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Maximal and Minimal Congruences on the Semigroup $T_E(X)$

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ABSTRACT

In semigroup theory, transformations play a crucial role. This paper explores a specific type of transformation semigroup, denoted by $T_E(X)$. Here, X is a non-empty set, and $T_E(X)$ consists of all transformations on X that preserve the equivalence classes established by an equivalence relation E on X. We delve into the internal structure of $T_E(X)$ by exploring how to partition its elements into the coarsest and finest possible partitions while preserving the validity of the transformation operation within each partition. These partitions correspond to maximal and minimal congruences on $T_E(X)$, respectively. We then address the existence of a specific type of congruence on $T_E(X)$ where each equivalence class forms a subsemigroup itself.

Keywords: Maximal congruences; Minimal congruences; Transformation semigroups

1. Introduction and Preliminaries

A non-identity congruence σ on a semigroup S is called maximal if, whenever $\sigma \subseteq \rho \subseteq S \times S$ for some congruence ρ on S, then $\rho = \sigma$ or $\rho = S \times S$. A non-identity congruence σ on S is called minimal if, whenever $\rho \subseteq \sigma$ for some congruence ρ on S, then ρ is the identity congruence or $\rho = \sigma$.

Let X be an arbitrary nonempty set and let $T(X)$ be the full transformation semigroup consisting of all mappings from X into X under composition. It is well-known that $T(X)$ is a regular semigroup, as shown in Reference [1]. Various subsemigroups of $T(X)$ have been studied in different years. In 2007, Sanwong and Sullivan [2] investigated all the maximal congruences on the semigroup $T(X)$. All the maximal congruences on the semigroup

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of all non-negative integers under multiplication were also described. In 2009, Sanwong, Sullivan and Singha [3] characterized all the maximal congruences on the semigroup $\mathbb{Z}_n$ under multiplication and they determined the maximal congruence for $T(X,Y)$, the semigroup of all elements of $T(X)$ whose range is contained in Y for arbitrary fixed subset Y of X. They shown that the maximal congruence on $T(X,Y)$ is unique. Moreover, they found all the minimal congruences on $T(X,Y)$.

Let E be an equivalence relation on a set X. Consider the following subset of $T(X)$:

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x,y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

It is obvious that if E is a non-trivial equivalence relation, then $T_E(X)$ is a proper subsemigroup of $T(X)$ and if E is the identity or universal relation, then $T_E(X)$ and $T(X)$ are identical. In 1994 and 1996, Pei discussed α-congruences and some regular subsemigroup inducing a certain lattice on $T_E(X)$ (see [4, 5]). In 2005, Pei [6] investigated regularity of elements and Green's relations on the semigroup $T_E(X)$. In [7], Pei determined the rank of the homeomorphism group and considered the rank of $T_E(X)$ when X is a finite set and each class of the equivalence E has the same cardinality. Furthermore, he also studied the rank of $\Gamma(X)$, the semigroup of all closed function α on a topological space X for which E classes form a basis. In 2008, Sun, Pei and Cheng [8] characterized the natural partial order on the semigroup $T_E(X)$. The compatibility of multiplication and all compatible elements were investigated. Moreover, they found maximal, minimal and covering elements with respect to the order. In 2011, Pei and Zhou [9] considered the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroup $T_E(X)$ and they gave the condition for the equivalence relation E under which $T_E(X)$ becomes abundant. In 2019, Sun [10] investigated the left and right compatibility with respect to the natural partial order on $T_E(X)$.

In this paper, we focus on some special congruences on $T_E(X)$. We start with a congruence γ which is defined in section 2. We prove that a semigroup $T(X,Y)$ is exactly γ-class whenever $Y \in X/E$. In section 3, we define a maximal congruence on $T_E(X)$ and show that a maximal congruence on $T_E(X)$ is not unique. In the last section, we also determine all the minimal congruences on $T_E(X)$.

