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Original research article

Involutive Weak Cubical ω-categories

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ABSTRACT

We investigate the notion of involutive weak cubical ω-categories via Penon's approach: as algebras for the monad induced by the free involutive strict ω-category functor on cubical ω-sets. A few examples of involutive weak cubical ω-categories are provided.

Keywords: ω-categories; Category theory; Higher category; Involutive category; Monad

1. Introduction

Motivated by research in algebraic topology, category theory, starting from Eilenberg-MacLane [1], developed into an independent mathematical subject. Although higher categories had been already implicit in the definition of natural transformations, the study of n-categories (both in their globular and cubical versions) was initiated in C.Ehresmann [2]. Strict ω-categories had been conjectured by J.Roberts (as later reported in J.Roberts [3]) and independently introduced and studied by Brown-Higgins [4]. The development of weak higher category theory (somehow implicit in the definition of monoidal category) probably started with the definition of bicategory in J.Bénabou [5] and n-category in R.Street [6] and is now a quite active area of research (see for example Cheng-Lauda [7], Leinster [8, 9]). Algebraic approaches to the definition of weak globular higher-categories have been developed by M.Batanin [10], Penon [11] and Leinster [9]. A similar study for the weak cubical higher categories, using Penon's technique, has been carried on by C.Kachour in several important recent works [12].

The notion of involution (duality) in category theory has a relatively "involved" history with concepts independently introduced by several authors in different contexts and generality (see [13] and [14, section 4] for some bibliographical details); a recent systematic treatment of the topic is contained in [15] where further references

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can be found.

Here we are specifically interested in a (vertical) categorification of the usual $*$-operation in operator algebras: the "$*$-categories" considered in [16, 17] and the "dagger categories" axiomatized in [18] and utilized in [19].

Strict involutive globular n-categories have been considered in [14]. Weak involutive globular ω-categories have been introduced, using Penon's contractions in [20, 21] and, in [22, 23], using Leinster's definition of globular ω-categories.

In the present work, we aim at a sufficiently general definition of *involutive weak cubical ω-category* following the C.Kachour algebraic notion of weak cubical Penon ω-category.

The organization of the paper is the following.

After this introduction, in section 2, we approach the study of strict involutive cubical ω-categories:

- following the ideas of [4] and [12], suitably general notions of cubical ω-quivers and cubical ω-sets are introduced in definitions 2.1 and 2.2,
- self-dualities on cubical ω-sets and the algebraic properties of cubical involutions are axiomatized, following the double category case in [24], in definitions 2.3 and 2.5,

The proof that the free strict involutive cubical ω-category of a cubical ω-set exists is postponed to section 3 in lemmata 3.3 and 3.4 and hence the associated monad is constructed in corollary 3.5.

In section 3 we deal with the involutive version of Penon-Kachour weak cubical ω-categories:

- we introduce in definition 3.1 a notion of Penon-Kachour contraction for our cubical ω-sets,
- in lemma 3.6 it is proved that the free contracted Penon-Kachour cubical involutive ω-contraction exists and hence, in theorem 3.7, we show that we have an associated monad,
- in definition 3.8 weak involutive cubical ω-categories are introduced (similarly to Kachour for cubical groupoids) as algebras for the previous monad,
- some examples of such weak involutive cubical ω-categories are suggested in subsection 3.1.

Finally in a brief outlook section 4 we examine some possible future direction of development of this work.

2. Strict (Involutive) Cubical ω-categories

The first definition only formalizes the idea that "n-dimensional cells" $x \in \mathcal{Q}^n$ are equipped with a family of "source/target" $(n-1)$-dimensional cells, indexed as the "faces of an n-dimensional hypercube". The sets D with cardinality $|D| = n$ indicate the possible "directions" of the n-dimensional cells, where the "directions" are selected via subsets (of cardinality n) in the infinite countable set $\mathbb{N}_0$. In this generality, morphisms are just a countable family of "dimension-preserving" maps compatible with sources and targets.

Definition 2.1. *A* ***cubical*** ω***-quiver*** *is a family* $\left(\mathfrak{Q}^n_{D-\{d\}} \xleftarrow{s^n_{D,d},\ t^n_{D,d}} \mathfrak{Q}^{n+1}_D\right)_{n\in\mathbb{N}}$ *of* ***source maps*** $s^n_{D,d}$ *and* ***target maps*** $t^n_{D,d}$ *indexed by* $n \in \mathbb{N}$*, by any* $D \subset \mathbb{N}_0$ *with cardinality* $|D| = n + 1$ *and any* $d \in D$*.*

A ***morphism of cubical*** ω***-quivers*** *is a family* $\mathfrak{Q}^n_D \xrightarrow{\phi^n_D} \hat{\mathfrak{Q}}^n_D$ *indexed by* $n \in \mathbb{N}$ *and* $D \subset \mathbb{N}$ *with* $|D| = n$*, such that*

- $\hat{s}^n_{D,d} \circ \phi^{n+1}_D = \phi^n_{D-\{d\}} \circ s^n_{D,d}$

  and

- $\hat{t}^n_{D,d} \circ \phi^{n+1}_D = \phi^n_{D-\{d\}} \circ t^n_{D,d}$,

for all $n \in \mathbb{N}$*,* $D \subset \mathbb{N}_0$ *with* $|D| = n$ *and* $d \in D$*.*

The actual n-dimensional "cubical shape" of n-cells is specified by the following axioms.

Definition 2.2. *A* ***cubical*** ω***-set*** *is a cubical* ω*-quiver* $\left(\mathfrak{Q}^n_{D-\{d\}} \xleftarrow{s^n_{D,d},\ t^n_{D,d}} \mathfrak{Q}^{n+1}_D\right)_{n\in\mathbb{N}}$ *satisfying the* ***cubical axioms****:*

for all $n \in \mathbb{N} : n \geq 2$*,* $D \subset \mathbb{N}_0$ *with* $|D| = n$ *and* $d \neq e \in D$*,*

$$s^{n-2}_{D-\{d,e\}} \circ s^{n-1}_{D-\{d\}} = s^{n-2}_{D-\{d,e\}} \circ s^{n-1}_{D-\{e\}},$$

$$t^{n-2}_{D-\{d,e\}} \circ t^{n-1}_{D-\{d\}} = t^{n-2}_{D-\{d,e\}} \circ t^{n-1}_{D-\{e\}},$$

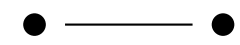
$$s^{n-2}_{D-\{d,e\}} \circ t^{n-1}_{D-\{d\}} = t^{n-2}_{D-\{d,e\}} \circ s^{n-1}_{D-\{e\}},$$

$$t^{n-2}_{D-\{d,e\}} \circ s^{n-1}_{D-\{d\}} = s^{n-2}_{D-\{d,e\}} \circ t^{n-1}_{D-\{e\}}.$$

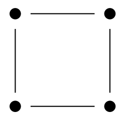
A ***morphism of cubical*** ω***-sets*** *is just a morphism of underlying cubical* ω*-quivers.*

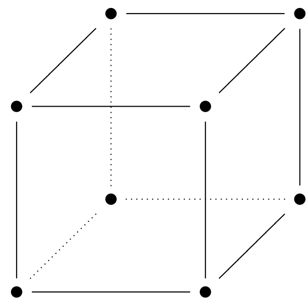
A pictorial description of cubical n-cells, for four cases $n = 0, D = \varnothing$; $n = 1, D = \{1\}$; $n = 2, D = \{1, 2\}$; $n = 3, D = \{1, 2, 3\}$ respectively, is here below:

case $1 : n = 0$ case $2 : n = 1$



case $3 : n = 2$ case $4 : n = 3$





Next we introduce three families of (binary, nullary, unary) operations on cubical n-cells.

Definition 2.3. *Given a cubical* ω*-set* $\mathfrak{Q}$*, we can introduce on it the following operations:*

a. ***binary compositions****:* $\forall n \in \mathbb{N}_0$*,* $\forall d \in D \subset \mathbb{N}_0 : |D| = n$*,*
$\circ^n_{D,d} : \mathfrak{Q}^n_D \times_{\mathfrak{Q}^{n-1}_{D-\{d\}}} \mathfrak{Q}^n_D \to \mathfrak{Q}^n_D$*, where*

$$\mathfrak{Q}^n_D \times_{\mathfrak{Q}^{n-1}_{D-\{d\}}} \mathfrak{Q}^n_D := \left\{(x, y) \mid s^{n-1}_{D,d}(x) = t^{n-1}_{D,d}(y)\right\}$$

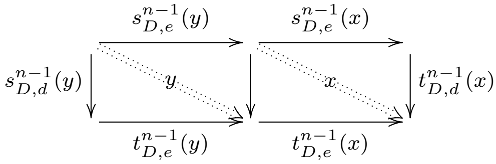
and we assume:

$$s^{n-1}_{D,d}(x \circ^n_{D,d} y) = s^{n-1}_{D,d}(y),$$
$$t^{n-1}_{D,d}(x \circ^n_{D,d} y) = t^{n-1}_{D,d}(x),$$

