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Interval-Valued Picture Fuzzy Ideals of Semigroups

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ABSTRACT

In this article, we define an interval-valued picture fuzzy subsemigroup and an interval-valued picture fuzzy left ideal[right ideal, ideal, bi-ideal, interior ideal, quasi-ideal] of a semigroup, as well as investigate some properties of an interval-valued picture fuzzy subsemigroup and various types of an interval-valued picture fuzzy ideal of a semigroup. Furthermore, we will study the relationship between each ideal of a semigroup and its interval-valued picture fuzzification.

Keywords: Interval-valued fuzzy sets; Interval-valued picture fuzzy sets; Interval-valued picture fuzzy ideals

1. Introduction and Preliminaries

For the purpose of completeness, we start by reviewing a few basic concepts. Throughout this paper, let X be a nonempty set, A and B nonempty subsets of X and let S be a semigroup. For nonempty subsets T and Y of S we define a *multiplication* $TY = \{ty \mid t \in T, y \in Y\}$. Let T be a nonempty subset of S. We call T a *subsemigroup* of S if $T^2 \subseteq T$, it is a *left ideal* of S if $ST \subseteq T$, and it is a *right ideal* of S if $TS \subseteq T$. By an *ideal*, we mean that it is both a left ideal and a right ideal of S. Again, we say that T is a *bi-ideal* of S if $TST \subseteq T$ and T must also be a subsemigroup of S. Additionally, if T is a subsemigroup of S and $STS \subseteq T$, then T is an *interior ideal* of S. In addition, we call T a *quasi-ideal* of S if $TS \cap ST \subseteq T$. If for each $s \in S$, there exists an element $t \in S$ such that $s = sts$, then S is said to be *regular*.

Theorem 1.1. [6] *For any right ideal R and left ideal L of S, we have $R \cap L = RL$ if and only if S is regular.*

Let α be a function from X to a close interval $[0, 1]$. Then we call α a *fuzzy sub-*

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set of X. It was launched by Zadeh [18] in 1965.

A decade later, Zadeh [19] presented the basic idea of an interval-valued fuzzy subset. It is a more general version of fuzzy subsets. We now review some concepts of interval numbers.

Let $CI[0,1]$ be the set of all closed subintervals within $[0,1]$, that is,

$$CI[0,1] = \{[x,y] \mid x \leq y \text{ and } x, y \in [0,1]\}.$$

We denote $[0,0]$ and $[1,1]$ by $\mathbf{0}$ and $\mathbf{1}$, respectively.

Let $[x_1, y_1]$ and $[x_2, y_2]$ be elements of $CI[0,1]$.

1. The *refined minimum* of $[x_1, y_1]$ and $[x_2, y_2]$ is defined by
$$\text{rmin}\{[x_1, y_1], [x_2, y_2]\} = [l_*, u_*],$$
where $l_* = \min\{x_1, x_2\}$ and $u_* = \min\{y_1, y_2\}$.

2. The *refined maximum* of $[x_1, y_1]$ and $[x_2, y_2]$, is defined by
$$\text{rmax}\{[x_1, y_1], [x_2, y_2]\} = [l^*, u^*],$$
where $l^* = \max\{x_1, x_2\}$ and $u^* = \max\{y_1, y_2\}$.

3. $[x_1, y_1] \succeq [x_2, y_2]$ iff
$$x_1 \geq x_2 \text{ and } y_1 \geq y_2.$$

4. $[x_1, y_1] \preceq [x_2, y_2]$ iff
$$x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

5. $[x_1, y_1] = [x_2, y_2]$ iff
$$x_1 = x_2 \text{ and } y_1 = y_2.$$

6. $[x_1, y_1] \succ [x_2, y_2]$ iff
$$[x_1, y_1] \succeq [x_2, y_2] \text{ and } [x_1, y_1] \neq [x_2, y_2].$$

7. $[x_1, y_1] \prec [x_2, y_2]$ iff
$$[x_1, y_1] \preceq [x_2, y_2] \text{ and } [x_1, y_1] \neq [x_2, y_2].$$

Let $\{[x_i, y_i] \mid i \in \Lambda\}$ be the collection of close subintervals of $[0,1]$. We define

$$\inf_{i\in\Lambda} [x_i, y_i] = \inf_{i\in\Lambda} x_i,$$
$$\sup_{i\in\Lambda} [x_i, y_i] = \sup_{i\in\Lambda} y_i,$$
$$\operatorname*{rinf}_{i\in\Lambda} [x_i, y_i] = \left[\inf_{i\in\Lambda} x_i, \inf_{i\in\Lambda} y_i\right]$$
$$\text{and } \operatorname*{rsup}_{i\in\Lambda} [x_i, y_i] = \left[\sup_{i\in\Lambda} x_i, \sup_{i\in\Lambda} y_i\right].$$

We call a function from X into $CI[0,1]$ an *interval-valued fuzzy subset (IvFS)* of X.

Let $\overline{\alpha}$ and $\overline{\varrho}$ be IvFSs of X. We define

1. $\overline{\alpha} \subseteq \overline{\varrho}$ iff $\overline{\alpha}(z) \preceq \overline{\varrho}(z)$ for all $z \in X$.

2. $\overline{\alpha} = \overline{\varrho}$ iff $\overline{\alpha} \subseteq \overline{\varrho}$ and $\overline{\varrho} \subseteq \overline{\alpha}$.

3. $(\overline{\alpha} \cup \overline{\varrho})(z) = \text{rmax}\{\overline{\alpha}(z), \overline{\varrho}(z)\}$ for all $z \in X$.

4. $(\overline{\alpha} \cap \overline{\varrho})(z) = \text{rmin}\{\overline{\alpha}(z), \overline{\varrho}(z)\}$ for all $z \in X$.

Proposition 1.2. [9] *Let $\overline{\sigma}$, $\overline{\varrho}$ and $\overline{\nu}$ be IvFSs of X.*

(1) *$\overline{\sigma} \subseteq \overline{\sigma} \cup \overline{\varrho}$ and $\overline{\varrho} \subseteq \overline{\sigma} \cup \overline{\varrho}$.*

(2) *$\overline{\sigma} \cap \overline{\varrho} \subseteq \overline{\sigma}$ and $\overline{\sigma} \cap \overline{\varrho} \subseteq \overline{\varrho}$.*

(3) *If $\overline{\sigma} \subseteq \overline{\varrho}$ and $\overline{\varrho} \subseteq \overline{\nu}$, then $\overline{\sigma} \subseteq \overline{\nu}$.*

(4) *If $\overline{\sigma} \subseteq \overline{\varrho}$, then $\overline{\sigma} \cup \overline{\nu} \subseteq \overline{\varrho} \cup \overline{\nu}$ and $\overline{\nu} \cup \overline{\sigma} \subseteq \overline{\nu} \cup \overline{\varrho}$.*

(5) *If $\overline{\sigma} \subseteq \overline{\varrho}$, then $\overline{\sigma} \cap \overline{\nu} \subseteq \overline{\varrho} \cap \overline{\nu}$ and $\overline{\nu} \cap \overline{\sigma} \subseteq \overline{\nu} \cap \overline{\varrho}$.*

Define interval-valued fuzzy subsets $\overline{\kappa}_A$ and $\overline{\kappa}'_A$ of X by

$$\overline{\kappa}_A(a) = \begin{cases} \mathbf{1} & \text{if } a \in A, \\ \mathbf{0} & \text{if } a \notin A, \end{cases}$$

and

$$\overline{\kappa}'_A(a) = \begin{cases} \mathbf{0} & \text{if } a \in A, \\ \mathbf{1} & \text{if } a \notin A. \end{cases}$$

Proposition 1.3. [9] *The following properties are true.*

(1) $A \subseteq B$ *if and only if* $\overline{\kappa}_A \subseteq \overline{\kappa}_B$.

(2) $\overline{\kappa}_{A\cup B} = \overline{\kappa}_A \cup \overline{\kappa}_B$.

(3) $\overline{\kappa}_{A\cap B} = \overline{\kappa}_A \cap \overline{\kappa}_B$.

Proposition 1.4. *The following properties are true.*

(1) $A \subseteq B$ *if and only if* $\overline{\kappa}'_B \subseteq \overline{\kappa}'_A$.

(2) $\overline{\kappa}'_{A\cup B} \subseteq \overline{\kappa}'_A \cup \overline{\kappa}'_B$.