In the rest of this section, we recall some notions that will be used in this paper. Let S be a semigroup and let ρ be an equivalence relation on S. The relation ρ is left compatible if $(a,b) \in \rho$ implies $(ca, cb) \in \rho$ for all $c \in S$. Similarly, ρ is right compatible if $(a,b) \in \rho$ implies $(ac, bc) \in \rho$ for all $c \in S$. If an equivalence ρ is both left and right compatible, then ρ is called a congruence on S. The quotient semigroup S/ρ is the semigroup whose elements are the congruence classes of ρ, and whose operation $*$ is defined by

$$(a)_\rho * (b)_\rho = (ab)_\rho,$$

for all $a, b \in S$ and $(a)_\rho$ is a ρ-class containing a.

A subset I of a semigroup S is called an ideal of S if both SI and IS are subsets of I, where $SI = \{sx : s \in S \text{ and } x \in I\}$ and similarly for IS. An ideal P of S is called prime if $P \neq S$ and for all $a, b \in S$, $ab \in P$ implies $a \in P$ or $b \in P$. Let I be an ideal of a semigroup S. We define the relation I^* on S by

$$xI^*y \text{ if either } x = y \text{ or both } x \text{ and } y \text{ are in } I.$$

Then I^* is an equivalence relation on S. The equivalence classes under I^* are the singleton sets $\{x\}$ with x not in I and the set I. Since I is an ideal of S, the relation I^* is a congruence on S. The quotient semigroup S/I^* is called the Rees factor semigroup of S modulo I.

Lemma 1.1. *[6] Let $\alpha \in T_E(X)$. Then for each $B \in X/E$, there exists $B' \in X/E$ such that $B\alpha \subseteq B'$. Consequently, for any $A \in X/E$, $A\alpha^{-1}$ is either $\emptyset$ or a union of some E-classes.*

2. Congruence γ

In this section, we will consider the relation γ on $T_E(X)$ which is defined by:

$$\alpha\ \gamma\ \beta \text{ if } X\alpha, X\beta \subseteq A \text{ for some } A \in X/E \text{ or } \alpha = \beta.$$

We will obtain that γ is a congruence on $T_E(X)$. And each γ-class is either a singleton or a subsemigroup of $T_E(X)$.

Theorem 2.1. *The relation γ is a congruence on $T_E(X)$.*

Proof. Clearly, γ is an equivalence relation on $T_E(X)$. Let $(\alpha, \beta) \in \gamma$ be such that $\alpha \neq \beta$ and let $\delta \in T_E(X)$. Then $X\alpha, X\beta \subseteq A$ for some $A \in X/E$. Since $\delta \in T_E(X)$ and Lemma 1.1, there exists $A' \in T_E(X)$ such that $A\delta \subseteq A'$. Hence $X\alpha\delta \subseteq A\delta \subseteq A'$ and $X\beta\delta \subseteq A\delta \subseteq A'$. Thus $(\alpha\delta, \beta\delta) \in \gamma$ which means that γ is right compatible. Note that $X\delta \subseteq X$. Therefore $X\delta\alpha \subseteq X\alpha \subseteq A$. Similarly, $X\delta\beta \subseteq A$. Hence $(\delta\alpha, \delta\beta) \in \gamma$ and so γ is left compatible. Therefore, we conclude that γ is a congruence on $T_E(X)$. □

Theorem 2.2. *Let $Q_1 = \{\alpha \in T_E(X) : X\alpha \subseteq A$ for some $A \in X/E\}$. Then Q_1 is an ideal of $T_E(X)$ and γ is a subcongruence of Q_1^*.*

Proof. Assume that $\alpha \in T_E(X)$ and $\beta \in Q_1$. Then $X\beta \subseteq A$ for some $A \in X/E$. By Lemma 1.1, there exists $A' \in X/E$ such that $A\alpha \subseteq A'$. From $X\beta \subseteq A$, we have $X\beta\alpha \subseteq A\alpha \subseteq A'$. Thus $\beta\alpha \in Q_1$. Since $X\alpha \subseteq X$, we obtain $X\alpha\beta \subseteq X\beta \subseteq A$. So $\alpha\beta \in Q_1$. Hence Q_1 is an ideal of $T_E(X)$. It is clear that $\gamma \subseteq Q_1^*$. □