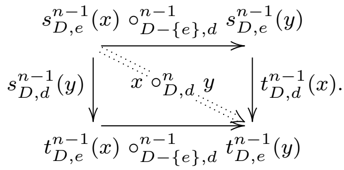
$$s^{n-1}_{D,e}(x \circ^n_{D,d} y) = s^{n-1}_{D,e}(x) \circ^{n-1}_{D-\{e\},d} s^{n-1}_{D,e}(y),$$

$$t^{n-1}_{D,e}(x \circ^n_{D,d} y) = t^{n-1}_{D,e}(x) \circ^{n-1}_{D-\{e\},d} t^{n-1}_{D,e}(y), \ \forall e \neq d.$$

The diagram of binary composition is depicted as follows;



maps to



b. ***nullary reflectors***

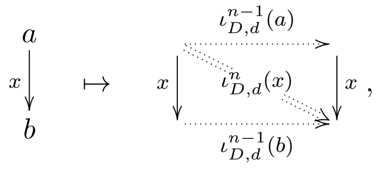
$$\iota^n_{D,d} : \mathcal{Q}^{n-1}_{D-\{d\}} \to \mathcal{Q}^n_D, \forall n \in \mathbb{N}_0,$$
$$\forall d \in D \subset \mathbb{N}_0 : |D| = n,$$

where the following structural axioms are assumed:

$$s^{n-1}_{D,d}(\iota^n_{D,d}(x)) = x = t^{n-1}_{D,d}(\iota^n_{D,d}(x)),$$
$$s^{n-1}_{D,e}(\iota^n_{D,d}(x)) = \iota^{n-1}_{D,d}(s^{n-2}_{D-\{d\},e}(x)),$$
$$t^{n-1}_{D,e}(\iota^n_{D,d}(x)) = \iota^{n-1}_{D,d}(t^{n-2}_{D-\{d\},e}(x)),$$

$\forall e \neq d$.

The following diagram is a ***nullary reflector****,*



where $a := s^{n-2}_{D-\{d\},e}(x)$,
$b := t^{n-2}_{D-\{d\},e}(x)$.

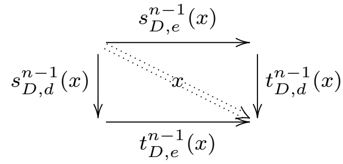
c. ***unary self-dualities***

$$*^n_{D,d} : \mathcal{Q}^n_D \to \mathcal{Q}^n_D, \forall n \in \mathbb{N}_0,$$
$$\forall d \in D \subset \mathbb{N}_0 : |D| = n,$$

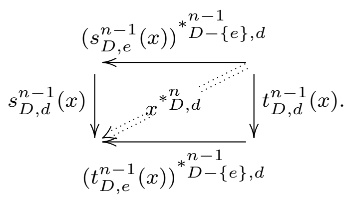
where we assume the following structural axioms:

$s^{n-1}_{D,e}(x^{*^n_{D,d}}) = (s^{n-1}_{D,e}(x))^{*^{n-1}_{D-\{e\},d}}$,
$t^{n-1}_{D,e}(x^{*^n_{D,d}}) = (t^{n-1}_{D,e}(x))^{*^{n-1}_{D-\{e\},d}}$,
$\forall e \neq d$,
$s^{n-1}_{D,d}(x^{*^n_{D,d}}) = t^{n-1}_{D,d}(x)$,
$t^{n-1}_{D,d}(x^{*^n_{D,d}}) = s^{n-1}_{D,d}(x)$.

The following diagram is a ***unary self-duality***



maps to



A ***reflective cubical*** ω***-set*** *is a cubical* ω*-set equipped with reflectors as in point (b.) of definition 2.3 above; a* ***self-dual cubical*** ω***-set*** *is a cubical* ω*-set equipped with the self-dualities, as in point (c.) of definition 2.3. A* ***cubical*** ω***-magma*** *is a cubical* ω*-set equipped with the binary compositions in point (a.) of definition 2.3; a* ***reflective (self-dual) cubical*** ω***-magma*** *is a cubical* ω*-set equipped with reflectors (self-dualities) and compositions.*

A ***morphism of reflective cubical*** ω***-sets*** *is a morphism* $(\phi^n_D)_{n\in\mathbb{N}, D\subset\mathbb{N}_0:|D|=n}$ *of cubical* ω*-sets that also satisfies:*
$\phi^n_D \circ \iota^n_{D,d} = \hat{\iota}^n_{D,d} \circ \phi^{n-1}_{D-\{d\}}$, *for all* $n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ *with* $|D| = n$, $d \in D$.

*A **morphism of self-dual cubical** ω-**sets** is a morphism $(\phi^n_D)_{n\in\mathbb{N},\ D\subset\mathbb{N}_0:|D|=n}$ of cubical ω-sets that also satisfies:*
$\phi^n_D \circ *^n_{D,d} = \hat{*}^n_{D,d} \circ \phi^n_D$, for all $n \in \mathbb{N}$, $D \subset \mathbb{N}_0$ with $|D| = n$, $d \in D$.

*A **morphism of cubical** ω-**magmas** is a morphism $(\phi^n_D)_{n\in\mathbb{N},\ D\subset\mathbb{N}_0:|D|=n}$ of cubical ω-sets that also satisfies:*
$\phi^n_D(x \circ^n_{D,d} y) = \phi^n_D(x)\hat{\circ}^n_{D,d}\phi^n_D(y)$, for all $n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ with $|D| = n$, $d \in D$ and $(x, y) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D}-\{d\}} \mathcal{Q}^n_D$.

To obtain strict cubical ω-categories we further impose the usual algebraic axioms.

Definition 2.4. *A **strict cubical** ω-**category** is a cubical reflective ω-magma such that the following algebraic axioms are satisfied:*

- ***associativity of compositions**: for all $n \in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and for all $d \in D$:*
  $$x \circ^n_{D,d} (y \circ^n_{D,d} z) = (x \circ^n_{D,d} y) \circ^n_{D,d} z,$$
  $$\forall (x, y, z) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{Q}^n_D,$$
- ***unitality of compositions**: for all $n \in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and for all $d \in D$:*
  $x \circ^n_{D,d} \iota^n_{D,d}(s^{n-1}_{D,d}(x)) = x$
  and
  $x = \iota^n_{D,d}(t^{n-1}_{D,d}(x)) \circ^n_{D,d} x$,
  $\forall x \in \mathcal{Q}^n_D$,
- ***functoriality of identities**: for all $n \in \mathbb{N}_0 - \{1\}$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and for all $e \neq d \in D$:*
  $$\iota^n_{D,d}(x \circ^{n-1}_{D-\{d\},e} y) = \iota^n_{D,d}(x) \circ^n_{D,e} \iota^n_{D,d}(y),$$
  $$\forall (x, y) \in \mathcal{Q}^{n-1}_D \times_{\mathcal{Q}^{n-2}_{D-\{d\}}} \mathcal{Q}^{n-1}_D,$$
- ***exchange property**: for all $n \in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and for all $e \neq f \in D$:*
  $$(x \circ^n_{D,e} y) \circ^n_{D,f} (w \circ^n_{D,e} z) = (x \circ^n_{D,f} w) \circ^n_{D,e} (y \circ^n_{D,f} z),$$
  $$\forall (x, y), (w, x) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{e\}}} \mathcal{Q}^n_D,$$
  $$\forall (x, w), (y, z) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{f\}}} \mathcal{Q}^n_D.$$

*A **covariant functor between cubical** ω-**categories** is just a morphism of reflective cubical ω-magmas.*

Definition 2.5. *A **strict involutive cubical** ω-**category** further requires these algebraic axioms:*

- ***involutivity**: for all $\in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and $d \in D$,*
  $$(x^{*^n_{D,d}})^{*^n_{D,d}} = x, \qquad \forall x \in \mathcal{Q}^n_D,$$
- ***commutativity of involutions**: for all $\in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$,*
  $$(x^{*^n_{D,e}})^{*^n_{D,f}} = (x^{*^n_{D,f}})^{*^n_{D,e}},$$
  $$\forall x \in \mathcal{Q}^n_D, \quad \forall e \neq f \in D,$$
- ***functoriality of involutions**: for all $\in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$,*
  $\forall d \in D$,
  $$(x \circ^n_{D,d} y)^{*^n_{D,d}} = (y^{*^n_{D,d}}) \circ^n_{D,d} (x^{*^n_{D,d}}),$$
  $\forall d \neq e \in D$,
  $$(x \circ^n_{D,d} y)^{*^n_{D,e}} = (x^{*^n_{D,e}}) \circ^n_{D,d} (y^{*^n_{D,e}}),$$
- ***Hermitianity of identities**: for all $\in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$,*
  $$\forall x \in \mathcal{Q}^n_D, (\iota^n_{D,d}(x))^{*^n_{D,d}} = \iota^n_{D,d}(x),$$
  $\forall x \in \mathcal{Q}^n_D, \forall d \neq e \in D$,
  $$(\iota^n_{D,d}(x))^{*^n_{D,e}} = \iota^n_{D,d}(x^{*^n_{D,e}}).$$

*A **covariant functor between involutive cubical** ω**-categories** is a morphism of self-dual reflective cubical ω-magmas.*

3. Penon Kachour Weak (Involutive) Cubical ω-categories

We proceed to define Penon-Kachour contractions in the cubical setting.