(3) $\overline{\kappa}'_A \cap \overline{\kappa}'_B \subseteq \overline{\kappa}'_{A\cap B}$.

(4) $\overline{\kappa}'_A \cup \overline{\kappa}'_B = \overline{\kappa}'_{A\cap B}$.

(5) $\overline{\kappa}'_A \cap \overline{\kappa}'_B = \overline{\kappa}'_{A\cup B}$.

Proof. The proofs of these five properties are straightforward. □

Let $\overline{\alpha}$ and $\overline{\varrho}$ be two IvFSs of S. Define $\overline{\alpha} \mathbin{\bar{\circ}} \overline{\varrho}$ and $\overline{\alpha} \mathbin{\bar{\bullet}} \overline{\varrho}$ of S by

$$(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\varrho})(s) = \begin{cases} \operatorname*{rsup}_{s=vt} \operatorname{rmin}\{\overline{\alpha}(v), \overline{\varrho}(t)\} & \text{if } s \in S^2, \\ \mathbf{0} & \text{if } s \notin S^2, \end{cases}$$

and

$$(\overline{\alpha} \mathbin{\bar{\bullet}} \overline{\varrho})(s) = \begin{cases} \operatorname*{rinf}_{s=vt} \operatorname{rmax}\{\overline{\alpha}(v), \overline{\varrho}(t)\} & \text{if } s \in S^2, \\ \mathbf{1} & \text{if } s \notin S^2. \end{cases}$$

It is well-known that the operations $\bar{\circ}$ and $\bar{\bullet}$ are associative.

Proposition 1.5. *Let $\overline{\sigma}$, $\overline{\varrho}$ and $\overline{\nu}$ be IvFSs of S. Then*

(1) *If $\overline{\varrho} \subseteq \overline{\nu}$, then*

$$\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho} \subseteq \overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu} \text{ and } \overline{\varrho} \mathbin{\bar{\circ}} \overline{\sigma} \subseteq \overline{\nu} \mathbin{\bar{\circ}} \overline{\sigma}.$$

(2) *If $\overline{\varrho} \subseteq \overline{\nu}$, then*

$$\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\varrho} \subseteq \overline{\sigma} \mathbin{\bar{\bullet}} \overline{\nu} \text{ and } \overline{\varrho} \mathbin{\bar{\bullet}} \overline{\sigma} \subseteq \overline{\nu} \mathbin{\bar{\bullet}} \overline{\sigma}.$$

Proof. The proofs these two statements are straightforward. □

Proposition 1.6. *Let $\overline{\sigma}$, $\overline{\varrho}$ and $\overline{\nu}$ be IvFSs of S. Thus the following properties list below are true.*

(1) $\overline{\sigma} \mathbin{\bar{\circ}} (\overline{\varrho} \cup \overline{\nu}) = (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho}) \cup (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu})$.

(2) $\overline{\sigma} \mathbin{\bar{\circ}} (\overline{\varrho} \cap \overline{\nu}) = (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho}) \cap (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu})$.

(3) $(\overline{\sigma} \cup \overline{\varrho}) \mathbin{\bar{\circ}} \overline{\nu} = (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu}) \cup (\overline{\varrho} \mathbin{\bar{\circ}} \overline{\nu})$.

(4) $(\overline{\sigma} \cap \overline{\varrho}) \mathbin{\bar{\circ}} \overline{\nu} = (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu}) \cap (\overline{\varrho} \mathbin{\bar{\circ}} \overline{\nu})$.

(5) $\overline{\sigma} \mathbin{\bar{\bullet}} (\overline{\varrho} \cup \overline{\nu}) = (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\varrho}) \cup (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\nu})$.

(6) $\overline{\sigma} \mathbin{\bar{\bullet}} (\overline{\varrho} \cap \overline{\nu}) = (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\varrho}) \cap (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\nu})$.

(7) $(\overline{\sigma} \cup \overline{\varrho}) \mathbin{\bar{\bullet}} \overline{\nu} = (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\nu}) \cup (\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\nu})$.

(8) $(\overline{\sigma} \cap \overline{\varrho}) \mathbin{\bar{\bullet}} \overline{\nu} = (\overline{\sigma} \mathbin{\bar{\bullet}} \overline{\nu}) \cap (\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\nu})$.

Proof. (1) Let $s \in S$.
Case 1: $s \notin S^2$. Then

$$\begin{aligned}\big(\overline{\sigma} \mathbin{\bar{\circ}} (\overline{\varrho} \cup \overline{\nu})\big)(s) &= \mathbf{0} \\ &= \big((\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho}) \cup (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\nu})\big)(s).\end{aligned}$$

Case 2: $s \in S^2$. So

$$\begin{aligned}&\big(\overline{\sigma} \mathbin{\bar{\circ}} (\overline{\varrho} \cup \overline{\nu})\big)(s) \\ &\quad= \operatorname*{rsup}_{s=vt} \operatorname{rmin}\big\{\overline{\sigma}(v), (\overline{\varrho} \cup \overline{\nu})(t)\big\} \\ &\quad= \operatorname*{rsup}_{s=vt} \operatorname{rmin}\big\{\overline{\sigma}(v), \operatorname{rmax}\{\overline{\varrho}(t), \overline{\nu}(t)\}\big\} \\ &\quad= \operatorname*{rsup}_{s=vt} \operatorname{rmax}\big\{\operatorname{rmin}\{\overline{\sigma}(v), \overline{\varrho}(t)\},\end{aligned}$$

$$\begin{aligned} &\qquad\qquad \text{rmin}\{\overline{\sigma}(v), \overline{v}(t)\}\} \\ &= \text{rmax}\big\{\underset{s=vt}{\text{rsup}}\, \text{rmin}\{\overline{\sigma}(v), \overline{\varrho}(t)\}, \\ &\qquad \underset{s=vt}{\text{rsup}}\, \text{rmin}\{\overline{\sigma}(v), \overline{v}(t)\}\big\} \\ &= \text{rmax}\{(\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho})(s), (\overline{\sigma} \mathbin{\bar{\circ}} \overline{v})(s)\} \\ &= \big((\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho}) \cup (\overline{\sigma} \mathbin{\bar{\circ}} \overline{v})\big)(s). \end{aligned}$$

Therefore, $\overline{\sigma} \mathbin{\bar{\circ}} (\overline{\varrho} \cup \overline{v}) = (\overline{\sigma} \mathbin{\bar{\circ}} \overline{\varrho}) \cup (\overline{\sigma} \mathbin{\bar{\circ}} \overline{v})$.

The proofs of (2)–(8) are the same manner as those of (1). □

Proposition 1.7. [9] *Let T and Y be nonempty subsets of S. Thus*

$$\overline{\kappa}_T \mathbin{\bar{\circ}} \overline{\kappa}_Y = \overline{\kappa}_{TY}.$$

Proposition 1.8. *Let T and Y be nonempty subsets of S. Thus*

$$\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y = \overline{\kappa}'_{TY}.$$

Proof. Let $s \in S$.
Case 1: $s \in TY$. Thus $s = ty$ for some $t \in T$ and $y \in Y$. Then

$$\begin{aligned} (\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y)(s) &= \underset{s=ne}{\text{rinf}}\, \text{rmax}\{\overline{\kappa}'_T(n), \overline{\kappa}'_Y(e)\} \\ &\preceq \text{rmax}\{\overline{\kappa}'_T(t), \overline{\kappa}'_Y(y)\} \\ &= \text{rmax}\{\mathbf{0}, \mathbf{0}\} = \mathbf{0}. \end{aligned}$$

Thus $(\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y)(s) = \mathbf{0} = \overline{\kappa}'_{TY}(s)$.
Case 2: $s \notin TY$.
If $s \in S^2$, then $s = ty$ for some $t, y \in S$ with $t \notin T$ or $y \notin Y$.

$$\begin{aligned} (\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y)(s) &= \underset{s=ne}{\text{rinf}}\, \text{rmax}\{\overline{\kappa}'_T(n), \overline{\kappa}'_Y(e)\} \\ &\preceq \text{rmax}\{\overline{\kappa}'_T(t), \overline{\kappa}'_Y(y)\} \\ &= \mathbf{1} = \overline{\kappa}'_{TY}(s). \end{aligned}$$

If $s \notin S^2$, then

$$(\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y)(s) = \mathbf{1} = \overline{\kappa}'_{TY}(s).$$

Therefore, $\overline{\kappa}'_T \mathbin{\bar{\bullet}} \overline{\kappa}'_Y = \overline{\kappa}'_{TY}$. □

In 1983, Atanassov [1] extended a fuzzy subset to an intuitionistic fuzzy set. For an intuitionistic fuzzy set (IFS), an element is expressed by a degree of membership and non-membership, where the summation of these two degrees of membership is always less than or equal to one. Later, Atanassov and Gargov [2, 3] proposed an interval-valued intuitionistic fuzzy set (IvIFS) based on an IFS and an IvFS. Mathemathecians studied a concept of an IvFS in various algebraic structures.