Theorem 2.3. *Let $(\alpha)_\gamma$ be a γ-congruence class containing α where $\alpha \in T_E(X)$. Then $|(\alpha)_\gamma| = 1$ or $(\alpha)_\gamma = T(X, A)$ for some $A \in X/E$.*

Proof. Suppose that $|(\alpha)_\gamma| > 1$. Let $\beta \in T_E(X)$ be such that $\alpha\ \gamma\ \beta$. Then $X\alpha, X\beta \subseteq A$ for some $A \in X/E$. This means that $T(X, A) = (\alpha)_\gamma$. □

From the definition of Q_1, we observe that Q_1 is the union of γ-classes, each of which is a subsemigroup of $T_E(X)$. Recall that a semigroup S is called right zero if for any x, y in S, $xy = y$. With this in mind, we obtain the following result.

Theorem 2.4. *Q_1/γ is a right zero subsemigroup of $T_E(X)/\gamma$.*

Proof. Let $\alpha, \beta \in Q_1$. Then $X\beta \subseteq A$ for some $A \in X/E$. We obtain that $(\alpha\beta, \beta) \in \gamma$ since $X\alpha\beta \subseteq A$. Hence $(\alpha)_\gamma(\beta)_\gamma = (\beta)_\gamma$, as required. □

3. Maximal congruences on $T_E(X)$

In this section, we let X/E be a finite set. We define a relation σ on $T_E(X)$ and show that σ is a maximal congruence but not only one maximal congruence on $T_E(X)$.

Theorem 3.1. *Let $Q_2 = \{\alpha \in T_E(X) : X\alpha \cap A = \emptyset$ for some $A \in X/E\}$. Then Q_2 is a prime ideal of $T_E(X)$.*

Proof. Assume that $\alpha \in T_E(X)$ and $\beta \in Q_2$. Let $X/E = \{A_1, A_2, \dots, A_n\}$ for some a natural number n. Without loss of generality, we may assume that $X\beta \cap A_n = \emptyset$. Since $X\alpha \subseteq X$, we have $X\alpha\beta \subseteq X\beta$. Thus $X\alpha\beta \cap A_n \subseteq X\beta \cap A_n = \emptyset$. So $\alpha\beta \in Q_2$. On the other hand, if $\alpha \in Q_2$, then $\beta\alpha \in Q_2$. Suppose that $\alpha \notin Q_2$. Then $X\alpha \cap A_i \neq \emptyset$ for all $i \in \{1, 2, \dots, n\}$. From $\alpha \in T_E(X)$ and $A_n \in X/E$, we have $A_n\alpha \subseteq A_j$ for some $j \in \{1, 2, \dots, n\}$. Since X/E is finite and Lemma 1.1, we get that $A_i\alpha \cap A_j = \emptyset$ for each $i \in \{1, 2, \dots, n-1\}$. Therefore $X\beta\alpha \subseteq (X \setminus A_n)\alpha$ which implies that $X\beta\alpha \cap A_j \subseteq (X \setminus A_n)\alpha \cap A_j = \emptyset$. This means that $\beta\alpha \in Q_2$. Thus Q_2 is an ideal of $T_E(X)$. Next, we will show that Q_2 is prime. Let $\alpha, \beta \in T_E(X)$ be such that $\alpha, \beta \notin Q_2$. Then $X\alpha \cap A_i \neq \emptyset$ and $X\beta \cap A_i \neq \emptyset$ for all $i \in \{1, 2, \dots, n\}$. Let $j \in \{1, 2, \dots, n\}$. There exists $m \in \{1, 2, \dots, n\}$ such that $A_m\alpha \subseteq A_j$. Since $X\beta \cap A_m \neq \emptyset$, there is $k \in \{1, 2, \dots, n\}$ such that $A_k\beta \subseteq A_m$. This implies that $A_k\beta\alpha \subseteq A_m\alpha \subseteq A_j$. Hence $\emptyset \neq A_k\beta\alpha \subseteq X\beta\alpha \cap A_j$. Therefore $\beta\alpha \notin Q_2$ and so Q_2 is prime. □

