) + p(\sigma, \eta) \preceq \epsilon. \end{aligned}$$

□

Example 3.5. Let (X, Γ) be a complex valued metric space, and let $\mathfrak{I}$ be continuous mapping from X into itself. Then a function $p : X \times X \to \mathbb{C}$ defined by $p(\zeta, \eta) = \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\}$ for every $\zeta, \eta \in X$, p is a ϖ-distance on X.

Proof. We see that (1) holds. We show (2). Let $\zeta, \eta, \sigma \in X$. If $\Gamma(\mathfrak{I}\zeta, \sigma) \succeq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\sigma)$, then

$$\begin{aligned} p(\zeta, \sigma) &\preceq \Gamma(\mathfrak{I}\zeta, \sigma) \\ &\preceq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta) + \Gamma(\mathfrak{I}\eta, \sigma) \\ &\preceq \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\} \\ &\quad + \max\{\Gamma(\mathfrak{I}\eta, \sigma), \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma)\} \\ &= p(\zeta, \eta) + p(\eta, \sigma). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} p(\zeta, \sigma) &\preceq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\sigma) \\ &\preceq \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta) + \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma) \\ &\preceq \max\{\Gamma(\mathfrak{I}\zeta, \eta), \Gamma(\mathfrak{I}\zeta, \mathfrak{I}\eta)\} \\ &\quad + \max\{\Gamma(\mathfrak{I}\eta, \sigma), \Gamma(\mathfrak{I}\eta, \mathfrak{I}\sigma)\} \\ &= p(\zeta, \eta) + p(\eta, \sigma). \end{aligned}$$

Hence (2) holds. Since $\mathfrak{I}$ is continuous, we have for all $\zeta \in X$, $p(\zeta, \cdot) : X \to \mathbb{C}$ is lower semicontinuous. Let $\epsilon \succ 0$. Choose $\delta = \frac{\epsilon}{2}$. Therefore, if $p(\zeta, \sigma) \preceq \delta$ and $p(\sigma, \eta) \preceq \delta$, then $p(\zeta, \mathfrak{I}\sigma) \preceq \delta$ and $p(\mathfrak{I}\sigma, \eta) \preceq \delta$. Hence $\Gamma(\zeta, \eta) \preceq \Gamma(\mathfrak{I}\sigma, \zeta) + \Gamma(\mathfrak{I}\sigma, \eta) \preceq \epsilon$. □

Next, we prove some basis lemma for ϖ-distance in complex-valued metric space.

Lemma 3.6. Let (X, Γ) be a complex valued metric spaces, and let p be an ϖ-distance on X. Let $\{\zeta_n\}, \{\eta_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{C}$ with $\alpha_n \succeq 0$ and $\beta_n \succeq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ convergent to zero and let $\zeta, \eta, \sigma, a \in X$. Then we have the following:

(*i*) if $p(\zeta_n, \eta_n) \preceq \alpha_n$ and $p(\zeta_n, \sigma) \preceq \beta_n$ for all $n \in \mathbb{N}$, then $\eta_n \to \sigma$;

(*ii*) if $p(\zeta_n, \eta) \preceq \alpha_n$ and $p(\zeta_n, \sigma) \preceq \beta_n$ for all $n \in \mathbb{N}$, then $\eta = \sigma$.

In paticular, if $p(\zeta, \eta) = 0$ and $p(\zeta, \sigma) = 0$, then $\eta = \sigma$;

(*iii*) if $p(\zeta_n, \zeta_m) \preceq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m \geq n$, then $\{\zeta_n\}$ is a Cauchy sequence;

(*iv*) if $p(\eta, \zeta_n) \preceq \alpha_n$ for all $n \in \mathbb{N}$, then $\{\zeta_n\}$ is a Cauchy sequence.