Many years later, in 2013, Cuong et. al. [4,5] presented a picture fuzzy set (PFS), which are extensions of a fuzzy subset and an IFS.

A *picture fuzzy set* of X is defined as the set

$$\Big\{\big(p, \alpha(p), \varrho(p), \nu(p)\big) \mid p \in X\Big\},$$

where α, ϱ, ν are fuzzy subsets of X, satisfying the following condition:

$$0 \leq \alpha(p) + \varrho(p) + \nu(p) \leq 1 \text{ for all } p \in X.$$

We refer to $\alpha(p)$, $\varrho(p)$, and $\nu(p)$ as the degree of positive membership, neutral membership, and negative membership of p, respectively. The degree of refusal membership of p in X is then defined as $1 - \alpha(p) - \varrho(p) - \nu(p)$.

In general, a situation in which human decisions necessitate a greater range of responses: yes, abstain, no, and refuse, a PFS can be applied. Additionally, a definition and properties of an interval-valued picture fuzzy set (IvPFS) was simultaneously created.

An abstention is a term in election procedure that refers to when a voter does not vote (on election day).

In 2015, Yang et al. [14] redefined picture fuzzy sets without mentioning the Cuong's defining of PFSs.

A *picture fuzzy set (PFS)* of X is described as the set

$$\Big\{(p, \alpha(p), \varrho(p), \nu(p)) \mid p \in X\Big\},$$

where α, ϱ, ν are fuzzy subsets of X, satisfying the following conditions:

$$0 \leq \alpha(p) + \nu(p) \leq 1 \text{ and}$$

$$0 \leq \alpha(p) + \varrho(p) + \nu(p) \leq 2 \text{ for all } p \in X.$$

A PFS of X is briefly denoted by (α, ϱ, ν).

An *interval-valued picture fuzzy set (IvPFS)* of X is defined by,

$$\Big\{(p, \overline{\alpha}(p), \overline{\varrho}(p), \overline{\nu}(p)) \mid p \in X\Big\},$$

where $\overline{\alpha}, \overline{\varrho}, \overline{\nu}$ are IvFSs of X, satisfying the following conditions:

$$0 \leq \sup\ \overline{\alpha}(p) + \sup\ \overline{\nu}(p) \leq 1 \text{ and}$$

$$0 \leq \sup\ \overline{\alpha}(p) + \sup\ \overline{\varrho}(p) +\ \overline{\nu}(p) \leq 2$$

for all $p \in X$. The IvPFS of X is briefly denoted by $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$.

Let $\mathcal{K}_1 = (\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $\mathcal{K}_2 = (\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ be two IvPFSs of X.

1. $\mathcal{K}_1 \subseteq \mathcal{K}_2$ iff $\overline{\alpha}_1 \subseteq \overline{\alpha}_2$, $\overline{\varrho}_2 \subseteq \overline{\varrho}_1$ and $\overline{\nu}_2 \subseteq \overline{\nu}_1$.
2. $\mathcal{K}_1 = \mathcal{K}_2$ iff $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $\mathcal{K}_2 \subseteq \mathcal{K}_1$.
3. We define a *union* of $\mathcal{K}_1$ and $\mathcal{K}_2$ by $\mathcal{K}_1 \cup \mathcal{K}_2 = (\overline{\alpha}_1 \cup \overline{\alpha}_2, \overline{\varrho}_1 \cap \overline{\varrho}_2, \overline{\nu}_1 \cap \overline{\nu}_2)$.
4. We define an *intersection* of $\mathcal{K}_1$ and $\mathcal{K}_2$ by $\mathcal{K}_1 \cap \mathcal{K}_2 = (\overline{\alpha}_1 \cap \overline{\alpha}_2, \overline{\varrho}_1 \cup \overline{\varrho}_2, \overline{\nu}_1 \cup \overline{\nu}_2)$.

The next results are following from Proposition 1.2.

Proposition 1.9. *Let $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ be IvPFSs of S. The following properties are valid.*

(1) $\mathcal{K}_1 \subseteq \mathcal{K}_1 \cup \mathcal{K}_2$ *and* $\mathcal{K}_2 \subseteq \mathcal{K}_1 \cup \mathcal{K}_2$.

(2) $\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \mathcal{K}_1$ *and* $\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \mathcal{K}_2$.

(3) *If* $\mathcal{K}_1 \subseteq \mathcal{K}_2$ *and* $\mathcal{K}_2 \subseteq \mathcal{K}_3$, *then* $\mathcal{K}_1 \subseteq \mathcal{K}_3$.

(4) *If* $\mathcal{K}_1 \subseteq \mathcal{K}_2$, *then* $\mathcal{K}_1 \cup \mathcal{K}_3 \subseteq \mathcal{K}_2 \cup \mathcal{K}_3$ *and* $\mathcal{K}_3 \cup \mathcal{K}_1 \subseteq \mathcal{K}_3 \cup \mathcal{K}_2$.

(5) *If* $\mathcal{K}_1 \subseteq \mathcal{K}_2$, *then* $\mathcal{K}_1 \cap \mathcal{K}_3 \subseteq \mathcal{K}_2 \cap \mathcal{K}_3$ *and* $\mathcal{K}_3 \cap \mathcal{K}_1 \subseteq \mathcal{K}_3 \cap \mathcal{K}_2$.

The *characteristic interval-valued picture fuzzy set (CIvPFS)* of A in X is defined by

$$\Big\{\Big(p, \overline{\kappa}_A(p), \overline{\kappa}'_A(p), \overline{\kappa}'_A(p)\Big) \mid p \in X\Big\}.$$

We denote the CIvPFS of A in X by $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A)$ and write $\mathcal{S}$ instead of $(\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S)$.

The next proposition is a direct consequence of Proposition 1.3 and Proposition 1.4.

Proposition 1.10. *Let $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A)$ and $(\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)$ be two CIvPFSs of subsets A and B of X, respectively. Then*

(1) $A \subseteq B$ *if and only if* $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A) \subseteq (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)$.

(2) $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A) \cup (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B) = (\overline{\kappa}_{A\cup B}, \overline{\kappa}'_{A\cup B}, \overline{\kappa}'_{A\cup B})$.

(3) $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A) \cap (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B) = (\overline{\kappa}_{A\cap B}, \overline{\kappa}'_{A\cap B}, \overline{\kappa}'_{A\cap B})$.

The *product* of two IvPFSs $\mathcal{K}_1 = (\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $\mathcal{K}_2 = (\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ of S, written $\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_2$, is defined by

$$(\overline{\alpha}_1 \,\bar{\circ}\, \overline{\alpha}_2, \overline{\varrho}_1 \,\bar{\bullet}\, \overline{\varrho}_2, \overline{\nu}_1 \,\bar{\bullet}\, \overline{\nu}_2).$$

Proposition 1.11. *Let $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ be IvPFSs of S. If $\mathcal{K}_2 \subseteq \mathcal{K}_3$, then $\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_2 \subseteq \mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_3$ and $\mathcal{K}_2 \,\bar{\circ}_p\, \mathcal{K}_1 \subseteq \mathcal{K}_3 \,\bar{\circ}_p\, \mathcal{K}_1$.*

The next results follow directly from Proposition 1.6.

Proposition 1.12. *Let $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ be IvPFSs of S. The properties list below are true.*

(1) $\mathcal{K}_1 \,\bar{\circ}_p\, (\mathcal{K}_2 \cap \mathcal{K}_3) = (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_2) \cap (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_3)$.

(2) $(\mathcal{K}_1 \cap \mathcal{K}_2) \,\bar{\circ}_p\, \mathcal{K}_3 = (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_3) \cap (\mathcal{K}_2 \,\bar{\circ}_p\, \mathcal{K}_3)$.

(3) $\mathcal{K}_1 \,\bar{\circ}_p\, (\mathcal{K}_2 \cup \mathcal{K}_3) = (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_2) \cup (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_3)$.