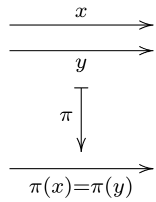
Definition 3.1. *Given a cubical (self-dual) reflective ω-magma $\mathcal{M}$, a strict cubical (involutive) ω-category $\mathcal{C}$ and a morphism of cubical (self-dual) reflective ω-magmas $\mathcal{M} \xrightarrow{\pi} \mathcal{C}$, a **Penon-Kachour** π**-contraction** is a family of maps*

$$\kappa^n_{D,d} : \mathcal{M}^{n-1}_D(\pi) \to \mathcal{M}^n_{D\cup\{d\}},$$

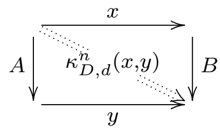
for all $n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ with $|D| = n$ and all $d \in \mathbb{N}_0 - D$ such that: $\mathcal{M}^{n-1}_D(\pi) := \big\{(x, y) \in \mathcal{M}^{n-1}_D \times \mathcal{M}^{n-1}_D \mid \pi(x) = \pi(y)\big\}$,

- $s^{n-1}_{D\cup\{d\},d}(\kappa^n_{D,d}(x, y)) = x$,
- $t^{n-1}_{D\cup\{d\},d}(\kappa^n_{D,d}(x, y)) = y$,
- $s^{n-1}_{D\cup\{d\},e}(\kappa^n_{D,d}(x, y)) = \kappa^{n-1}_{D-\{e\},d}\left(s^{n-2}_{D,e}(x), s^{n-2}_{D,e}(y)\right)$,
- $t^{n-1}_{D\cup\{d\},e}(\kappa^n_{D,d}(x, y)) = \kappa^{n-1}_{D-\{e\},d}\left(t^{n-2}_{D,e}(x), t^{n-2}_{D,e}(y)\right)$, $\forall e \in D$,
- $\pi^n_{D\cup\{d\}}(\kappa^n_{D,d}(x, y)) = \iota^n_{D\cup d,d}(\pi^{n-1}_D(x)) = \iota^n_{D\cup d,d}(\pi^{n-1}_D(y))$,
- $x = y \in \mathcal{M}^{n-1}_D \Longrightarrow \kappa^n_{D,d}(x, y) = \iota^n_{D,d}(x)$,

In pictures, under the following condition of "parallelism" of $(n-1)$-arrows



*we have the following "shape" for the **Penon-Kachour contraction** n-cells*



where $A = \kappa^{n-1}_{D-\{e\},d}(s^{n-2}_{D,e}(x), s^{n-2}_{D,e}(y))$ and $B = \kappa^{n-1}_{D-\{e\},d}(t^{n-2}_{D,e}(x), t^{n-2}_{D,e}(y))$.

*A **morphism of cubical Penon-Kachour contractions** $(\mathcal{M} \xrightarrow{\pi} \mathcal{C}, \kappa) \xrightarrow{(\phi,\Phi)} (\hat{\mathcal{M}} \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}, \hat{\kappa})$ is given by a covariant morphism of reflexive (self-dual) ω-magmas $\mathcal{M} \xrightarrow{\Phi} \hat{\mathcal{M}}$, a covariant (involutive) functor $\mathcal{C} \xrightarrow{\phi} \hat{\mathcal{C}}$ such that:*

$$\hat{\pi} \circ \Phi = \phi \circ \pi, \qquad \Phi \circ \kappa = \hat{\kappa} \circ \phi.$$

With some abuse of notation, we denote by $\mathfrak{U}$ forgetful functors, without explicitly indicating the categories (that will be clear from the context).

Definition 3.2. *A **free (self-dual, reflective) cubical** ω**-magma over a cubical** ω**-set** $\mathcal{Q}$ is a morphism of cubical ω-sets $\mathcal{Q} \xrightarrow{\eta} \mathfrak{U}(\mathcal{M}(\mathcal{Q}))$, into a (self-dual, reflective) cubical ω-magma $\mathcal{M}(\mathcal{Q})$, such that the following universal factorization property holds: for any other morphism of cubical ω-sets $\mathcal{Q} \xrightarrow{\phi} \mathfrak{U}(\mathcal{M})$ into another (self-dual, reflective) cubical ω-magma, there exists a* *unique morphism of (self-dual, reflective) ω-magmas $\mathcal{M}(\mathcal{Q}) \xrightarrow{\hat{\phi}} \mathcal{M}$ such that $\phi = \mathfrak{U}(\hat{\phi}) \circ \eta$.*

*A **free (involutive) cubical ω-category over a cubical ω-set** $\mathcal{Q}$ is a morphism of cubical ω-sets $\mathcal{Q} \xrightarrow{\eta} \mathfrak{U}(\mathcal{C}(\mathcal{Q}))$, into an (involutive) cubical ω-category $\mathcal{C}(\mathcal{Q})$, such that the following universal factorization property holds: for any other morphism of cubical ω-sets $\mathcal{Q} \xrightarrow{\phi} \mathfrak{U}(\mathcal{C})$ into another (involutive) cubical ω-category, there exists a unique morphism of (involutive) ω-categories $\mathcal{C}(\mathcal{Q}) \xrightarrow{\hat{\phi}} \mathcal{C}$ such that $\phi = \mathfrak{U}(\hat{\phi}) \circ \eta$.*

*A **free (self-dual) cubical Penon-Kachour ω-contraction over a cubical ω-set** $\mathcal{Q}$ is a morphism of cubical ω-sets $\mathcal{Q} \xrightarrow{\eta} \mathfrak{U}(\mathcal{M})$ into the underlying cubical ω-set $\mathfrak{U}(\mathcal{M})$ of the magma of a (self-dual) Penon-Kachour contraction $(\mathcal{M} \xrightarrow{\pi} \mathcal{C}, \kappa)$, such that the following universal factorization property holds: for any other morphism $\mathcal{Q} \xrightarrow{\phi} \mathfrak{U}(\hat{\mathcal{M}})$ of cubical ω-sets into the underlying cubical ω-set $\mathfrak{U}(\hat{\mathcal{M}})$ of the magma of another (self-dual) Penon-Kachour contraction $(\hat{\mathcal{M}} \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}, \hat{\kappa})$, there exists a unique morphism of (self-dual) Penon-Kachour contractions*

$$(\mathcal{M} \xrightarrow{\pi} \mathcal{C}, \kappa) \xrightarrow{(\hat{\phi}, \hat{\Phi})} (\hat{\mathcal{M}} \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}, \hat{\kappa})$$

such that $\phi = \mathfrak{U}((\hat{\phi}, \hat{\Phi})) \circ \eta$.

The uniqueness of free structures, up to a unique isomorphism compatible with the universal factorization property, is assured from the definition. The existence is proved in lemma 3.3 below.

Lemma 3.3. *There exists a free self-dual reflective cubical ω-magma over a cubical ω-set $\mathcal{Q}$.*

Proof. The following proof follows the recursive construction strategy in [21, proposition 3.1], also recalled in [23, proposition 3.2 point a.], adapted to our specific cubical ω-set definition.

We start with a given cubical ω-set $\left(\mathcal{Q}^n_{D-\{d\}} \xleftarrow{s^n_{D,d},\, t^n_{D,d}} \mathcal{Q}^{n+1}_D\right)$, with $n \in \mathbb{N}$, $D \subset \mathbb{N}_0$ such that $|D| = n$ and $d \in D$.

We are going to construct a self-dual reflective cubical ω-magma $\left(\mathcal{M}(\mathcal{Q})^n_{D-\{d\}} \xleftarrow{\hat{s}^n_{D,d},\, \hat{t}^n_{D,d}} \mathcal{M}(\mathcal{Q})^{n+1}_D\right)$, with compositions $\circ^n_{D,d}$, self-dualities $*^n_{D,d}$ and reflectors $\iota^n_{D,d}$ as in definition 2.3; and a morphism of cubical ω-sets $\left(\mathcal{Q}^n_D \xrightarrow{\eta^n_D} \mathcal{M}(\mathcal{Q})^n_D\right)$ that satisfies the universal factorization property in the first part of definition 3.2.

We start, for $n := 0$ and necessarily $D := \varnothing$, defining $\mathcal{M}(\mathcal{Q})^0_D := \mathcal{Q}^0_D$ and $\mathcal{Q}^0_D \xrightarrow{\eta^0_D} \mathcal{M}(\mathcal{Q})^0_D$ as the identity map.

The construction of "free 1-arrows" starts defining free 1-identities, in every direction $D := \{d\}$ with $d \in \mathbb{N}_0$, corresponding to the already available objects in $\mathcal{M}(\mathcal{Q})^0_\varnothing$: we set, for all $d \in \mathbb{N}_0$ and 1-direction $D := \{d\}$, $d(\mathcal{Q}^0) := \{(x, d) \mid x \in \mathcal{Q}^0\}$ and $\mathcal{M}(\mathcal{Q})^1[0]^0_D := \mathcal{Q}^1_D \cup d(\mathcal{Q}^0)$; furthermore we extend the definition of sources and targets for the extra identity 1-arrows: $\mathcal{M}(\mathcal{Q})^0_\varnothing \xleftarrow{s^0_{D,d},\, t^0_{D,d}} d(\mathcal{Q}^0)$ by $s^0_{D,d}(x, d) := x =: t^0_{D,d}(x, d)$.