Proof. First, we show (*i*). Let $\epsilon \succ 0$. From the definition of ϖ-distance , there exists a $\delta \succ 0$ such that $p(\mu, \nu) \preceq \delta$ and $p(\mu, \sigma) \preceq \delta$ imply $d(\nu, \sigma) \preceq \epsilon$. Since $\{\alpha_n\}$ and $\{\beta_n\}$ are converging to zero, we have $n_0 \in \mathbb{N}$ such that $\alpha_n \preceq \delta$ and $\beta_n \preceq \delta$ for all $n \geq n_0$. Then we get, for any $n \geq n_0$,

$$p(\zeta_n, \eta_n) \preceq \alpha_n \preceq \delta,$$

$$p(\zeta_n, \sigma) \preceq \beta_n \preceq \delta.$$

So, $d(\eta_n, \sigma) \preceq \epsilon$, and then $\{\eta_n\}$ converges to σ. (*ii*) holds, because using (*i*).

Now, to show that (*iii*) is true. Let $\epsilon \succ 0$. As in the proof of (*ii*), choose $\delta \succ 0$, and then $n_0 \in \mathbb{N}$. Then, for any $m, n \geq n_0 + 1$

$$p(\zeta_{n_0}, \zeta_n) \preceq \alpha_{n_0} \preceq \delta,$$

$$p(\zeta_{n_0}, \zeta_m) \preceq \alpha_{n_0} \preceq \delta.$$

So, $\Gamma(\zeta_n, \zeta_m) \preceq \epsilon$. Therefore ζ_n is a Cauchy sequence. As in proof of (*iii*), we can prove (*iv*). □

4. Fixed point theorems

In this section, we will prove some fixed point theorems for mappings satisfying some appropriate inequalities in complete complex-valued metric spaces. Moreover, we deduce new fixed point results in complete complex-valued metric spaces and provide some examples to illustrate the usability of the obtained results. We suppose $(X, \leq)$ is a partially ordered set and $\Im$ is a mapping of X into itself. We say that $\Im$ is non-decreasing if for $\zeta, \eta \in X$, $\zeta \leq \eta$ implies $\Im\zeta \leq \Im\eta$.

Theorem 4.1. Let (X, Γ) be a complete complex-valued metric spaces. Assume that $\Im$ is a mapping from X into itself and a function $p : X \times X \to [0, \infty)$ is ϖ-distance on X. Suppose that the following conditions are satisfied.

(*i*) $(X, \leq)$ is a partially ordered set and $\Im$ non-decreasing mapping;

(*ii*) for any fixed $\zeta \in X$ with $\zeta \leq \Im\zeta$

$$\inf_{\eta \in X}\{p(\zeta, \eta) + p(\zeta, \Im\zeta)\} \succ 0, \quad (4.1)$$

with $\eta \neq \Im\eta$;

(*iii*) there exists an $\zeta_0 \in X$ with $\zeta_0 \leq \Im\zeta_0$;

(*iv*) there exists $\alpha, \beta \in [0, \frac{1}{2})$ such that

$$p(\Im\zeta, \Im^2\eta) \precsim \alpha p(\zeta, \Im\eta) + \beta\frac{p(\zeta, \Im\zeta)p(\Im\eta, \Im^2\eta)}{1 + p(\zeta, \Im\eta)}, \quad (4.2)$$

for all $\zeta \leq \Im\zeta$ and any $\eta \in X$ and $\alpha + \beta < 1$.

Then $\Im$ has a fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. Let ζ_0 be an arbitrary point in X, we define $\zeta_{n+1} = \Im\zeta_n$, for all $n \in \mathbb{N}_0$. If $\Im\zeta_0 = \zeta_0$, then the proof is complete. On the other hand, suppose that $\Im\zeta_0 \neq \zeta_0$. Since $\zeta_0 \leq \Im\zeta_0$ and $\Im$ is non-decreasing, we have

$$\zeta_0 \leq \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \dots \leq \zeta_n \leq \zeta_{n+1} \leq \dots .$$