Theorem 3.2. *A relation $\sigma = (Q_2 \times Q_2) \cup (T_E(X) \setminus Q_2 \times T_E(X) \setminus Q_2)$ is a maximal congruence on $T_E(X)$.*

Proof. Clearly, σ is an equivalence relation on $T_E(X)$. We will show that σ is compatible. Let $(\alpha, \beta) \in \sigma$ and $\gamma \in T_E(X)$.

Case 1 : $(\alpha, \beta) \in Q_2 \times Q_2$. Since Q_2 is an ideal of $T_E(X)$, we obtain that $(\alpha\gamma, \beta\gamma) \in Q_2 \times Q_2$.

Case 2 : $(\alpha, \beta) \in T_E(X) \setminus Q_2 \times T_E(X) \setminus Q_2$. If $\gamma \in Q_2$, then $(\alpha\gamma, \beta\gamma) \in Q_2 \times Q_2$. If $\gamma \notin Q_2$, then $(\alpha\gamma, \beta\gamma) \in T_E(X) \setminus Q_2 \times T_E(X) \setminus Q_2$ since Q_2 is prime. This means that σ is right compatible. Similarly, σ is left compatible. Hence σ is a congruence on $T_E(X)$. We will show that σ is a maximal congruence on $T_E(X)$. Assume that ρ is a congruence on $T_E(X)$ with $\sigma \subsetneq \rho$. Then there exists $(\alpha, \beta) \in \rho \setminus \sigma$. Without loss of generality, we may assume that $\alpha \in Q_2$ and $\beta \notin Q_2$. Note that $\beta, id_X \notin Q_2$. Thus $(\beta, id_X) \in \sigma \subseteq \rho$. We let δ be arbitrary constant mapping in $T_E(X)$. Then $\delta \in Q_2$. Hence $(\delta, \alpha) \in \sigma \subseteq \rho$ and so $(\delta, id_X) \in \rho$ from the transitivity of ρ. Let $\eta, \theta \in T_E(X)$. Consider $(\eta\delta, \eta id_X) = (\delta, \eta) \in \rho$ since ρ is left compatible. Similarly, $(\theta\delta, \theta id_X) = (\delta, \theta) \in \rho$. From ρ is an equivalence relation, we get that $(\eta, \theta) \in \rho$. It follows that ρ is the universal congruence on $T_E(X)$. Therefore σ is maximal. □

Theorem 3.3. *Let ρ be a maximal congruence on $T_E(X)$. Then all constants in $T_E(X)$ are ρ-equivalent.*

Proof. If $\rho = \sigma$, then the proof is done. Assume that $\rho \neq \sigma$. Let $(\alpha, \beta) \in \rho \setminus \sigma$. We may assume that $\alpha \in Q_2$ and $\beta \notin Q_2$. Then there exists $A \in X/E$ such that $X\alpha \cap A = \emptyset$. Let $\theta, \lambda \in T_E(X)$ be such that $X\theta = \{a\}$ and $X\lambda = \{b\}$. Define $\delta : X \to X$ by

$$x\delta = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise,} \end{cases}$$

for all $x \in X$. It is easy to verify that $\delta \in T_E(X)$. From $\beta \notin Q_2$, we choose $z \in A\beta^{-1}$ and $y \in X \setminus A\beta^{-1}$. Let $C \in X/E$ be such that $a \in C$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} z & \text{if } x \in C, \\ y & \text{otherwise,} \end{cases}$$

for all $x \in X$. Clearly, $\gamma \in T_E(X)$ and then

$$x\gamma\beta\delta = \begin{cases} a & \text{if } x \in C, \\ b & \text{otherwise,} \end{cases}$$

and $\gamma\alpha\delta = \lambda$. From the compatibility of ρ, we have $(\lambda, \gamma\beta\delta) \in \rho$. From ρ is left compatible, we obtain that $(\lambda, \theta) \in \rho$. □