We also introduce the structural map $\eta^1_D : \mathcal{Q}^1_D \to \mathcal{M}(\mathcal{Q})^1[0]^0_D$ as the inclusion of $\mathcal{Q}^1_D$.

We now further introduce arbitrary free duals (in the already available direction) of the 1-arrows in $\mathcal{M}(\mathcal{Q})^1[0]^0$ by the following iterative procedure: suppose that

$\mathcal{M}(\mathcal{Q})^1[0]^j$ has been already constructed; [1] for all $d \in \mathbb{N}_0$ and $D := \{d\}$ we provide

$$\mathcal{M}(\mathcal{Q})^1[0]^{j+1}_D := \{(x, \gamma_d) \mid x \in \mathcal{M}(\mathcal{Q})^1[0]^j_D\};$$

furthermore, we extend the source and target maps to the new extra free dual 1-arrows: $s^0_{D,d}(x, \gamma_d) := t^0_{D,d}(x)$ and $t^0_{D,d}(x, \gamma_d) := s^0_{D,d}(x)$, $\forall x \in \mathcal{M}(\mathcal{Q})^1[0]^j_D$ and $D = \{d\}$ with $d \in \mathbb{N}_0$. We then take $\mathcal{M}(\mathcal{Q})^1[0]_D := \bigcup_{j\in\mathbb{N}} \mathcal{M}(\mathcal{Q})^1[0]^j_D$ with the given source and targets.

The next step consists in introducing free "concatenations" (in the only available direction) of the previous 1-arrows (and their source/target maps). Suppose that we already got $\mathcal{M}(\mathcal{Q})^1[m]$ for all $0 \leq m \leq k$; for all $d \in \mathbb{N}_0$, $D := \{d\}$, we recursively introduce: [2]

$\mathcal{M}(\mathcal{Q})^1[k+1]^0_D$ that is defined by $\big\{(x, d, y) \mid (x, y) \in \mathcal{M}(\mathcal{Q})^1[i]_D \times \mathcal{M}(\mathcal{Q})^1[j]_D\big\}$, where $i + j = k + 1$, $s^0_{D,d}(x) = s^0_{D,d}(y)$;

we also recursively extend the source and target maps to the newly introduced free concatenations:

$$s^0_{D,d}(x, d, y) := s^0_{D,d}(y),$$
$$t^0_{D,d}(x, d, y) := t^0_{D,d}(x),$$
$$\forall (x, d, y) \in \mathcal{M}(\mathcal{Q})^1[k+1]_D.$$

The family $\mathcal{M}(\mathcal{Q})^1[k+1]_D$ is defined by $\bigcup_{j\in\mathbb{N}} \mathcal{M}(\mathcal{Q})^1[k+1]^j_D$, for $D := \{d\}$ and $d \in \mathbb{N}_0$, with its source and target maps into $\mathcal{M}(\mathcal{Q})^0$, is obtained repeating the iteration construction of duals.

[1]Notice that the running index $j \in \mathbb{N}$ is here denoting the number of successive iterations of a given duality, here denoted by the symbol γ_d, applied to an element $x \in \mathcal{M}(\mathcal{Q})^1[0]^0$.

[2]Notice that here the running index $m \in \mathbb{N}_0$ denotes the level of concatenations, corresponding to the number of compositions in the given direction d.

Then we introduce $\mathcal{M}(\mathcal{Q})^1_D$ by $\bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathcal{Q})^1[k]_D$ with the already disjointly defined sources and targets.

As a final recursive step, suppose now that we already defined $\mathcal{Q}^n_{D'} \xrightarrow{\eta^n_{D'}} \mathcal{M}(\mathcal{Q})^n_{D'}$, for $D' \subset \mathbb{N}_0$ with $|D'| = n$, and, for all $d \in D'$, also all the source and target maps $\mathcal{M}(\mathcal{Q})^{n-1}_{D'-\{d\}} \xleftarrow{s^{n-1}_{D',d},\, t^{n-1}_{D',d}} \mathcal{M}(\mathcal{Q})^n_{D'}$, we proceed to define the next stage $\mathcal{M}(\mathcal{Q})^n_{D-\{d\}} \xleftarrow{s^n_{D,d},\, t^n_{D,d}} \mathcal{M}(\mathcal{Q})^{n+1}_D$, for all $D \subset \mathbb{N}_0$ with $|D| = n+1$ and $d \in D$, with the structural maps $\eta^{n+1}_D : \mathcal{Q}^{n+1}_D \to \mathcal{M}(\mathcal{Q})^{n+1}_D$.

We start setting $\mathcal{M}(\mathcal{Q})^{n+1}[0]^0_D$ by $\mathcal{Q}^{n+1}_D \cup \Big(\bigcup_{d\in D} d(\mathcal{Q}^n_{D-\{d\}})\Big)$, where, $d(\mathcal{Q}^n_{D-\{d\}}) := \{(x, d) \mid x \in \mathcal{Q}^n_{D-\{d\}}\}$, for all $D \subset \mathbb{N}_0$ with $|D| = n+1$ and $d \in D$. We also extend the source and target maps to each set $d(\mathcal{Q}^n_{D-\{d\}})$, for $d \in D$, via $s^n_{D,d}(x, d) := x =: t^n_{D,d}(x, d)$ and, whenever $e \neq d \in D$, with

$$s^n_{D,e}(x, d) = (s^{n-1}_{D-\{d\},e}(x), e),$$
$$t^n_{D,e}(x, d) = (t^{n-1}_{D-\{d\},e}(x), e).$$

Then we recursively introduce $\mathcal{M}(\mathcal{Q})^{n+1}[0]^{j+1}_D$ via $\{(x, \gamma_d) \mid x \in \mathcal{M}(\mathcal{Q})^{n+1}[0]^j_D,\ d \in D\}$; we further extend the source and target maps as $s^n_{D,d}(x, \gamma_d) := t^n_{D,d}(x)$, $t^n_{D,d}(x, \gamma_d) := s^n_{D,d}(x)$ and, whenever $d \neq e \in D$, via $s^n_{D,e}(x, \gamma_d) := s^n_{D,e}(x)$ and $t^n_{D,e}(x, \gamma_d) := t^n_{D,e}(x)$; finally we set $\mathcal{M}(\mathcal{Q})^{n+1}[0]_D := \bigcup_{j\in\mathbb{N}} \mathcal{M}(\mathcal{Q})^{n+1}[0]^j_D$, for all $D \subset \mathbb{N}_0$ with $|D| = n+1$ with the already introduced source and target maps.

At last we suppose already defined all $\mathcal{M}(\mathcal{Q})^{n+1}[m]_D$, for all $0 \leq m \leq k$, with their source and target maps and we are going to introduce $\mathcal{M}(\mathcal{Q})^{n+1}[k+1]^0_D$ via

(x, d, y), where (x, y) is in Cartesian product of $\mathcal{M}(\mathcal{Q})^{n+1}[i]_D$ and $\mathcal{M}(\mathcal{Q})^{n+1}[j]_D$, for $i + j = k + 1,\ d \in D,\ s^n_{D,d}(x) = t^n_{D,d}(y)$ defining

$s^n_{D,d}(x, d, y) = s^n_{D,d}(y)$,

$t^n_{D,d}(x, d, y) = t^n_{D,d}(x)$ and, whenever $e \neq d \in D$,
$s^n_{D,e}(x, d, y) = (s^n_{D,e}(x), d, s^n_{D,e}(y))$,
$t^n_{D,d}(x, d, y) = (t^n_{D,e}(x), d, t^n_{D,e}(y))$;
setting $\mathcal{M}(\mathcal{Q})^{n+1}[k]_D$ via $\bigcup_{j\in\mathbb{N}_0} \mathcal{M}(\mathcal{Q})^{n+1}[k]^j_D$, with the same previous recursion strategy freely adding dual $(n + 1)$-arrows, we finally define $\mathcal{M}(\mathcal{Q})^{n+1}_D := \bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathcal{Q})^{n+1}[k]_D$, with already locally well-defined source and target maps. We also define η^{n+1}_D is a morphism $\mathcal{Q}^{n+1}_D \to \mathcal{M}(\mathcal{Q})^{n+1}_D$ as the inclusion into $\mathcal{M}(\mathcal{Q})^{n+1}[0]^0_D \subset \mathcal{M}(\mathcal{Q})^{n+1}_D$.

Up to this point we managed to recursively define a morphism $\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q})$ of cubical ω-sets.

The nullary, unary and binary operations on the cubical ω-set $\mathcal{M}(\mathcal{Q})$ are readily available as follows:

$$\iota^n_{D,d} : \mathcal{M}(\mathcal{Q})^{n-1}_{D-\{d\}} \to \mathcal{M}(\mathcal{Q})^n_D, \quad x \mapsto (x, d),$$
$$*^n_{D,d} : \mathcal{M}(\mathcal{Q})^n_D \to \mathcal{M}(\mathcal{Q})^n_D, (x)^{*^n_{D,d}} := (x, \gamma_d),$$
$$\circ^n_{D,d} : \mathcal{M}(\mathcal{Q})^n_D \times_{\mathcal{M}\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{M}(\mathcal{Q})^n_D \to \mathcal{M}(\mathcal{Q})^n_D,$$

where $(x \circ^n_{D,d} y) := (x, d, y)$.