(4) $(\mathcal{K}_1 \cup \mathcal{K}_2) \,\bar{\circ}_p\, \mathcal{K}_3 = (\mathcal{K}_1 \,\bar{\circ}_p\, \mathcal{K}_3) \cup (\mathcal{K}_2 \,\bar{\circ}_p\, \mathcal{K}_3)$.

Proposition 1.7 and Proposition 1.8 yield the following result.

Proposition 1.13. *Let $(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T)$ and $(\overline{\kappa}_Y, \overline{\kappa}'_Y, \overline{\kappa}'_Y)$ be two CIvPFSs of nonempty subsets T and Y of S, respectively.*

$$(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T) \,\bar{\circ}_p\, (\overline{\kappa}_Y, \overline{\kappa}'_Y, \overline{\kappa}'_Y) = (\overline{\kappa}_{TY}, \overline{\kappa}'_{TY}, \overline{\kappa}'_{TY}).$$

The idea of a fuzzy subset was applied to a group by Rosenfeld [13], to that rings by Liu [11], and to that semigroups by Kuroki [8, 9].

Let α be a fuzzy subset of S. Then we call α a *fuzzy subsemigroup* of S if it follows a condition

$$\alpha(tw) \geq \min\{\alpha(t), \alpha(w)\},$$

for all $t, w \in S$, it is a *fuzzy left ideal* of S if it satisfies the following condition

$$\alpha(tw) \geq \alpha(w),$$

for all $t, w \in S$, and it is a *fuzzy right ideal* of S if

$$\alpha(tw) \geq \alpha(t),$$

for all $t, w \in S$. If α is both a fuzzy left ideal and a fuzzy right ideal, then it is said to be a *fuzzy ideal* of S. Let α be a fuzzy subsemigroup of S. We call α a *fuzzy bi-ideal* of S if

$$\alpha(tws) \geq \min\{\alpha(t), \alpha(s)\},$$

for all $t, w, s \in S$, and α is called *interior ideal* of S if

$$\alpha(tws) \geq \alpha(w),$$

for all $t, w, s \in S$. A fuzzy subset α of S is a *fuzzy quasi-ideal* of S if

$$(\alpha \cap \kappa_S) \circ (\kappa_S \cap \alpha) \subseteq \alpha,$$

where κ_S is a characteristic function of S.

In 2003, Kuroki et. al. studied some properties of a fuzzy left ideal, a fuzzy right ideal, a fuzzy ideal, and a fuzzy bi-ideal of a semigroup. For the IvFS, Narayanan and Manikantan [12] gave the notions of an interval-valued fuzzy subsemigroup and various tpyes of an interval-valued fuzzy ideal of a semigroup in 2006. Furthermore, an IFS and an IvIFS were studied of semigroups. Recently, Iampan et. al. [7, 10, 17] discussed PFSs and IvPFSs in UP-algebras. In addition, Yiarayong [15, 16] applied the concept of a PFS to a semigroup and studied a picture fuzzy subsemigroup, a picture fuzzy left ideal, a picture right ideal, a picture fuzzy ideal, and a picture fuzzy bi-ideal of a semigroup.

In this paper, we introduce an interval-valued picture fuzzy subsemigroup and various types of interval-valued picture fuzzy ideal (left ideal, right ideal, ideal, bi-ideal, interior ideal, quasi-ideal) of a semigroup and study some properties of an interval-valued picture fuzzy subsemigroup and each types of an interval-valued picture fuzzy ideal of a semigroup. Moreover, we will investigate the relationship between each ideal of semigroups and its interval-valued picture fuzzification.

2. Main Result

We begin this section by introducing an interval-valued picture fuzzy subsemigroup and giving some properties of them.

Definition 2.1. An interval-valued picture fuzzy set $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is called an *interval-valued picture fuzzy subsemigroup (IvPF-subsemigroup)* of S if it satisfies:

1. $\overline{\alpha}(st) \succeq \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}$,
2. $\overline{\varrho}(st) \preceq \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}$,
3. $\overline{\nu}(st) \preceq \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}$,

for all $s, t \in S$.

Example 2.2. Consider the semigroup $S = \{a, b, c, d\}$ with a multiplication table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	b

and define the IvPFSs $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ of S by

$$\begin{aligned}(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) &= \Big\{(a, [0.8, 1.0], [0.0, 0.0], [0.0, 0.0]),\\ &\quad (b, [0.7, 0.7], [0.0, 0.2], [0.1, 0.1]),\\ &\quad (c, [0.5, 0.8], [0.0, 0.0], [0.0, 0.2]),\\ &\quad (d, [0.3, 1.0], [0.0, 0.0], [0.0, 0.0])\Big\},\end{aligned}$$

$$\begin{aligned}(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2) &= \Big\{(a, [0.7, 1.0], [0.0, 0.0], [0.0, 0.0]),\\ &\quad (b, [0.5, 0.5], [0.3, 0.3], [0.1, 0.2]),\\ &\quad (c, [0.4, 0.5], [0.0, 0.3], [0.1, 0.2]),\\ &\quad (d, [0.1, 0.2], [0.0, 0.0], [0.2, 0.8])\Big\}.\end{aligned}$$

Then $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ are IvPF-subsemigroups of S.

Lemma 2.3. *Let $\mathcal{T} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus $\mathcal{T}$ is an IvPF-subsemigroup of S if and only if*

$$\mathcal{T} \mathbin{\bar{\circ}_p} \mathcal{T} \subseteq \mathcal{T}.$$

Proof. Firstly, we suppose that $\mathcal{T}$ is an IvPF-subsemigroup of S.
Let s be an element of S.
Case 1: $s \notin S^2$. Then $(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\alpha})(s) = \mathbf{0}$, $(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\varrho})(s) = \mathbf{1}$, and $(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\nu})(s) = \mathbf{1}$. So $(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\alpha})(s) \preceq \overline{\alpha}(s)$, $\overline{\varrho}(s) \preceq (\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\varrho})(s)$, and $\overline{\nu}(s) \preceq (\overline{\nu} \mathbin{\bar{\bullet}} \overline{\nu})(s)$.

Case 2: $s \in S^2$. Since $\mathcal{T}$ is an IvPF-subsemigroup of S, we have

$$\begin{aligned}(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\alpha})(s) &= \operatorname*{rsup}_{s=vt} \text{rmin}\{\overline{\alpha}(v), \overline{\alpha}(t)\}\\ &\preceq \operatorname*{rsup}_{s=vt} \overline{\alpha}(vt)\\ &= \overline{\alpha}(s).\end{aligned}$$

Moreover, we have

$$\begin{aligned}(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\varrho})(s) &= \operatorname*{rinf}_{s=vt} \text{rmax}\{\overline{\varrho}(v), \overline{\varrho}(t)\}\\ &\succeq \operatorname*{rinf}_{s=vt} \overline{\varrho}(vt)\\ &= \overline{\varrho}(s).\end{aligned}$$

Similarly, we get $(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\nu})(s) \succeq \overline{\nu}(s)$.
Then we conclude that $\mathcal{T} \mathbin{\bar{\circ}_p} \mathcal{T} \subseteq \mathcal{T}$.

Conversely, we first suppose that

$$\mathcal{T} \mathbin{\bar{\circ}_p} \mathcal{T} \subseteq \mathcal{T}.$$

Let $s, t \in S$. Thus

$$\overline{\alpha}(st) \succeq (\overline{\alpha} \mathbin{\bar{\circ}} \overline{\alpha})(st) \succeq \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}$$

and

$$\overline{\varrho}(st) \preceq (\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\varrho})(st) \preceq \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}.$$

Similarly, we have $\overline{\nu}(st) \preceq \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}$.
Hence, we get $\mathcal{T}$ is an IvPF-subsemigroup of S. □

Theorem 2.4. *Let $\emptyset \neq T \subseteq S$. Thus T is a subsemigroup of S if and only if $(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T)$ is an IvPF-subsemigroup of S.*

Proof. Let T be a subsemigroup of S. Thus $T^2 \subseteq T$. By Propositions 1.13 and 1.10, we have that

$$\begin{aligned}(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T) \;\bar{\circ}_p\; (\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T) &= (\overline{\kappa}_{T^2}, \overline{\kappa}'_{T^2}, \overline{\kappa}'_{T^2}) \\ &\subseteq (\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T).\end{aligned}$$

Therefore, by Lemma 2.3, we obtain that $(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T)$ is an IvPF-subsemigroup of S.