Theorem 3.4. *Let* $Q_3 = \{\alpha \in T_E(X) : |A\alpha| = 1$ *for all* $A \in X/E\}$. *Then* Q_3 *is an ideal of* $T_E(X)$.

Proof. Assume that $\alpha \in Q_3$ and $\beta \in T_E(X)$. Then $|A\alpha| = 1$ for all $A \in X/E$. Let $A \in X/E$ and then $A\alpha = \{a\}$ for some $a \in X$. Thus $A\alpha\beta = \{a\beta\}$ and hence $\alpha\beta \in Q_3$. Note that $\beta \in T_E(X)$, then there exists $A' \in X/E$ such that $A\beta \subseteq A'$. Therefore $1 \leq |A\beta\alpha| \leq |A'\alpha| = 1$ since $\alpha \in Q_3$. Hence Q_3 is an ideal of $T_E(X)$. □

Theorem 3.5. *Let* ρ *be a maximal congruence on* $T_E(X)$. *If* $\rho \neq \sigma$, *then* $Q_3^* \subseteq \rho$.

Proof. Let $(\alpha, \beta) \in \rho \setminus \sigma$. Assume that $\alpha \in Q_2$ and $\beta \notin Q_2$. Let $A \in X/E$ such that $X\alpha \cap A = \emptyset$. For each $B \in X/E$, there exists a unique $B' \in X/E$ such that $B'\beta \subseteq B$. For each $B \in X/E$, we fix $b \in B$. Define $\delta : X \to X$ by

$$x\delta = b \text{ if } x \in B \text{ and } B \in X/E,$$

for all $x \in X$. Clearly, $\delta \in T_E(X)$. Note that $B\delta \cap B \neq \emptyset$ which implies that $\delta \notin Q_2$. Since Q_2 is prime, we obtain that $\beta\delta \notin Q_2$ and $\alpha\delta \in Q_2$. Now, we assume that $X\alpha\delta = \{b_1, b_2, \ldots, b_k\}$ where $k < n = |X/E|$ and choose $c_i \in b_i(\alpha\delta)^{-1}$ for each $i \in \{1, 2, \ldots, k\}$. Let $\mathcal{B} = \{B \in X/E : c_i \notin B$ for all $i = 1, 2, \ldots, k\}$ and $\mathcal{B}' = \{B \in X/E : b_i \notin B$ for all $i = 1, 2, \ldots, k\}$. From $(c_i, c_j) \notin E$ if $i \neq j$, we then have $|\mathcal{B}| = |\mathcal{B}'| = n - k$. Assume that $\mathcal{B} = \{B_1, B_2, \ldots, B_{n-k}\}$ and $\mathcal{B}' = \{B'_1, B'_2, \ldots, B'_{n-k}\}$. Choose $d_j \in B_j$ for all $j = 1, 2, \ldots, n-k$. Now, we rewrite X/E as the set $\{A_1, ..., A_n\}$ where $b_i \in A_i$ for $i = 1, ..., k$ and $A_i = B'_{i-k}$ for $i = k+1, ..., n$. Define $\theta : X \to X$ by

$$x\theta = \begin{cases} c_{i+1} & \text{if } x \in A_i; i < k, \\ d_1 & \text{if } x \in A_k, \\ d_{i-k+1} & \text{if } x \in A_i; k < i < n, \\ c_1 & \text{otherwise}, \end{cases}$$