Using the inequality (4.2) and $p(\zeta_n, \Im\zeta_n) \prec 1 + p(\zeta_n, \Im\zeta_n)$, we obtain that

$$\begin{aligned} p(\zeta_{n+1}, \zeta_{n+2}) &= p(\Im\zeta_n, \Im^2\zeta_n) \\ &\precsim \alpha p(\zeta_n, \Im\zeta_n)+ \\ &\beta\frac{p(\zeta_n, \Im\zeta_n)p(\Im\zeta_n, \Im^2\zeta_n)}{1 + p(\zeta_n, \Im\zeta_n)} \\ &\precsim \alpha p(\zeta_n, \zeta_{n+1})+ \\ &\beta p(\zeta_{n+1}, \zeta_{n+2}). \end{aligned} \quad (4.3)$$

Thus,

$$\begin{aligned} p(\zeta_{n+1}, \zeta_{n+2}) &\precsim \gamma p(\zeta_n, \zeta_{n+1}) \\ &\precsim \gamma^2 p(\zeta_{n-1}, \zeta_n) \\ &\precsim \vdots \\ &\precsim \gamma^{n+1} p(\zeta_0, \zeta_1)., \end{aligned} \quad (4.4)$$

where $\gamma = \frac{\alpha}{1-\beta}$. Then, for any $n \in \mathbb{N}_0$ with $m > n$, using the triangle inequality, we obtain that

$$\begin{aligned} p(\zeta_n, \zeta_m) &\precsim p(\zeta_n, \zeta_{n+1}) + p(\zeta_{n+1}, \zeta_{n+2})+ \\ &\dots + p(\zeta_{m-1}, \zeta_m) \\ &\precsim \gamma^n p(\zeta_0, \zeta_1) + \gamma^{n+1} p(\zeta_0, \zeta_1)+ \\ &\dots + \gamma^{m-1} p(\zeta_0, \zeta_1) \\ &\precsim (\gamma^n + \gamma^{n+1} + \dots + \gamma^{m-1}) p(\zeta_0, \zeta_1) \\ &\precsim (\frac{\gamma^n}{1-\gamma}) p(\zeta_0, \zeta_1). \end{aligned}$$

By using Lemma 3.6, we obtain that $\{\zeta_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\eta \in X$ such that $\zeta_n \to \eta$. For any $n \in \mathbb{N}_0$. Then since $\{\zeta_n\}$ converges to η in (X, Γ) and $p(\zeta_n, \cdot)$ is lower semi-continuous, we obtain that $p(\zeta_n, \eta) \leq \liminf_{m\to\infty} p(\zeta_n, \zeta_m) \leq \frac{\gamma^n}{1-\gamma} p(\zeta_0, \zeta_1)$. Now, we show that η is a fixed point of $\Im$, (i.e. $\Im\eta = \eta$). If $\Im\eta \neq \eta$, then by using condition (4.1) and $\zeta_n \leq \zeta_{n+1}$, we get

$$\begin{aligned} 0 &\prec \inf_{\eta\in X} \{p(\zeta_n, \eta) + p(\zeta_n, \zeta_{n+1})\} \\ &\leq \inf_{\eta\in X} \{\frac{\gamma^n}{1-\gamma} p(\zeta_0, \zeta_1) + \gamma^n p(\zeta_0, \zeta_1)\}. \end{aligned}$$

It implies that $0 < 0$, which is a contradiction. Therefore $\eta = \Im\eta$. Moreover, by the inequality (4.2), we have $p(\eta, \eta) = 0$. □

Now, we will prove the unique fixed point as follows:

Corollary 4.2. Let (X, Γ) be a complete complex-valued metric space. Assume that the function $p : X \times X \to [0, \infty)$ is ϖ-distance on X and $\Im$ is a mapping from X into itself are satisfying the conditions $(i) - (iv)$ in Theorem 4.1 and $X \neq F(\Im)$. Then $\Im$ has a unique fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. We will show that $\Im$ has a unique fixed point. Assume that σ, η in X are a fixed point of $\Im$. If $\sigma \neq \eta$, then by assumption and using condition (4.1), there exists $\varsigma \in X$ such that $\Im\varsigma \neq \varsigma$, we get

$$0 \prec \inf_{\varsigma\in X} \{p(\eta, \varsigma) + p(\eta, \Im\eta)\}$$

and

$$0 \prec \inf_{\varsigma\in X} \{p(\sigma, \varsigma) + p(\sigma, \Im\sigma)\}.$$