With such definition and the already provided recursive definition of source and target maps, the cubical ω-set $\mathcal{M}(\mathcal{Q})$ becomes a self-dual reflective cubical ω-magma.

We only need to check the universal factorization property of the morphism $\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q})$.

Given a morphism $\mathcal{Q} \xrightarrow{\phi} \mathcal{M}$ into the underlying cubical ω-set of a self-dual reflective cubical ω-magma $\mathcal{M}$, the requirement $\phi = \hat{\phi} \circ \eta$ already implies that the restriction of $\hat{\phi}$ to the cubical ω-subset $\mathcal{Q}$ must coincide with ϕ.

Since $\mathcal{M}(\mathcal{Q}) \xrightarrow{\hat{\phi}} \mathcal{M}$ must be a morphism of self-dual reflective cubical ω-magmas, we necessarily have

$$\hat{\phi}(\iota^{n+1}_{D,d}(x)) = \iota^{n+1}_{D,d}(\hat{\phi}^n_D(x)),$$

hence $(x, d) \mapsto (\phi(x), d)$;

similarly $\hat{\phi}(x^{*^n_{D,d}}) = (\hat{\phi}(x))^{*^n_{D,d}}$ and finally $\hat{\phi}(x \circ^n_{D,d} y) = \hat{\phi}(x) \circ^n_{D,d} \hat{\phi}(y)$ and hence the morphism $\hat{\phi}$ is uniquely determined by our recursive construction, once it has been fixed (as in this case) on $\eta(\mathcal{Q})$. □

Instead of giving a direct recursive proof, the following lemma 3.4 is obtained with the same "quotient by congruences" technique as in [21, section 3.2]. In order to do so, we briefly recall the necessary preliminary material on congruences in the present setting of cubical ω-magmas:

- The category of morphisms of cubical ω-sets/magmas admits finite products (it is actually complete). Given two cubical ω-magmas $\mathcal{M}, \mathcal{N}$, their **product ω-magma** $\mathcal{M} \times \mathcal{N}$ can be constructed via Cartesian products $(\mathcal{M} \times \mathcal{N})^n_D := \mathcal{M}^n_D \times \mathcal{N}^n_D, \forall n \in \mathbb{N}$, $D \subset \mathbb{N}_0$ with cardinality of D equal n, equipped with componentwise defined sources/target maps, reflectors, self-dualities and compositions.

- A **congruence $\mathcal{R}$ in a cubical ω-magma** $\mathcal{M}$ is a cubical ω-magma $\mathcal{R}$ such that $\mathcal{R}^n_D \subset \mathcal{M}^n_D \times \mathcal{M}^n_D$, for all $n \in \mathbb{N}$ and all $D \subset \mathbb{N}_0$ with $|D| = n$, and such that the inclusion $\left(\mathcal{R}^n_D \xrightarrow{\nu^n_D} \mathcal{M}^n_D \times \mathcal{M}^n_D\right)$ is a morphism of cubical ω-magmas, from $\mathcal{R}$ into the product cubical ω-magma $\mathcal{M} \times \mathcal{M}$. [3]

- Given a congruence $\mathcal{R}$ in a cubical ω-magma $\mathcal{M}$, we define the **quotient** ω-**magma** $\mathcal{M}/\mathcal{R}$ and the **quotient morphism** $\left(\mathcal{M}^n_D \xrightarrow{\pi^n_D} (\mathcal{M}/\mathcal{R})^n_D\right)$, for $n \in \mathbb{N}$, $D \subset \mathbb{N}_0$ with $|D| = n$, as follows:

  the quotient sets $(\mathcal{M}/\mathcal{R})^n_D := \mathcal{M}^n_D/\mathcal{R}^n_D$ are a cubical ω-magma with well-defined sources/targets: $[x]_{\mathcal{R}^n_D} \mapsto [s^n_{D,d}(x)]_{\mathcal{R}^n_{D-\{d\}}}$,

  $[x]_{\mathcal{R}^n_D} \mapsto [t^n_{D,d}(x)]_{\mathcal{R}^n_{D-\{d\}}}$,

  $\forall x \in \mathcal{M}^{n+1}_D, \quad d \in D$;
  and one gets a (self-dual reflective) cubical ω-magma with the well-defined operations:

  $[x]_{\mathcal{R}^n_D} \hat{\circ}^n_{D-\{d\}} [y]_{\mathcal{R}^n_D}$ defined by $[x \circ^n_{D-\{d\}} y]_{\mathcal{R}^n_D}$,

  $([x]_{\mathcal{R}^n_D})^{\hat{*}^n_{D,d}} := [x^{*^n_{D-\{d\}}}]_{\mathcal{R}^n_D}$,

  $\hat{\iota}^{n+1}_{D,d}([x]_{\mathcal{R}^n_D}) := [\iota^{n+1}_{D,d}(x)]_{\mathcal{R}^{n+1}_D}$,

  $\forall x, y \in \mathcal{M}^n_D$.

  the maps $\pi^n_D : x \mapsto [x]_{\mathcal{R}^n_D}$, for $x \in \mathcal{M}^n_D$, provide the quotient morphisms between cubical ω-magmas.

- Every morphism $\mathcal{M} \xrightarrow{\phi} \mathcal{C}$ of self-dual reflective cubical ω-magmas induces a **kernel congruence** of self-dual reflective ω-magmas $\mathcal{K}_\phi \subset \mathcal{M}\times\mathcal{M}$ defined by:

  $$\mathcal{K}_\phi := \{(x, y) \in \mathcal{M}\times\mathcal{M} \mid \phi(x) = \phi(y)\}.$$

- Let $\mathcal{M} \xrightarrow{\phi} \mathcal{C}$ be a morphism of self-dual reflective cubical ω-magmas, given another congruence in $\mathcal{M}$ with $\mathcal{E} \subset \mathcal{K}_\phi$, there exists a unique morphism $\mathcal{M}/\mathcal{E} \xrightarrow{\hat{\phi}} \mathcal{C}$ of self-dual reflective cubical ω-magmas such that $\phi = \hat{\phi} \circ \pi_\mathcal{E}$, where $\mathcal{M} \xrightarrow{\pi_\mathcal{E}} \mathcal{M}/\mathcal{E}$ is the quotient morphism. The well-defined morphism $\hat{\phi}$ is uniquely determined by the relation $\hat{\phi}([x]_\mathcal{E}) := \phi(x)$, for all $x \in \mathcal{M}$.

Lemma 3.4. *There exists a free involutive cubical strict ω-category over a cubical ω-set $\mathcal{Q}$.* [4]

Proof. Starting with the cubical ω-set $\mathcal{Q}$, we first utilize lemma 3.3 to produce $\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q})$, a free self-dual reflective cubical ω-magma over $\mathcal{Q}$. For $n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ with $|D| = n$, then we consider the family of relations $\mathcal{X}^n_D \subset \mathcal{M}(\mathcal{Q})^n_D \times \mathcal{M}(\mathcal{Q})^n_D$ consisting of all the pairs of elements corresponding to the "missing cubical categorical axioms equalities" within terms of $\mathcal{M}(\mathcal{Q})$;
in practice $\mathcal{X}^n_D$ is obtained as the union of the following families of subsets of $\mathcal{M}(\mathcal{Q})^n_D \times \mathcal{M}(\mathcal{Q})^n_D$:

$$\bigcup \left\{ \left(x \circ^n_{D,d} (y \circ^n_{D,d} z),\ (x \circ^n_{D,d} y) \circ^n_{D,d} z\right) \right\},$$

such that $d \in D$ and
$\forall (x, y, z) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{Q}^n_D$,

$$\bigcup \{(A,\ x) \mid x \in \mathcal{Q}^n_D\} \cup \{(x,\ B) \mid x \in \mathcal{Q}^n_D\},$$

[3] Equivalently $\mathcal{R}$ is a cubical ω-subset of the product cubical ω-set $\mathcal{M} \times \mathcal{M}$ that is algebraically closed under all the nullary reflectors, unary self-dualities and binary composition operations in the cubical ω-magma $\mathcal{M}$.