for all $x \in X$. Clearly, $\theta \in T_E(X)$ and $\theta \notin Q_2$. Note that $X\alpha\delta\theta\alpha\delta \subsetneq X\alpha\delta$ and $\beta\delta\theta\beta\delta \notin Q_2$. From the right compatibility of ρ that $(\alpha\delta\theta, \beta\delta\theta), (\alpha\delta, \beta\delta) \in \rho$. Therefore $(\alpha\delta\theta\alpha\delta, \beta\delta\theta\beta\delta) \in \rho$ since ρ is a congruence. It follows from the construction of θ that there exists $m \in \mathbb{N}$ such that $(\alpha\delta)(\theta\alpha\delta)^m$ is a constant mapping. This implies that $((\alpha\delta)(\theta\alpha\delta)^m, (\beta\delta)(\theta\beta\delta)^m) \in \rho$. Let $\gamma = (\alpha\delta)(\theta\alpha\delta)^m$ and $\phi = (\beta\delta)(\theta\beta\delta)^m$. Assume that $X\gamma = \{a\}$. Let $X\phi = \{x_1, x_2, \ldots, x_n\}$ and $b \in X\phi \setminus \{a\}$. Clearly, $x_i\phi^{-1} \in X/E$ for each $i \in \{1, 2, \ldots, n\}$. Suppose that $(a, x_1) \in E$. Let $P \subseteq X$ with $a \in P$ and $|P| \leq n$. For arbitrary mapping $\pi : X\phi \to P$ with $x_1\pi = a$. Define $\xi : X \to X$ by $x\xi = x_i\pi$ where $(x, x_i) \in E$ for some $x_i \in X\phi$. Thus $\xi \in T_E(X)$ and $(\gamma, \phi\xi) = (\gamma\xi, \phi\xi) \in \rho$. It follows that $(\gamma, \tau) \in \rho$ for all $\tau \in Q_3$ and $a \in X\tau$. From this fact and right compatibility of ρ, we deduce that each constant map γ' and $\tau' \in Q_3$ such that $X\gamma' = \{b\}$ and $b \in X\tau'$ will satisfy $(\gamma', \tau') \in \rho$. Finally, from Theorem 3.3 and ρ is an equivalence relation, we obtain that $Q_3 \times Q_3 \subseteq \rho$. □

Finally, we give a particular example which show that there exists a maxmimal congruence on $T_E(X)$ which is not equal to σ.

Example 3.6. Let X be a finite set and $B = \{\alpha \in T_E(X) : \alpha$ is a bijection$\}$. Define $\rho = (B \times B) \cup (T_E(X) \setminus B \times T_E(X) \setminus B)$. Since X is finite, we have $T_E(X) \setminus B$ is a prime ideal and hence ρ is a maximal congruence on $T_E(X)$ such that $\rho \neq \sigma$.

4. Minimal Congruence

In this section, we show that there is only one minimal congruence on $T_E(X)$. Now, we let the relation σ on $T_E(X)$ be defined as follows:

$$\alpha\ \sigma\ \beta \text{ if } |X\alpha| = |X\beta| = 1 \text{ and } X\alpha, X\beta \subseteq A \\ \text{for some } A \in X/E \text{ or } \alpha = \beta.$$

We get that σ is a congruence on $T_E(X)$.