Thus, we obtain that

$$\begin{aligned} 0 &\prec \inf_{\varsigma\in X} \{p(\eta, \varsigma) + p(\eta, \Im\eta)\} \\ &+ \inf_{\varsigma\in X} \{p(\sigma, \varsigma) + p(\sigma, \Im\sigma)\} \\ &\leq p(\eta, \Im\eta) + p(\sigma, \Im\sigma) = 0, \end{aligned} \quad (4.5)$$

which a contradiction. Hence $\sigma = \eta$. Therefore $\Im$ has a unique fixed point. □

Corollary 4.3. Let (X, Γ) be a complete complex-valued metric spaces. Assume that the function $p : X \times X \to [0, \infty)$ is ϖ-distance on X and $\Im$ is a mapping from X into itself are satisfied the conditions $(i)-(iv)$ in Theorem 4.1 and, if $p(\zeta, \eta) = 0$, then $\zeta = \eta$ for all $\zeta, \eta \in X$.
Then $\Im$ has a unique fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. We will to show that $\Im$ has unique fixed point. Assume that σ, η in X are a fixed point of $\Im$. If $\sigma \neq \eta$, then using inequality (4.2), we have

$$\begin{aligned} p(\eta, \sigma) &= p(\Im\eta, \Im^2\sigma) \\ &\leq \alpha p(\eta, \Im\sigma) \\ &\quad + \beta\frac{p(\eta, \Im\eta)p(\Im\sigma, \Im^2\sigma)}{1 + p(\eta, \Im\sigma)} \\ &= \alpha p(\eta, \sigma). \end{aligned}$$

Since $0 \leq \alpha < 1$, we have $\sigma = \eta$. Therefore $\Im$ has unique fixed point. □

Example 4.4. Let $X = \mathbb{C}$, and let $|\cdot|$ be an absolute value of $\mathbb{R}$. Defined

$$p(\zeta, \eta) = |Re(\eta)| + i|Im(\eta)|,$$

for all $\zeta, \eta \in X$. Then p is ϖ-distance on X, by Example 3.3. Consider the function $\Im : X \to X$ defined by $\Im\zeta = \frac{\zeta}{2}$. Thus,

$$\begin{aligned} p(\Im\zeta, \Im^2\eta) &= |\Im^2\eta| = |Re(\frac{\eta}{4})| + i|Im(\frac{\eta}{4})| \\ &= \frac{1}{2}(|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|) \\ &\precsim \frac{1}{2}(|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|)+ \\ &\frac{1}{3}\frac{(|Re(\frac{\zeta}{2})| + i|Im(\frac{\zeta}{2})|)(|Re(\frac{\eta}{4})| + i|Im(\frac{\eta}{4})|)}{1 + (|Re(\frac{\eta}{2})| + i|Im(\frac{\eta}{2})|)} \\ &= \alpha p(\zeta, \Im\eta) + \beta\frac{p(\zeta, \Im\zeta)p(\Im\eta, \Im^2\eta)}{1 + p(\zeta, \Im\eta)}, \end{aligned}$$

where $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Then the conditions of Theorem 4.1 hold and the fixed point of $\Im$ is $0 + 0i$ and $P(0, 0) = 0$. Moreover, $\Im$ has a unique fixed point, because $X \neq F(\Im)$.

Next, we will replace the inequality (4.2) by the inequality (4.7), which have a same the result of theorem 4.1 and prove the fixed point theorem as follows:

Theorem 4.5. Let (X, Γ) be a complete complex-valued metric spaces. Assume that $\Im$ is a mapping from X into itself and a function $p : X \times X \to [0, \infty)$ is ϖ-distance on X. Suppose that the following conditions are satisfied.