[4] For simplicity, we omit in the following the explicit indication of the forgetful functors.

where A is $x \circ^n_{D,d} \iota^n_{D,d}(s^{n-1}_{D,d}(x))$,
B is $\iota^n_{D,d}(t^{n-1}_{D,d}(x)) \circ^n_{D,d} x$ and $d \in D$,

$$\bigcup \left\{ \left(\iota^n_{D,d}(C),\ \iota^n_{D,d}(x) \circ^n_{D,e} \iota^n_{D,d}(y)\right) \right\},$$

where C is $x \circ^{n-1}_{D-\{d\},e} y$, $e \neq d \in D$ and $\forall (x,y) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{d\}}} \mathcal{Q}^n_D$,

$$\bigcup_{e \neq f \in D} \{(A_i,\ A_{ii})\},$$

where A_i is $(x \circ^n_{D,e} y) \circ^n_{D,f} (w \circ^n_{D,e} z)$ and

A_{ii} is $(x \circ^n_{D,f} w) \circ^n_{D,e} (y \circ^n_{D,f} z)$
such that $(x,y),(w,x) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{e\}}} \mathcal{Q}^n_D$,

and $(x,w),(y,z) \in \mathcal{Q}^n_D \times_{\mathcal{Q}^{n-1}_{D-\{f\}}} \mathcal{Q}^n_D$

$$\bigcup_{d \in D} \left\{ \left((x^{*^n_{D,d}})^{*^n_{D,d}},\ x \right) \mid x \in \mathcal{Q}^n_D \right\} \quad (3.1)$$

$$\bigcup \left\{ \left((x^{*^n_{D,e}})^{*^n_{D,f}},\ (x^{*^n_{D,f}})^{*^n_{D,e}} \right) \mid x \in \mathcal{Q}^n_D \right\},$$

where $e \neq f \in D$,

$$\bigcup \left\{ \left((x \circ^n_{D,d} y)^{*^n_{D,d}}, (y^{*^n_{D,d}}) \circ^n_{D,d} (x^{*^n_{D,d}}) \right) \right\},$$

where $(x,y) \in \mathcal{Q}^n_{D-\{d\}} \times \mathcal{Q}^n_{D-\{d\}}$
and $d \in D$,

$$\bigcup \left\{ \left((x \circ^n_{D,d} y)^{*^n_{D,e}},\ (x^{*^n_{D,e}}) \circ^n_{D,d} (y^{*^n_{D,e}}) \right) \right\},$$

where $(x,y) \in \mathcal{Q}^n_{D-\{d\}} \times \mathcal{Q}^n_{D-\{d\}}$
and $e \neq d \in D$,

$$\bigcup_{d \in D} \left\{ \left((\iota^n_{D,d}(x))^{*^n_{D,d}},\ \iota^n_{D,d}(x) \right) \mid x \in \mathcal{Q}^n_D \right\},$$

$$\bigcup_{d \neq e \in D} \left\{ \left((\iota^n_{D,d}(x))^{*^n_{D,e}},\ \iota^n_{D,d}(x^{*^n_{D,e}}) \right) \right\},$$

such that $x \in \mathcal{Q}^n_D$.

The **congruence $\mathcal{R}_\mathcal{X}$ generated by the cubical ω-relation $\mathcal{X}$ in** $\mathcal{M}(\mathcal{Q})$ is the smallest congruence in $\mathcal{M}(\mathcal{Q})$ containing $\mathcal{X}$ and is obtained taking the intersection of the family of all the congruences in $\mathcal{M}(\mathcal{Q})$ containing $\mathcal{X}$.

The quotient self-dual reflective cubical ω-magma $\mathcal{M}(\mathcal{Q})/\mathcal{R}_\mathcal{X}$ by the congruence $\mathcal{R}_\mathcal{X}$ turns out to be a strict involutive cubical ω-category, since $\mathcal{X} \subset \mathcal{R}_\mathcal{X}$.

$\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q}) \xrightarrow{\pi} \mathcal{M}(\mathcal{Q})/\mathcal{R}_\mathcal{X}$, the composition of the quotient morphism of self-dual reflective cubical ω-magmas $\mathcal{M}(\mathcal{Q}) \xrightarrow{\pi} \mathcal{M}(\mathcal{Q})/\mathcal{R}_\mathcal{X}$ with the natural inclusion of cubical ω-sets $\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q})$, is a morphism of cubical ω-sets that satisfies the universal factorization property defining free involutive cubical ω-categories: given $\mathcal{Q} \xrightarrow{\phi} \mathcal{C}$ a morphism of cubical ω-sets into the underlying cubical ω-set of an involutive cubical ω-category $\mathcal{C}$, by the universal factorization property of the free self-dual reflective cubical ω-magma $\mathcal{Q} \xrightarrow{\eta} \mathcal{M}(\mathcal{Q})$, there exists a unique morphism of self-dual reflective cubical ω-magmas $\mathcal{M}(\mathcal{Q}) \xrightarrow{\tilde{\phi}} \mathcal{C}$ such that $\phi = \tilde{\phi} \circ \eta$.
The kernel relation $\mathcal{K}_{\tilde{\phi}} \subset \mathcal{M}(\mathcal{Q}) \times \mathcal{M}(\mathcal{Q})$, induced by the morphism $\tilde{\phi}$, is a congruence of self-dual reflective cubical ω-magma and it necessarily satisfies $\mathcal{X} \subset \mathcal{K}_{\tilde{\phi}}$ and hence $\mathcal{R}_\mathcal{X} \subset \mathcal{K}_{\tilde{\phi}}$. It follows that there exists a unique morphism of involutive cubical ω-categories $\mathcal{M}(\mathcal{Q})/\mathcal{R}_\mathcal{X} \xrightarrow{\hat{\phi}} \mathcal{C}$ such that $\tilde{\phi} = \hat{\phi} \circ \pi$ and so $\phi = \tilde{\phi} \circ \eta = \hat{\phi} \circ \pi \circ \eta$. □

Corollary 3.5. *There is a free ω-category monad obtained by composing the free involutive ω-category functor with the forgetful fuctor into the category of ω-sets.*

The subsequent lemma is obtained recursively, as done for the globular case in [21, proposition 3.3], introducing an intermediate construction of "free cubical contraction n-cells" at each stage n of the construction of free self-dual reflective magmas, in lemma 3.3, and of their quotient free involutive categories over a given cubical ω-set in lemma 3.4.

Lemma 3.6. *There exists a free self-dual cubical Penon-Kachour contraction over a cubical ω-set $\mathcal{Q}$.*

Proof. Starting with a cubical ω-set $\mathcal{Q}$, we will recursively construct a free self-dual cubical Penon-Kachour contraction $(\mathcal{M}^\kappa(\mathcal{Q}) \xrightarrow{\pi} \mathcal{C}^\kappa(\mathcal{Q}), \kappa)$ over $\mathcal{Q}$. Notice that the self-dual relfective cubical ω-magma $\mathcal{M}^\kappa(\mathcal{Q})$ and the involutive cubical ω-category $\mathcal{C}^\kappa(\mathcal{Q})$ differ from the free cubical ω-magma $\mathcal{M}(\mathcal{Q})$ and the free involutive cubical ω-category $\mathcal{C}(\mathcal{Q})$ already introduced in lemmata 3.3 and 3.4, since further "free-contraction n-cells" (and consequently further congruence terms) are introduced at every level $n \in \mathbb{N}$ of the procedure.

For $n = 0$, define $\mathcal{M}^\kappa(\mathcal{Q})^0 := \mathcal{Q}^0$; consider $\mathcal{X}^0 := \varnothing \subset \mathcal{Q}^0 \times \mathcal{Q}^0$ (the empty relation) and its generated equivalence relation $\mathcal{R}^0_\mathcal{X} = \Delta_{\mathcal{Q}^0}$ (the identity equivalence relation in $\mathcal{Q}^0$), obtaining
$\mathcal{C}^\kappa(\mathcal{Q})^0 := \mathcal{M}^\kappa(\mathcal{Q})^0/\mathcal{R}^0_\mathcal{X}$ and the bijective quotient map $\mathcal{M}^\kappa(\mathcal{Q})^0 \xrightarrow{\pi^0} \mathcal{C}^\kappa(\mathcal{Q})^0$. There are no object-valued free-contractions in $\mathcal{M}^\kappa(\mathcal{Q})^0$. The structural inclusion
$\mathcal{Q}^0 \xrightarrow{\eta^0} \mathcal{M}^\kappa(\mathcal{Q})^0$ is just the identity map.

Passing now to the case $n = 1$, in principle, we should modify the construction in lemma 3.3 of the "level-1" free self-dual reflective magma $\mathcal{M}(\mathcal{Q})^1$, introducing as input (for the arbitrary composition of self-dualities and concatenations) not only all the 1-cells in $\mathcal{Q}^1$ and the free identities $\cup_{d\in\mathbb{N}_0} d(\mathcal{Q}^0)$, but also the free 1-cells $\kappa^1(\pi^0)$ coming from the contractions induced by the map π^0.