Theorem 4.1. *A relation σ is the only minimal congruence on $T_E(X)$.*

Proof. Clearly, σ is an equivalence relation on $T_E(X)$. Let $(\alpha, \beta) \in \sigma$ be such that $\alpha \neq \beta$ and let $\gamma \in T_E(X)$. Then $|X\alpha| = |X\beta| = 1$ and $X\alpha, X\beta \subseteq A$ for some $A \in X/E$. Thus $X\alpha = \{a\}$ and $X\beta = \{b\}$ where $a, b \in A$. This implies that $X\alpha\gamma = \{a\gamma\}$ and $X\beta\gamma = \{b\gamma\}$. Since $\gamma \in T_E(X)$ and by Lemma 1.1, there exists $A' \in T_E(X)$ such that $A\gamma \subseteq A'$. Consider $X\alpha\gamma \subseteq A\gamma \subseteq A'$. Similarly, $X\beta\gamma \subseteq A\gamma \subseteq A'$. Thus $(\alpha\gamma, \beta\gamma) \in \sigma$ and so σ is right compatible. Note that $X\gamma \subseteq X$. Therefore $X\gamma\alpha = \{a\}$ and $X\gamma\beta \subseteq X\beta = \{b\}$. Hence $(\gamma\alpha, \gamma\beta) \in \sigma$ and so σ is left compatible. We conclude that σ is a congruence on $T_E(X)$. We will show that σ is a minimal congruence on $T_E(X)$. Assume that ρ is a non-identity congruence on $T_E(X)$ with $\rho \subseteq \sigma$. Then there exists $(\alpha, \beta) \in \rho$ and $\alpha \neq \beta$. So, there is $x \in X$ such that $x\alpha \neq x\beta$. Since $(\alpha, \beta) \in \rho \subseteq \sigma$, we have $|X\alpha| = |X\beta| = 1$ and $X\alpha, X\beta \subseteq A$ for some $A \in X/E$. Let $X\alpha = \{a\}$ and $X\beta = \{b\}$ where $a \neq b$. Let $(\delta, \gamma) \in \sigma$ with $\delta \neq \gamma$. Then $|X\delta| = |X\gamma| = 1$ and $X\delta, X\gamma \subseteq B$ for some $B \in X/E$. Let $X\delta = \{c\}$ and $X\gamma = \{d\}$ where $c \neq d$. Define $\theta : X \to X$ by

$$x\theta = \begin{cases} c & \text{if } x = a, \\ d & \text{otherwise,} \end{cases}$$

for all $x \in X$. Clearly that $\theta \in T_E(X)$ since $X\theta \subseteq B$. Then $X\alpha\theta = \{c\} = X\delta$ and $X\beta\theta = \{d\} = X\gamma$. From the right compatible of ρ, we obtain $(\delta, \gamma) = (\alpha\theta, \beta\theta) \in \rho$. Therefore $\sigma \subseteq \rho$ and so $\rho = \sigma$. Hence σ is minimal. Next, we will show that σ is the only minimal congruence on $T_E(X)$. Suppose that ρ is a minimal congruence on $T_E(X)$. From ρ is not the identity, there exists $(\alpha, \beta) \in \rho$ such that $\alpha \neq \beta$. Thus $z\alpha \neq z\beta$ for some $z \in X$. We claim that $\sigma \subseteq \rho$. Let $(\delta, \gamma) \in \sigma$ be such that $\delta \neq \gamma$. Then $|X\delta| = |X\gamma| = 1$ and $X\delta, X\gamma \subseteq A$ for some $A \in X/E$. Let $X\delta = \{c\}$ and $X\gamma = \{d\}$ where $c \neq d$. Define $\theta : X \to X$ by $x\theta = z$ for all $x \in X$. Thus $X\theta\alpha = \{z\alpha\}$ and $X\theta\beta = \{z\beta\}$. Define $\phi : X \to X$ by

$$x\phi = \begin{cases} c & \text{if } x = z\alpha, \\ d & \text{otherwise,} \end{cases}$$

for all $x \in X$. Since $X\phi \subseteq A$, we have $\phi \in T_E(X)$. Therefore $X\theta\alpha\phi = \{c\} = X\delta$ and $X\theta\beta\phi = \{d\} = X\gamma$. From the compatibility of ρ, we get $(\delta, \gamma) = (\theta\alpha\phi, \theta\beta\phi) \in \rho$. So $\sigma \subseteq \rho$. By minimality of ρ, we obtain that $\rho = \sigma$. Hence σ is the only minimal congruence on $T_E(X)$. □

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