Conversely, assume that $(\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T)$ is an IvPF-subsemigroup of S. Then

$$\begin{aligned}&(\overline{\kappa}_{T^2}, \overline{\kappa}'_{T^2}, \overline{\kappa}'_{T^2}) \\ &\quad= (\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T) \;\bar{\circ}_p\; (\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T) \\ &\quad\subseteq (\overline{\kappa}_T, \overline{\kappa}'_T, \overline{\kappa}'_T).\end{aligned}$$

Hence, $T^2 \subseteq T$ by Proposition 1.10. Consequently, T is a subsemigroup of S. □

Theorem 2.5. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are IvPF-subsemigroups of S, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also an IvPF-subsemigroup of S.*

Proof. Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two IvPF-subsemigroups of S. Then $\mathcal{T}_1 \;\bar{\circ}_p\; \mathcal{T}_1 \subseteq \mathcal{T}_1$ and $\mathcal{T}_2 \;\bar{\circ}_p\; \mathcal{T}_2 \subseteq \mathcal{T}_2$ by Lemma 2.3. By Proposition 1.12,

$$\begin{aligned}&(\mathcal{T}_1 \cap \mathcal{T}_2) \;\bar{\circ}_p\; (\mathcal{T}_1 \cap \mathcal{T}_2) \\ &= \Big(\mathcal{T}_1 \;\bar{\circ}_p\; (\mathcal{T}_1 \cap \mathcal{T}_2)\Big) \cap \Big(\mathcal{T}_2 \;\bar{\circ}_p\; (\mathcal{T}_1 \cap \mathcal{T}_2)\Big) \\ &= \Big((\mathcal{T}_1 \;\bar{\circ}_p\; \mathcal{T}_1) \cap (\mathcal{T}_1 \;\bar{\circ}_p\; \mathcal{T}_2)\Big) \\ &\qquad \cap \Big((\mathcal{T}_2 \;\bar{\circ}_p\; \mathcal{T}_1) \cap (\mathcal{T}_2 \;\bar{\circ}_p\; \mathcal{T}_2)\Big) \\ &\subseteq \mathcal{T}_1 \cap (\mathcal{T}_1 \;\bar{\circ}_p\; \mathcal{T}_2) \cap (\mathcal{T}_2 \;\bar{\circ}_p\; \mathcal{T}_1) \cap \mathcal{T}_2 \\ &\subseteq \mathcal{T}_1 \cap \mathcal{T}_2.\end{aligned}$$

Therefore, by Lemma 2.3, we have $\mathcal{T}_1 \cap \mathcal{T}_2$ is an IvPF-subsemigroup of S. □

Example 2.6. Consider the semigroup $(\mathbb{N}, +)$ where $+$ is the usual addition. By Theorem 2.4, $(\overline{\kappa}_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}})$ and $(\overline{\kappa}_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}})$ are IvPF-subsemigroups of $\mathbb{N}$. We have

$$\begin{aligned}&(\overline{\kappa}_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}}) \cup (\overline{\kappa}_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}}) \\ &\qquad= (\overline{\kappa}_{2\mathbb{N}\cup 3\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}\cup 3\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}\cup 3\mathbb{N}}).\end{aligned}$$

Since $2\mathbb{N} \cup 3\mathbb{N}$ is not a subsemigroup of $(\mathbb{N}, +)$, it follows from Theorem 2.4 that

$$(\overline{\kappa}_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}}, \overline{\kappa}'_{2\mathbb{N}}) \cup (\overline{\kappa}_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}}, \overline{\kappa}'_{3\mathbb{N}})$$

is not an IvPF-subsemigroup of $\mathbb{N}$.

Example 2.6 shows that in general, the union of two IvPF-subsemigroups of S need not be an IvPF-subsemigroup of S.

Definition 2.7. Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of X and $I_1, I_2, I_3 \in CI[0,1]$. Define

$$\begin{aligned}&(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \\ &\quad= \big\{x \in X \mid \overline{\alpha}(x) \succeq I_1, \overline{\varrho}(x) \preceq I_2, \overline{\nu}(x) \preceq I_3\big\}.\end{aligned}$$

We call $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ an *interval-valued picture $\{I_1, I_2, I_3\}$-level set.*

Theorem 2.8. *Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-subsemigroup of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S.*

Proof. Suppose that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-subsemigroup of S.
Let $I_1, I_2, I_3 \in CI[0,1]$ such that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$.
Let $s, t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\begin{aligned}\overline{\alpha}(s) \succeq I_1, \overline{\varrho}(s) \preceq I_2, \overline{\nu}(s) \preceq I_3, \\ \overline{\alpha}(t) \succeq I_1, \overline{\varrho}(t) \preceq I_2, \overline{\nu}(t) \preceq I_3.\end{aligned}$$

Thus

$$\overline{\alpha}(st) \succeq \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\} \succeq I_1,$$

$$\overline{\varrho}(st) \preceq \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\} \preceq I_2,$$
$$\overline{\nu}(st) \preceq \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\} \preceq I_3.$$

Hence, $st \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S.

Conversely, we assume that for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S. Let $s, t \in S$. Given

$$I_1 = \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\},$$
$$I_2 = \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\},$$
$$I_3 = \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}.$$

Then $I_1, I_2, I_3 \in CI[0,1]$. It is clear that, $s, t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. So, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S, by assumption. So $st \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\overline{\alpha}(st) \succeq I_1 = \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\},$$
$$\overline{\varrho}(st) \preceq I_2 = \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\},$$
$$\overline{\nu}(st) \preceq I_3 = \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}.$$

Hence, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-subsemigroup of S. □

Next, we define an interval-valued picture fuzzy left ideal and an interval-valued picture fuzzy right ideal of a semigroup.

Definition 2.9. An interval-valued picture fuzzy set $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is an *interval-valued picture fuzzy left* [*right*] *ideal (IvPF-left[right] ideal)* of S if it satisfies

1. $\overline{\alpha}(st) \succeq \overline{\alpha}(t)\ \left[\ \overline{\alpha}(st) \succeq \overline{\alpha}(s)\ \right]$,
2. $\overline{\varrho}(st) \preceq \overline{\varrho}(t)\ \left[\ \overline{\varrho}(st) \preceq \overline{\varrho}(s)\ \right]$,
3. $\overline{\nu}(st) \preceq \overline{\nu}(t)\ \left[\ \overline{\nu}(st) \preceq \overline{\nu}(s)\ \right]$,

for all $s, t \in S$ and if an IvPFS $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is both an IvPF-left ideal and an IvPF-right ideal, then we call it an *interval-valued picture fuzzy ideal (IvPF-ideal)* of S.

Example 2.10. Let S be a semigroup in an Example 2.2. Define the IvPFSs $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ of S by

$$\begin{aligned}(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) = \Big\{&(a, [1,1], [0,0], [0,0]),\\ &(b, [1,1], [0,0], [0,0]),\\ &(c, [0,0], [1,1], [1,1]),\\ &(d, [0,0], [1,1], [1,1])\Big\}, \text{ and}\end{aligned}$$

$$\begin{aligned}(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) = \Big\{&(a, [1,1], [0,0], [0,0]),\\ &(b, [1,1], [0,0], [0,0]),\\ &(c, [1,1], [0,0], [0,0]),\\ &(d, [0,0], [1,1], [1,1])\Big\}.\end{aligned}$$

Then $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ are IvPF-ideals of S because they are both IvPF-left ideals and IvPF-right ideals of S.