Since π^0 is bijective, we have $\mathcal{M}^\kappa(\mathcal{Q})(\pi)^0 := \{(x,y) \mid \pi^0(x) = \pi^0(y)\} = \Delta_{\mathcal{Q}^0}$ for all $(x,y) \in \mathcal{M}^\kappa(\mathcal{Q})^0 \times \mathcal{M}^\kappa(\mathcal{Q})^0$ and hence, from the last axiom in the definition of cubical Penon-Kachour contraction $\kappa^1_{\varnothing,d} : \mathcal{M}^\kappa(\mathcal{Q})^0 \to \mathcal{M}^\kappa(\mathcal{Q})^1$, we obtain $\kappa^1_{\varnothing,d}(x,y) = \iota^1_{\varnothing,d}(x) = \iota^1_{\varnothing,d}(y)$, for all $(x,y) \in \mathcal{M}^\kappa(\mathcal{Q})(\pi)^0$ and for all $d \in \mathbb{N}_0$. Hence, in the case $n = 1$ the free-contraction cells are coinciding with the already defined free level-1 identities in $\mathcal{M}(\mathcal{Q})^1$. Hence we simply define $\mathcal{M}^\kappa(\mathcal{Q})^1 := \mathcal{M}(\mathcal{Q})^1$ and, taking $\mathcal{R}^1_\mathcal{X}$ as the equivalence relation in $\mathcal{M}(\mathcal{Q})^1$ generated by all the "axioms" $\mathcal{X}^1$ listed in the equations Eq. (3.1), we define $\mathcal{C}^\kappa(\mathcal{Q})^1 := \mathcal{C}(\mathcal{Q})^1 := \mathcal{M}(\mathcal{Q})^1/\mathcal{R}^1_\mathcal{X}$ with $\mathcal{M}^\kappa(\mathcal{Q})^1 \xrightarrow{\pi^1} \mathcal{C}^\kappa(\mathcal{Q})^1$ the quotient map and contraction $\kappa^1 : \mathcal{M}^\kappa(\mathcal{Q})(\pi)^0 \to \mathcal{M}^\kappa(\mathcal{Q})^1$ as $\kappa^1_{\varnothing,d}(x,y) := \iota^1_{\varnothing,d}(x) = \iota^1_{\varnothing,d}(y)$, for all $d \in \mathbb{N}_0$. Finally we also define the structural free-inclusion
$\mathcal{Q}^1 \xrightarrow{\eta^1} \mathcal{M}^\kappa(\mathcal{Q})^1 = \mathcal{M}(\mathcal{Q})^1$ as in lemma 3.3.

Suppose now, by recursion, that we already constructed, for $n \in \mathbb{N}$, a morphism of self-dual reflective cubical n-magmas $\mathcal{M}^\kappa(\mathcal{Q})^n \xrightarrow{\pi^n} \mathcal{C}^\kappa(\mathcal{Q})^n$ onto the involutive cubical n-category $\mathcal{C}^\kappa(\mathcal{Q})^n$, with cubical Penon-Kachour contraction $\mathcal{M}^\kappa(\mathcal{Q})^{n-1}(\pi^n) \xrightarrow{\kappa^n} \mathcal{M}^\kappa(\mathcal{Q})^n$ and with structural morphism of cubical n-sets $\mathcal{Q}^n \xrightarrow{\eta^n} \mathcal{M}^\kappa(\mathcal{Q})^n$.

The projection π^n determines the domain set $\mathcal{M}^\kappa(\mathcal{Q})(\pi)^n$ defined by $\{(x,y) \mid \pi^n(x) = \pi^n(y)\}$ for all (x,y) in $\mathcal{M}^\kappa(\mathcal{Q})^n \times \mathcal{M}^\kappa(\mathcal{Q})^n$of the free-contraction κ^{n+1}. We consider, as in lemma 3.3, the $(n+1)$-cells $\mathcal{Q}^{n+1}_D \cup \left(\bigcup_{d\in D} d(\mathcal{Q}^n_{D-\{d\}})\right)$ (containing already the "freely generated"

$(n+1)$-identities) and we further add the "freely-generated" $(n+1)$-contractions $\kappa_{D,d}(\mathcal{Q}^n) := [x,d,y]^{n+1}_D$ such that $\pi^n_{D-\{d\}}(x) = \pi^n_{D-\{d\}}(y)$, where (x,y) is in $\mathcal{M}^\kappa(\mathcal{Q})^n_{D-\{d\}} \times \mathcal{M}^\kappa(\mathcal{Q})^n_{D-\{d\}}$, $x \neq y$, $D \subset \mathbb{N}_0$ with $|D| = n+1$ and $d \in D$.
In this way, we introduce $\mathcal{M}^\kappa(\mathcal{Q})^{n+1}_D[0]^0$ defined by $\mathcal{Q}^{n+1}_D \cup \left(\bigcup_{d\in D} d(\mathcal{Q}^n_{D-\{d\}})\right) \cup \left(\bigcup_{d\in D} \kappa_{D,d}(\mathcal{Q}^n)\right)$, extending the definition of sources and targets to the extra free-contractions as required by the axioms of Penon-Kachour contraction:

$$s^n_{D,d}([x,d,y]^{n+1}_D) := x,$$
$$t^n_{D,d}([x,d,y]^{n+1}_D) := y$$

and, $\forall e \in D$ with $e \neq d$, we proceed to define $s^n_{D,e}([x,d,y]^{n+1}_D)$ by $\kappa^n_{D-\{e\},d}(s^{n-1}_{D-\{d\},e}(x), s^{n-1}_{D-\{d\},e}(y))$, and to define $t^n_{D,e}([x,d,y]^{n+1}_D)$ by $\kappa^n_{D-\{e\},d}(t^{n-1}_{D-\{d\},e}(x), t^{n-1}_{D-\{d\},e}(y))$.
The Penon-Kachour contraction is defined as $\kappa^{n+1}_{D,d}(x,y) := [x,d,y]^{n+1_D}$, for all $(x,y) \in \mathcal{M}^\kappa(\mathcal{Q})(\pi)^n$ with $x \neq y$ and by $\kappa^{n+1}_{D,d}(x,y) := \iota^{n+1}_{D,d}(x) = \iota^{n+1}_{D,d}(y)$, whenever $x = y$.

The iterative construction of the sets $\mathcal{M}^\kappa(\mathcal{Q})^{n+1}[k]^j$ and $\mathcal{M}^\kappa(\mathcal{Q})^{n+1}$, and the nullary, unary and binary operations of cubical $(n+1)$-magma, proceeds at this point exactly as in lemma 3.3; similarly, the new binary relation,
$\mathcal{X}^{n+1} \subset \mathcal{M}^\kappa(\mathcal{Q})^{n+1} \times \mathcal{M}^\kappa(\mathcal{Q})^{n+1}$ is obtained using the same type of pairs, as in equation Eq. (3.1), but with terms from the bigger set $\mathcal{M}^\kappa(\mathcal{Q})^{n+1}$; furthermore we set $\mathcal{C}^\kappa(\mathcal{Q})^{n+1} := \mathcal{M}^\kappa(\mathcal{Q})^{n+1}/\mathcal{R}^{n+1}_{\mathcal{X}}$, where $\mathcal{R}^{n+1}_{\mathcal{X}}$ is the congruence relation generated by $\mathcal{X}^{n+1}$ in the cubical $(n+1)$-magma $\mathcal{M}^\kappa(\mathcal{Q})^{n+1}$ and with

$$\mathcal{M}^\kappa(\mathcal{Q})^{n+1} \xrightarrow{\pi^{n+1}} \mathcal{C}^\kappa(\mathcal{Q})^{n+1}$$

we denote the quotient map into the cubical involutive $(n+1)$-category $\mathcal{C}^\kappa(\mathcal{Q})^{n+1}$.

Now that the recursive construction of the cubical Penon-Kachour contraction $(\mathcal{M}^\kappa(\mathcal{Q}) \xrightarrow{\pi} \mathcal{C}^\kappa(\mathcal{Q}), \kappa)$ has been completed, we only need to show that it satisfies the universal factorization property.

For any morphism $\mathcal{Q} \xrightarrow{\phi} \hat{\mathcal{M}}$ of cubical ω-magmas into the cubical ω-magma $\hat{\mathcal{M}}$ of another cubical Penon-Kachour contraction $(\hat{\mathcal{M}} \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}, \hat{\kappa})$, we need to show the existence of a unique morphism of Penon-Kachour contractions

$$(\mathcal{M}^\kappa(\mathcal{Q}) \xrightarrow{\pi} \mathcal{C}^\kappa(\mathcal{Q}), \kappa) \xrightarrow{(\hat{\phi},\hat{\Phi})} (\hat{\mathcal{M}} \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}, \hat{\kappa})$$

such that $\hat{\Phi} \circ \kappa = \hat{\kappa} \circ (\phi, \phi)$.

Since $\hat{\Phi}$ is already fixed as $\Phi(\eta(x)) := \phi(x)$ on $\eta(\mathcal{Q}) \subset \mathcal{M}^\kappa(\mathcal{Q})$, and since $\hat{\Phi}$ must be a morphism of cubical self-dual reflective ω-magmas compatible with the contractions $\hat{\Phi}([x,d,y]^{n+1}_D) = \hat{\kappa}^{n+1}_{D,d}(\phi^n(x), \phi^n(y))$; we see that $\hat{\Phi}^{n+1}$ is uniquely determined inductively by $\hat{\Phi}^n$ and ϕ^{n+1}, for all $n \in \mathbb{N}$.