(i) $(X, \leq)$ is a partially ordered set and $\Im$ non-decreasing mapping;

(ii) for any fixed $\zeta \in X$ with $\zeta \leq \Im\zeta$

$$\inf_{\eta \in X}\{p(\zeta, \eta) + p(\zeta, \Im\zeta)\} \succ 0, \tag{4.6}$$

with $\eta \neq \Im\eta$;

(iii) there exists an $\zeta_0 \in X$ with $\zeta_0 \leq \Im\zeta_0$;

(iv) there exists $\alpha, \beta \in [0, \frac{1}{2})$ such that

$$p(\Im\zeta, \Im^2\eta) \precsim \alpha p(\zeta, \Im\eta) + \beta\frac{p(\zeta, \Im\zeta)p(\eta, \Im^2\eta)}{1 + p(\zeta, \Im\eta)}, \tag{4.7}$$

for all $\zeta \leq \Im\zeta$ and any $\eta \in X$ and $\alpha + 2\beta < 1$. Then $\Im$ has a fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. Let ζ_0 be an arbitrary point in X, we define $\zeta_{n+1} = \Im\zeta_n$, for all $n \in \mathbb{N}_0$. If $\Im\zeta_0 = \zeta_0$, then the proof is complete. On the other hand, suppose that $\Im\zeta_0 \neq \zeta_0$. Since $\zeta_0 \leq \Im\zeta_0$ and $\Im$ is non-decreasing, we have

$$\zeta_0 \leq \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \ldots \leq \zeta_n \leq \zeta_{n+1} \leq \ldots.$$

Using the inequality (4.7) and $p(\zeta_n, \Im\zeta_n) \prec 1 + p(\zeta_n, \Im\zeta_n)$, we obtain that

$$p(\zeta_{n+1}, \zeta_{n+2}) = p(\Im\zeta_n, \Im^2\zeta_n)$$

$$\precsim \alpha p(\zeta_n, \Im\zeta_n) + \beta \frac{p(\zeta_n, \Im\zeta_n)p(\zeta_n, \Im^2\zeta_n)}{1 + p(\zeta_n, \Im\zeta_n)}$$
$$\leq \alpha p(\zeta_n, \zeta_{n+1}) + \beta p(\zeta_n, \zeta_{n+2}),$$
$$\leq \alpha p(\zeta_n, \zeta_{n+1}) + \beta p(\zeta_n, \zeta_{n+1})+ \beta p(\zeta_{n+1}, \zeta_{n+2}). \quad (4.8)$$

Thus,

$$\begin{aligned} p(\zeta_{n+1}, \zeta_{n+2}) &\precsim \gamma p(\zeta_n, \zeta_{n+1}) \\ &\precsim \gamma^2 p(\zeta_{n-1}, \zeta_n) \\ &\precsim \vdots \\ &\precsim \gamma^{n+1} p(\zeta_0, \zeta_1), \end{aligned} \quad (4.9)$$

where $\gamma = \frac{\alpha+\beta}{1-\beta}$. Then, for any $n \in \mathbb{N}_0$ with $m > n$, using the triangle inequality, we obtain that

$$\begin{aligned} p(\zeta_n, \zeta_m) &\precsim p(\zeta_n, \zeta_{n+1}) + p(\zeta_{n+1}, \zeta_{n+2})+ \dots + p(\zeta_{m-1}, \zeta_m) \\ &\precsim \gamma^n p(\zeta_0, \zeta_1) + \gamma^{n+1} p(\zeta_0, \zeta_1)+ \dots + \gamma^{m-1} p(\zeta_0, \zeta_1) \\ &\precsim (\gamma^n + \gamma^{n+1} + \dots + \gamma^{m-1}) p(x_0, x_1) \\ &\precsim (\frac{\gamma^n}{1-\gamma}) p(\zeta_0, \zeta_1). \end{aligned} \quad (4.10)$$

By using Lemma 3.6, we obtain that $\{\zeta_n\}$ is a Cauchy sequence in X. As in proof of Theorem 4.1, we obtain that $\Im$ has a fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$. □

Corollary 4.6. Let (X, Γ) be a complete complex-valued metric space. Assume that the function $p : X \times X \to [0, \infty)$ is ϖ-distance on X and $\Im$ is a mapping from X into itself are satisfied conditions $(i) - (iv)$ in Theorem *4.5* and $X \neq F(\Im)$.
Then $\Im$ has a unique fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. The same proof of Corollary 4.2. □

Corollary 4.7. Let (X, Γ) be a complete complex-valued metric space. Assume that the function $p : X \times X \to [0, \infty)$ is ϖ-distance on X and $\Im$ is a mapping from X into itself are satisfied conditions $(i) - (iv)$ in Theorem *4.5* and, if $p(\zeta, \eta) = 0$, then $\zeta = \eta$ for all $\zeta, \eta \in X$.
Then $\Im$ has a unique fixed point. Moreover, if $\nu = \Im\nu$, then $p(\nu, \nu) = 0$.