Lemma 2.11. *Let $\mathcal{L}, \mathcal{R}$ and $\mathcal{I}$ be IvPFSs of S. The following statements are true.*

(1) *$\mathcal{L}$ is an IvPF-left ideal of S if and only if*
$$\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{L} \subseteq \mathcal{L}.$$

(2) *$\mathcal{R}$ is an IvPF-right ideal of S if and only if*
$$\mathcal{R} \mathbin{\bar{\circ}_p} \mathcal{S} \subseteq \mathcal{R}.$$

(3) *$\mathcal{I}$ is an IvPF-ideal of S if and only if*
$$\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{I} \subseteq \mathcal{I} \text{ and } \mathcal{I} \mathbin{\bar{\circ}_p} \mathcal{S} \subseteq \mathcal{I}.$$

Proof. (1) Assume that $\mathcal{L} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-left ideal of S. Let $l \in S$.
Case 1: $l \notin S^2$. Then
$(\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(l) = \mathbf{0}$, $(\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(l) = \mathbf{1}$, and $(\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(l) = \mathbf{1}$. So
$(\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(l) \preceq \overline{\alpha}(l)$, $\overline{\varrho}(l) \preceq (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(l)$, and $\overline{\nu}(l) \preceq (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(l)$.
Case 2: $l \in S^2$. Since $\mathcal{L}$ is an IvPF-left ideal of S, we get

$$(\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(l) = \underset{l=ab}{\text{rsup}}\ \text{rmin}\{\overline{\kappa}_S(a), \overline{\alpha}(b)\}$$

$$\preceq \operatorname{rsup}_{l=ab} \operatorname{rmin}\{\mathbf{1}, \overline{\alpha}(ab)\}$$
$$= \operatorname{rsup} \overline{\alpha}(l)$$
$$= \overline{\alpha}(l),$$

and

$$(\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(l) = \operatorname{rinf}_{l=ab} \operatorname{rmax}\{\overline{\kappa}'_S(a), \overline{\varrho}(b)\}$$
$$\succeq \operatorname{rinf}_{l=ab} \operatorname{rmax}\{\mathbf{0}, \overline{\varrho}(ab)\}$$
$$= \operatorname{rinf} \overline{\varrho}(l)$$
$$= \overline{\varrho}(l).$$

Similarly, we can get $(\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(l) \succeq \overline{\nu}(l)$. Then we conclude that $\mathcal{S} \,\bar{\circ}_p\, \mathcal{L} \subseteq \mathcal{L}$.

To prove the converse, we suppose that $\mathcal{S} \,\bar{\circ}_p\, \mathcal{L} \subseteq \mathcal{L}$. Let $s, t \in S$. Then

$$\overline{\alpha}(st) \succeq (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(st)$$
$$\succeq \operatorname{rmin}\{\overline{\kappa}_S(s), \overline{\alpha}(t)\}$$
$$= \operatorname{rmin}\{\mathbf{1}, \overline{\alpha}(t)\}$$
$$= \overline{\alpha}(t),$$

and

$$\overline{\varrho}(st) \preceq (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(st)$$
$$\preceq \operatorname{rmax}\{\overline{\kappa}'_S(s), \overline{\varrho}(t)\}$$
$$= \operatorname{rmax}\{\mathbf{0}, \overline{\varrho}(t)\}$$
$$= \overline{\varrho}(t).$$

In a similar way, we have $\overline{\nu}(st) \preceq \overline{\nu}(t)$. Hence, $\mathcal{L}$ is an IvPF-left ideal of S.

(2) This proof is similar to (1).

(3) This result is following from Statements (1) and (2). □

Theorem 2.12. *Let L, R and I be nonempty subsets of S. The following statements list below are valid.*

(1) *L is a left ideal of S if and only if $(\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$ is an IvPF-left ideal of S.*

(2) *R is a right ideal of S if and only if $(\overline{\kappa}_R, \overline{\kappa}'_R, \overline{\kappa}'_R)$ is an IvPF-right ideal of S.*

(3) *I is an ideal of S if and only if $(\overline{\kappa}_I, \overline{\kappa}'_I, \overline{\kappa}'_I)$ is an IvPF-ideal of S.*

Proof. (1) Let L be a left ideal of S. Then $SL \subseteq L$. By Proposition 1.13 and 1.10, we have that

$$(\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S) \,\bar{\circ}_p\, (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$$
$$= (\overline{\kappa}_{SL}, \overline{\kappa}'_{SL}, \overline{\kappa}'_{SL})$$
$$\subseteq (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$$

It follows from Lemma 2.11 that $(\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$ is an IvPF-left ideal of S.

On the other hand, suppose that $(\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$ is an IvPF-left ideal of S. Then

$$(\overline{\kappa}_{SL}, \overline{\kappa}'_{SL}, \overline{\kappa}'_{SL})$$
$$= (\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S) \,\bar{\circ}_p\, (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$$
$$\subseteq (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L).$$

Hence, $SL \subseteq L$. So that L is a left ideal of S.

The proof of (2) is similar to (1), and (3) follows from (1) and (2). □

Theorem 2.13. *All of the following properties are valid.*

(1) *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are IvPF-left ideals of S, then $\mathcal{L}_1 \cap \mathcal{L}_2$ is also an IvPF-left ideal of S.*

(2) *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are IvPF-right ideals of S, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is also an IvPF-right ideal of S.*

(3) *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are IvPF-ideals of S, then $\mathcal{I}_1 \cap \mathcal{I}_2$ is also an IvPF-ideal of S.*

Proof. To prove (1), let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two IvPF-left ideals of S. Then by Lemma 2.11, $\mathcal{S} \bar{\circ}_p \mathcal{L}_1$ and $\mathcal{S} \bar{\circ}_p \mathcal{L}_2$. By Proposition 1.12,

$$\mathcal{S} \bar{\circ}_p \left(\mathcal{L}_1 \cap \mathcal{L}_2\right) = \left(\mathcal{S} \bar{\circ}_p \mathcal{L}_1\right) \cap \left(\mathcal{S} \bar{\circ}_p \mathcal{L}_2\right) \subseteq \mathcal{L}_1 \cap \mathcal{L}_2.$$

Therefore, by Lemma 2.11, we have that $\mathcal{L}_1 \cap \mathcal{L}_2$ is an IvPF-left ideal of S.

The proof of (2) is similar to the proof of (1) and the proof of (3) follows from (1) and (2). □

Theorem 2.14. *Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus*

(1) *$(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-left ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a left ideal of S.*

(2) *$(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-right ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a right ideal of S.*

(3) *$(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is an ideal of S.*

Proof. (1) Assume that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-left ideal of S. Let $I_1, I_2, I_3 \in CI[0,1]$ such that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$. Let $s \in S$ and $t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then $\overline{\alpha}(t) \succeq I_1$, $\overline{\varrho}(t) \preceq I_2$, and $\overline{\nu}(t) \preceq I_3$. Thus

$$\begin{aligned}\overline{\alpha}(st) &\succeq \overline{\alpha}(t) \succeq I_1,\\ \overline{\varrho}(st) &\preceq \overline{\varrho}(t) \preceq I_2,\\ \overline{\nu}(st) &\preceq \overline{\nu}(t) \preceq I_3.\end{aligned}$$

Hence, $st \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a left ideal of S.

On the other hand, we assume that for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a left ideal of S. Let $s, t \in S$ and let $I_1 = \overline{\alpha}(t)$, $I_2 = \overline{\varrho}(t)$, and $I_3 = \overline{\nu}(t)$. Then $I_1, I_2, I_3 \in CI[0,1]$. It is easy to see that $t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. By assumption $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a left ideal of S. Then $st \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\begin{aligned}\overline{\alpha}(st) &\succeq I_1 = \overline{\alpha}(t),\\ \overline{\varrho}(st) &\preceq I_2 = \overline{\varrho}(t),\\ \overline{\nu}(st) &\preceq I_3 = \overline{\nu}(t).\end{aligned}$$

Hence, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-left ideal of S.

We can prove (2) similar to (1) and (3) follows from (1) and (2). □

Next, an interval-valued picture fuzzy bi-ideal of a semigroup will be defined and its some properties will be studied.

Definition 2.15. An IvPF-subsemigroup $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is called an *interval-valued picture fuzzy bi-ideal (IvPF-bi-ideal)* of S if it satisfies the following conditions given below:

1. $\overline{\alpha}(svt) \succeq \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}$,
2. $\overline{\varrho}(svt) \preceq \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}$,
3. $\overline{\nu}(svt) \preceq \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}$,

for all $s, v, t \in S$.

Example 2.16. Define the IvPFS $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of the semigroup S in an Example 2.2 by

$$\begin{aligned}(\overline{\alpha}, \overline{\varrho}, \overline{\nu}) = \Big\{&(a, [0.5, 0.6], [0,0], [0,0]),\\ &(b, [0.5, 0.6], [0,0], [0,0]),\\ &(c, [0,0], [0.9,1], [0.9,1]),\\ &(d, [0,0], [0.7,0.7], [0.7,0.7])\Big\}.\end{aligned}$$

Since $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ satisfies all conditions of an IvPF-bi-ideal of S, it follows that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-bi-ideal of S.