The existence of the unique morphism $\mathcal{C}^\kappa(\mathcal{Q}) \xrightarrow{\hat{\phi}} \hat{\mathcal{C}}$ of involutive cubical ω-categories such that $\hat{\pi} \circ \hat{\Phi} = \hat{\phi} \circ \pi$ follows immediately from the fact that the kernel relation of $\hat{\pi} \circ \hat{\Phi}$ is a congruence of cubical ω-magma in $\mathcal{M}^\kappa(\mathcal{Q})$ containing the set $\mathcal{X}$ and hence its generated congruence $\mathcal{R}_{\mathcal{X}}$, so that there exists a unique well-defined involutive functor $\mathcal{C}^\kappa(\mathcal{Q}) \xrightarrow{\hat{\phi}} \hat{\mathcal{C}}$ given by $\hat{\phi}([x]_{\mathcal{R}_{\mathcal{X}}}) := \hat{\pi}(\Phi(x))$, for all $x \in \mathcal{M}^\kappa(\mathcal{Q})$. ☐

Theorem 3.7. *There is an adjunction* $\mathscr{Q} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{K}$, $F \dashv U$ *between the category of morphisms of cubical* ω*-sets and the category of morphisms of contractions of cubical reflective (self-dual)* ω*-magmas, where*

U is the forgetful functor associating to every contraction $(\mathcal{M} \xrightarrow{\pi} \mathcal{C}, \kappa)$ the underlying cubical ω-set of $\mathcal{M}$ and F associates to every cubical ω-set $\mathcal{Q}$ the free contraction as constructed above in lemma 3.6.

Proof. The existence of a left adjoint functor F and an adjunction $F \dashv U$ is a standard consequence of the already proved universal factorization property for the free Penon-Kachour contraction over cubical ω-sets (see for example [25, section 2.3 and theorem 2.3.6]). □

As a consequence of the existence of any adjunction $F \dashv U$, with unit η and counit ϵ, we have an associated monad $(U \circ F, \eta, F \circ \epsilon \circ U)$, where the unit η of the adjunction takes the role of the monadic unit for the monad endofunctor $U \circ F$ and the monadic multiplication $F \circ \epsilon \circ U$ is obtained from the co-unit ϵ of the adjunction (see for example [26, section 5.1 and lemma 5.1.3]).

After all this preliminary work, we finally arrive at our definition of involutive weak cubical ω-category.

Definition 3.8. *An **involutive weak cubical ω-category** is an algebra for the monad $U \circ F$ associated to the adjunction $F \dashv U$.*

3.1 Examples

Every weak cubical ω-groupoid as already studied in [12] becomes an example of weak involutive ω-category, simply considering as involutions of n-arrows the "directional inverses" of the cubical n-arrows.

As a notable special example of weak cubical ω-groupoid, we can consider the weak ω-groupoid of homotopies (without fixed extrema) of a topological space.

Every strict involutive cubical ω-category is of course an example of weak involutive cubical ω-category.

Also in this trivial strict case, the specific definition of cubical ω-sets that we have adopted in the present paper is sufficiently general to allow the usage of different classes $\mathcal{Q}^n_D$, depending on the choice of the "n-direction" D: for example a countable family of involutive 1-categories $(\mathcal{C}_n, s_n, t_n, \circ_n, \iota_n, *_n)$, $n \in \mathbb{N}_0$, produces a *product strict involutive cubical ω-category* $\mathcal{D} := \Pi_{n\in\mathbb{N}_0}\mathcal{C}_n$ specified as follows:

- for $n \in \mathbb{N}$ and $D \subset \mathbb{N}_0$ with $|D| = n$, we define:

  $$\mathcal{D}^n_D := \{(x_j)_{j\in\mathbb{N}_0}\} \quad \text{where}$$

  $\forall j \in D$ such that $x_j \in \mathcal{C}^1_j$, $\forall j \notin D$ such that $x_j \in \mathcal{C}^0_j$,

- for all $n \in \mathbb{N}_0$, for all $D \subset \mathbb{N}_0$ with $|D| = n$ and $d \in D$, sources and targets are defined by:

  $$s^{n-1}_{D,d} : (x_j)_{j\in\mathbb{N}_0} \mapsto (\hat{x}_j)_{j\in\mathbb{N}_0},$$
  $$\text{where} \quad \hat{x}_j := \begin{cases} x_j & j \neq d, \\ s_d(x_j) & j = d, \end{cases}$$
  $$t^{n-1}_{D,d} : (x_j)_{j\in\mathbb{N}_0} \mapsto (\tilde{x}_j)_{j\in\mathbb{N}_0},$$
  $$\text{where} \quad \tilde{x}_j := \begin{cases} x_j & j \neq d, \\ t_d(x_j) & j = d, \end{cases}$$

- for all $n \in \mathbb{N}$, $D \subset \mathbb{N}_0$ with $|D| = n$ and $d \in D$, identities are given by:

  $$\iota^n_{D,d} : (x_j)_{j\in\mathbb{N}_0} \mapsto (\bar{x}_j)_{j\in\mathbb{N}_0},$$
  $$\text{where} \quad \bar{x}_j := \begin{cases} x_j & j \neq d, \\ \iota_d(x_j) & j = d, \end{cases}$$

- $\forall n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ with $|D| = n$, $d \in D$ composition are defined via:

  $$(x_j)_{j\in\mathbb{N}_0} \circ^n_{D,d} (y_j)_{j\in\mathbb{N}_0} := (z_j)_{j\in\mathbb{N}_0},$$
  $$\text{where} \quad z_j := \begin{cases} x_j = y_j & j \neq d, \\ x_j \circ_d y_j & j = d, \end{cases}$$

- $\forall n \in \mathbb{N}_0$, $D \subset \mathbb{N}_0$ with $|D| = n$, $d \in D$, involutions are provided by:

$$((x_j)_{j\in\mathbb{N}_0})^{*^n_{D,d}} := (w_j)_{j\in\mathbb{N}_0},$$
$$\text{where}\quad w_j := \begin{cases} x_j & j \neq d, \\ x_j^{*_d} & j = d. \end{cases}$$

Whenever we substitute the sequence of strict involutive 1-categories above, with a sequence of weak involutive 1-categories, one immediately obtains some non-trivial examples of weak involutive cubical ω-categories (for example using as morphisms bimodules over different pairs of involutive monoids).

Making full use of the material on involutions of multimodules recently developed in [27], one can immediately obtain weak cubical involutive ω-categories that are analogs of the example of product cubical ω-categories, by considering a family $\mathcal{O}$ of objects consisting of involutive monoids and n-arrows in the direction D as left-D-right-D-multimodules between finite families (with cardinality D) of the monoids in $\mathcal{O}$; the compositions in the direction d will consists in tensor products of multimodules over a single monoid in position d and involutions will consist in duals of multimodules with respect to the involutive monoids in position d.

Interestingly, the previous "product" examples of strict/weak involutive cubical ω-categories suggests an immediate generalization of the formalism of higher categories to the case $(\mathcal{C}_\gamma)_\Gamma$ of indexes labeled by well-ordered sets Γ of arbitrary cardinality (beyond the countable case $\mathbb{N}$); we will not pursue here such directions.

A similar cubical product strict/weak involutive ω-category, can actually be defined for any (countable) family of strict/weak globular involutive n-categories simply taking sequences $(x_j)_{j\in\mathbb{N}_0}$ of globular n-cells.

More interesting examples can be obtained considering "higher multimodules" as in this inductive construction:

- as objects ($n = 0$), we consider involutive monoids (or more generally involutive 1-categories) $\mathcal{A}, \mathcal{B}, \dots,$
- as 1-morphisms, we take all the bimodules ${}_\mathcal{A}\mathcal{M}_\mathcal{B}$ over the already defined objects: compositions will be the usual tensor products of bimodules ${}_\mathcal{A}\mathcal{M}_\mathcal{B} \otimes_\mathcal{B} {}_\mathcal{B}\mathcal{N}_\mathcal{C}$ and involution of 1-morphisms will be the usual notion of contragradient bimodule ${}_\mathcal{B}\hat{\mathcal{M}}_\mathcal{A}$,
- given a 1-morphism bimodule ${}_\mathcal{A}\mathcal{M}_\mathcal{B}$, and its contragradient ${}_\mathcal{B}\hat{\mathcal{M}}_\mathcal{A}$, one constructs their generated free involutive category $\mathcal{A}(\mathcal{M})^{[1]}$ with two objects $\mathcal{A}, \mathcal{B}$,
- one iterates the construction with the above generated involutive categories $\mathcal{A}^{[1]}, \mathcal{B}^{[1]}, \dots,$ in place of the original involutive monoids, obtaining bimodules of level-2 and so on, $\dots,$
- given a square (not necessarily commutative) diagram of the level-1 bimodules, cubical 2-arrows can be defined as level-2 multimodules over the pairs of level-1 bimodules of the diagram,
- proceeding recursively, given an n-dimensional cubical diagram of level-$(n-1)$ multimodules, one can introduce n-arrows as level-n multimodules with n-source and n-targets consisting of the level-$(n-1)$ multimodules appearing in the diagram,

- the operations of composition are iterated as tensor products of level-n multimodules over the involutive categories generated by level-$(n-1)$ multimodules and involutions are provided by the controgradient construction.

4. Outlook

The present paper is only a starting point in the study of involutions suitable for the definition of operator algebraic structures in the weak infinite vertically categorified (cubical) case (see the introduction of [14] for motivations).

It might be of interest to try to formulate a similar definition of weak involutive cubical ω-category using M.Batanin and T.Leinster's operadic techniques, as already done for the globular case in [23].

A more ambitious future goal will be the exploration of equivalences between weak globular involutive ω-categories in [21] and the present weak cubical involutive ω-categories, extending to the involutive weak category case famous results in [28]. In this direction, one must first generalize to the strict ω-category environment the (already quite involved) results obtained for strict involutive double categories and strict involutive globular 2-categories in [24].

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