Proof. We will show that $\Im$ has unique fixed point. Assume that σ, η in X are a fixed point of $\Im$. If $\sigma \neq \eta$, then using inequality (4.7), we have

$$\begin{aligned} p(\eta, \sigma) &= p(\Im\eta, \Im^2\sigma) \\ &\leq \alpha p(\eta, \Im\sigma) + \beta \frac{p(\eta, \Im\eta)p(\sigma, \Im^2\sigma)}{1 + p(\eta, \Im\sigma)} \\ &= \alpha p(\eta, \sigma). \end{aligned}$$

Since $0 \leq \alpha < 1$, we have $\sigma = \eta$. Therefore $\Im$ has unique fixed point. □

Example 4.8. Let $X = \mathbb{C}$, and define a mapping $\Gamma : X \times X \to \mathbb{C}$ by $|\zeta - \eta|$ for all $\zeta, \eta \in X$, then (X, Γ) can be easily verified as a complete complex-valued metric spaces. Defined $p(\zeta, \eta) = |Re(\eta)| + i|Im(\eta)|$, for all $\zeta, \eta \in X$. Then p is ϖ-distance on X, consider the function $\Im : X \to X$ defined by

$$\Im(\zeta + i\eta) = \begin{cases} 0, & \zeta, \eta \in Q \\ 3 + 3i, & \zeta, \eta \in Q^c \\ 3, & \zeta \in Q^c, \eta \in Q \\ 3i, & \zeta \in Q, \eta \in Q^c. \end{cases} \quad (4.11)$$

Now for $\zeta = \frac{1}{\sqrt{3}}$ and $\eta = 0$ we get $\Gamma(\Im(\frac{1}{\sqrt{3}}), \Im(0)) = \Gamma(3, 0) = 3 \precsim \lambda\Gamma(\frac{1}{\sqrt{3}}, 0) = \lambda\frac{1}{\sqrt{3}}$. Thus $\lambda \geq 3\sqrt{3}$, which is a contradiction as $0 \leq \lambda < 1$. However, notice that $\Im^2\sigma = 0$, so that $0 = \Gamma(\Im^2\sigma_1, \Im^2\sigma_2) \precsim \lambda\Gamma(\sigma_1, \sigma_2)$,
which shows that $\Im^2$ satisfies the requirement of Bryant Theorem and $\sigma = 0$ is the unique fixed point of $\Im$.

$$p(\Im\zeta, \Im^2\eta) = |\Im^2\eta| = |Re(\frac{\eta}{9})| + i|Im(\frac{\eta}{9})|$$
$$= \frac{1}{3}(|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|)$$
$$\precsim \frac{1}{3}(|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|)+$$
$$\frac{1}{4}\frac{(|Re(\frac{\zeta}{3})| + i|Im(\frac{\zeta}{3})|)(|Re(\frac{\eta}{9})| + i|Im(\frac{\eta}{9})|)}{1 + (|Re(\frac{\eta}{3})| + i|Im(\frac{\eta}{3})|)}$$
$$= \alpha p(\zeta, \Im\eta) + \beta\frac{p(\zeta, \Im\zeta)p(\eta, \Im^2\eta)}{1 + p(\zeta, \Im\eta)},$$

where $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{4}$. Then the conditions of Theorem 4.5 hold and the fixed point of $\Im$ is $0 + 0i$ and $p(0, 0) = 0$. Moreover, $\Im$ has a unique fixed point, because $X \neq F(\Im)$, ($F(\Im)$ is means that the set of fixed point of $\Im$).

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