Lemma 2.17. *Let $\mathcal{B} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Then $\mathcal{B}$ is an IvPF-bi-ideal of S if and only if*

$$\mathcal{B} \,\bar{\circ}_p\, \mathcal{S} \,\bar{\circ}_p\, \mathcal{B} \subseteq \mathcal{B}.$$

Proof. Assume that $\mathcal{B} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-bi-ideal of S. Let $b \in S$.
Case 1: $b \notin S^2$. Then
$(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(b) = \mathbf{0}$, $(\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(b) = \mathbf{1}$, and $(\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(b) = \mathbf{1}$. So
$(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(b) \preceq \overline{\alpha}(b)$, $\overline{\varrho}(b) \preceq (\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(b)$, and $\overline{\nu}(b) \preceq (\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(b)$.
Case 2: $b \in S^2$. If $b \in S^3$, then

$$\begin{aligned}
&(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(b)\\
&\quad= \operatorname*{rsup}_{b=skw} \operatorname{rmin}\{\overline{\alpha}(s), \overline{\kappa}_S(k), \overline{\alpha}(w)\}\\
&\quad\preceq \operatorname*{rsup}_{b=skw} \operatorname{rmin}\{\mathbf{1}, \overline{\alpha}(skw)\}\\
&\quad= \operatorname{rsup} \overline{\alpha}(b)\\
&\quad= \overline{\alpha}(b),
\end{aligned}$$

and

$$\begin{aligned}
&(\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(b)\\
&\quad= \operatorname*{rinf}_{b=skw} \operatorname{rmax}\{\overline{\varrho}(s), \overline{\kappa}'_S(k), \overline{\varrho}(w)\}\\
&\quad\succeq \operatorname*{rinf}_{b=skw} \operatorname{rmax}\{\mathbf{0}, \overline{\varrho}(skw)\}\\
&\quad= \operatorname{rinf} \overline{\varrho}(b)\\
&\quad= \overline{\varrho}(b).
\end{aligned}$$

Similarly, we get $(\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(b) \succeq \overline{\nu}(b)$.
If $b \notin S^3$, then

$$\begin{aligned}
&(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(b)\\
&\quad= \operatorname*{rsup}_{b=sk} \operatorname{rmin}\{\overline{\alpha}(s), (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(k)\}\\
&\quad= \operatorname*{rsup}_{b=sk} \operatorname{rmin}\{\overline{\alpha}(s), \mathbf{0}\}\\
&\quad= \mathbf{0} = \overline{\alpha}(b),
\end{aligned}$$

and

$$\begin{aligned}
&(\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(b)\\
&\quad= \operatorname*{rinf}_{b=sk} \operatorname{rmax}\{\overline{\varrho}(s), (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(k)\}\\
&\quad= \operatorname*{rinf}_{b=sk} \operatorname{rmax}\{\overline{\varrho}(s), \mathbf{1}\}\\
&\quad= \mathbf{1} = \overline{\varrho}(b).
\end{aligned}$$

Similarly, we get $(\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(b) = \overline{\nu}(b)$. Then we conclude that $\mathcal{B} \,\bar{\circ}_p\, \mathcal{S} \,\bar{\circ}_p\, \mathcal{B} \subseteq \mathcal{B}$.

For the converse, assume that $\mathcal{B} \,\bar{\circ}_p\, \mathcal{S} \,\bar{\circ}_p\, \mathcal{B} \subseteq \mathcal{B}$. Let s, v, t be elements of S. Then

$$\begin{aligned}
\overline{\alpha}(svt) &\succeq (\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(svt)\\
&\succeq \operatorname{rmin}\{\overline{\alpha}(s), \overline{\kappa}_S(v), \overline{\alpha}(t)\}\\
&= \operatorname{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\varrho}(svt) &\preceq (\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(svt)\\
&\preceq \operatorname{rmax}\{\overline{\varrho}(s), \overline{\kappa}'_S(v), \overline{\varrho}(t)\}\\
&= \operatorname{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}.
\end{aligned}$$

Similarly, we get

$$\overline{\nu}(svt) \preceq \operatorname{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}.$$

Hence, $\mathcal{B}$ is an IvPF-bi-ideal of S. □

Theorem 2.18. *Let B be a nonempty subset of S. Thus B is a bi-ideal of S if and only if $(\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)$ is an IvPF-bi-ideal of S.*

Proof. Suppose that B is a bi-ideal of S. Then $BSB \subseteq A$. By Proposition 1.13 and 1.10, we have that

$$\begin{aligned}
&(\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B) \,\bar{\circ}_p\, \mathcal{S} \,\bar{\circ}_p\, (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)\\
&\quad= (\overline{\kappa}_{BS}, \overline{\kappa}'_{BS}, \overline{\kappa}'_{BS}) \,\bar{\circ}_p\, (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)\\
&\quad= (\overline{\kappa}_{BSB}, \overline{\kappa}'_{BSB}, \overline{\kappa}'_{BSB})\\
&\quad\subseteq (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B).
\end{aligned}$$

By Lemma 2.17, we get $(\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)$ is an IvPF-bi-ideal of S.

On the other hand, assume that $(\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B)$ is an IvPF-bi-ideal of S. Then

$$\begin{aligned}(\overline{\kappa}_{BSB}, \overline{\kappa}'_{BSB}, \overline{\kappa}'_{BSB}) &= (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B) \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B) \\ &\subseteq (\overline{\kappa}_B, \overline{\kappa}'_B, \overline{\kappa}'_B).\end{aligned}$$

Hence, $BSB \subseteq B$. This implies that B is a bi-ideal of S. □

Theorem 2.19. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are IvPF-bi-ideals of S, then $\mathcal{B}_1 \cap \mathcal{B}_2$ is also an IvPF-bi-ideal of S.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two IvPF-bi-ideal of S. By Proposition 1.12 and Lemma 2.17,

$$\begin{aligned}&(\mathcal{B}_1 \cap \mathcal{B}_2) \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} (\mathcal{B}_1 \cap \mathcal{B}_2) \\ &= \Big((\mathcal{B}_1 \mathbin{\bar{\circ}_p} \mathcal{S}) \cap (\mathcal{B}_2 \mathbin{\bar{\circ}_p} \mathcal{S})\Big) \mathbin{\bar{\circ}_p} (\mathcal{B}_1 \cap \mathcal{B}_2) \\ &= \Big((\mathcal{B}_1 \mathbin{\bar{\circ}_p} \mathcal{S}) \mathbin{\bar{\circ}_p} (\mathcal{B}_1 \cap \mathcal{B}_2)\Big) \\ &\qquad \cap \Big((\mathcal{B}_2 \mathbin{\bar{\circ}_p} \mathcal{S}) \mathbin{\bar{\circ}_p} (\mathcal{B}_1 \cap \mathcal{B}_2)\Big) \\ &= \Big((\mathcal{B}_1 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_1) \cap (\mathcal{B}_1 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_2)\Big) \\ &\quad \cap \Big((\mathcal{B}_2 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_1) \cap (\mathcal{B}_2 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_2)\Big) \\ &\subseteq \Big(\mathcal{B}_1 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_1\Big) \cap \Big(\mathcal{B}_2 \mathbin{\bar{\circ}_p} \mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{B}_2\Big) \\ &\subseteq \mathcal{B}_1 \cap \mathcal{B}_2.\end{aligned}$$

Therefore, $\mathcal{B}_1 \cap \mathcal{B}_2$ is an IvPF-bi-ideal of S by Lemma 2.17. □

Theorem 2.20. *Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-bi-ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a bi-ideal of S.*

Proof. Firstly, suppose that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-bi-ideal of S. Let $I_1, I_2, I_3 \in CI[0,1]$ such that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$. By Theorem 2.8, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S. Let $v \in S$ and $s, t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\overline{\alpha}(s) \succeq I_1, \overline{\varrho}(s) \preceq I_2, \overline{\nu}(s) \preceq I_3,$$

$$\overline{\alpha}(t) \succeq I_1, \overline{\varrho}(t) \preceq I_2, \overline{\nu}(t) \preceq I_3.$$

Thus

$$\begin{aligned}\overline{\alpha}(svt) &\succeq \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\} \succeq I_1, \\ \overline{\varrho}(svt) &\preceq \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\} \preceq I_2, \\ \overline{\nu}(svt) &\preceq \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\} \preceq I_3.\end{aligned}$$

Hence, $svt \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a bi-ideal of S.

Next, we will prove the converse. Suppose that for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a bi-ideal of S. $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-subsemigroup of S, by Theorem 2.8 . Let $s, v, t \in S$ and let

$$\begin{aligned}I_1 &= \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}, \\ I_2 &= \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}, \\ I_3 &= \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}.\end{aligned}$$

Then $I_1, I_2, I_3 \in CI[0,1]$. It is evident that $s, t \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. By assumption $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a bi-ideal of S. Then $svt \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\begin{aligned}\overline{\alpha}(svt) &\succeq I_1 = \text{rmin}\{\overline{\alpha}(s), \overline{\alpha}(t)\}, \\ \overline{\varrho}(svt) &\preceq I_2 = \text{rmax}\{\overline{\varrho}(s), \overline{\varrho}(t)\}, \\ \overline{\nu}(svt) &\preceq I_3 = \text{rmax}\{\overline{\nu}(s), \overline{\nu}(t)\}.\end{aligned}$$

Hence, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-bi-ideal of S. □

Next, we introduce an interval-valued picture fuzzy interior ideal of a semigroup and give some properties of it.

Definition 2.21. We call an IvPF-subsemigroup $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S an *interval-valued picture fuzzy interior ideal (IvPF-interior ideal)* of S if it satisfies the following conditions:

1. $\overline{\alpha}(svt) \succeq \overline{\alpha}(v)$,
2. $\overline{\varrho}(svt) \preceq \overline{\varrho}(v)$,

3. $\overline{\nu}(svt) \preceq \overline{\nu}(v)$,

for all $s, v, t \in S$.

Example 2.22. From the semigroup S in an Example 2.2, define the IvPFS $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S by

$$\begin{aligned}(\overline{\alpha}, \overline{\varrho}, \overline{\nu}) = \Big\{&(a, [0.1, 0.1], [0, 0], [0, 0]),\\ &(b, [0.1, 0.1], [0, 0], [0, 0]),\\ &(c, [0.1, 0.1], [0, 0], [0, 0]),\\ &(d, [0, 0], [0.9, 1], [0.9, 1])\Big\}.\end{aligned}$$

We have that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-interior ideal of S because $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ satisfies all conditions of IvPF-interior ideals of S.

Lemma 2.23. *Let $\mathcal{M}$ be an IvPFS of S. Thus $\mathcal{M}$ is an IvPF-interior ideal of S if and only if*

$$\mathcal{S} \,\bar{\circ}_p\, \mathcal{M} \,\bar{\circ}_p\, \mathcal{S} \subseteq \mathcal{M}.$$

Proof. The proof is similar to that for Lemma 2.17. □

Theorem 2.24. *Let M be a nonempty subset of S. Thus M is an interior ideal of S if and only if $(\overline{\kappa}_M, \overline{\kappa}'_M, \overline{\kappa}'_M)$ is an IvPF-interior ideal of S.*

Proof. The proof is the same fashion to that for Theorem 2.18. □

Theorem 2.25. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are IvPF-interior ideals of S, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also an IvPF-interior ideal of S.*

Proof. This proof is the same fashion as the proof of Theorem 2.19. □

Theorem 2.26. *Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-interior ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is an interior ideal of S.*

Proof. We assume that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-interior ideal of S. Let $I_1, I_2, I_3 \in CI[0,1]$ such that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$. By Theorem 2.8, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a subsemigroup of S. Let s, t be elements of S and v be an element of $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\overline{\alpha}(v) \succeq I_1, \overline{\varrho}(v) \preceq I_2, \overline{\nu}(v) \preceq I_3.$$

Thus

$$\begin{aligned}\overline{\alpha}(svt) &\succeq \overline{\alpha}(v) \succeq I_1,\\ \overline{\varrho}(svt) &\preceq \overline{\varrho}(v) \preceq I_2,\\ \overline{\nu}(svt) &\preceq \overline{\nu}(v) \preceq I_3.\end{aligned}$$

Hence, $svt \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is an interior ideal of S.

Conversely, suppose that for each $I_1, I_2, I_3 \in CI[0,1]$, if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} \neq \emptyset$, then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is an interior ideal of S. By Theorem 2.8, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-subsemigroup of S. Let $s, v, t \in S$ and let

$$I_1 = \overline{\alpha}(v), I_2 = \overline{\varrho}(v), I_3 = \overline{\nu}(v).$$

Then $I_1, I_2, I_3 \in CI[0,1]$. It is easy to see that $v \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. By assumption $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is an interior ideal of S. Then $svt \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\begin{aligned}\overline{\alpha}(svt) &\succeq I_1 = \overline{\alpha}(v),\\ \overline{\varrho}(svt) &\preceq I_2 = \overline{\varrho}(v),\\ \overline{\nu}(svt) &\preceq I_3 = \overline{\nu}(v).\end{aligned}$$

Hence, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-interior ideal of S. □

We define an interval-valued picture fuzzy quasi-ideal of a semigroup as follows

Definition 2.27. An IvPFS $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is called an *interval-valued picture fuzzy quasi-ideal (IvPF-quasi-ideal)* of S if for each $q \in S$,

1. $\overline{\alpha}(q) \succeq \text{rmin}\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\}$
2. $\overline{\varrho}(q) \preceq \text{rmax}\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\}$
3. $\overline{\nu}(q) \preceq \text{rmax}\{(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(q)\}$

Example 2.28. Consider the semigroup S in an Example 2.2, define the IvPFSs $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ of S by

$$\begin{aligned}(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) = \Big\{&(a, [1,1], [0,0], [0,0]),\\ &(b, [0,0], [1,1], [1,1]),\\ &(c, [0,0], [1,1], [1,1]),\\ &(d, [0,0], [1,1], [1,1])\Big\},\\ (\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) = \Big\{&(a, [1,1], [0,0], [0,0]),\\ &(b, [1,1], [0,0], [0,0]),\\ &(c, [1,1], [0,0], [0,0]),\\ &(d, [0,0], [1,1], [1,1])\Big\}.\end{aligned}$$

By the conditions of IvPF-quasi-ideal of S, $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ are IvPF-quasi-ideals of S.

Proposition 2.29. *Assume that for each $q \in S$, there exist $t, h, e, b \in S$ such that $q = th = eb$. Then an IvPFS $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ of S is an IvPF-quasi-ideal of S if and only if*

1. $\overline{\alpha}(q) \succeq \text{rmin}\{\overline{\alpha}(t), \overline{\alpha}(b)\}$,
2. $\overline{\varrho}(q) \preceq \text{rmax}\{\overline{\varrho}(t), \overline{\varrho}(b)\}$,
3. $\overline{\nu}(q) \preceq \text{rmax}\{\overline{\nu}(t), \overline{\nu}(b)\}$,

for all $q \in S$.

Proof. Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPF-quasi-ideal of S and let $q \in S$. Thus $q = th = eb$ for some $t, h, e, b \in S$. So that

$$\begin{aligned}(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q) &\succeq \text{rmin}\{\overline{\alpha}(t), \overline{\kappa}_S(h)\}\\ &= \text{rmin}\{\overline{\alpha}(t), \mathbf{1}\} = \overline{\alpha}(t),\\ (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q) &\succeq \text{rmin}\{\overline{\kappa}_S(e), \overline{\alpha}(b)\}\\ &= \text{rmin}\{\mathbf{1}, \overline{\alpha}(b)\} = \overline{\alpha}(b),\\ (\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q) &\preceq \text{rmax}\{\overline{\varrho}(t), \overline{\kappa}'_S(h)\}\\ &= \text{rmax}\{\overline{\varrho}(t), \mathbf{0}\} = \overline{\varrho}(t),\\ (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q) &\preceq \text{rmax}\{\overline{\kappa}'_S(e), \overline{\varrho}(b)\}\\ &= \text{rmax}\{\mathbf{0}, \overline{\varrho}(b)\} = \overline{\varrho}(b),\\ (\overline{\nu} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q) &\preceq \text{rmax}\{\overline{\nu}(t), \overline{\kappa}'_S(h)\}\\ &= \text{rmax}\{\overline{\nu}(t), \mathbf{0}\} = \overline{\nu}(t),\\ (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(q) &\preceq \text{rmax}\{\overline{\kappa}'_S(e), \overline{\nu}(b)\}\\ &= \text{rmax}\{\mathbf{0}, \overline{\nu}(b)\} = \overline{\nu}(b).\end{aligned}$$

Hence,

$$\begin{aligned}\overline{\alpha}(q) &\succeq \text{rmin}\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\}\\ &\succeq \text{rmin}\{\overline{\alpha}(t), \overline{\alpha}(b)\},\\ \overline{\varrho}(q) &\preceq \text{rmax}\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\}\\ &\preceq \text{rmax}\{\overline{\varrho}(t), \overline{\varrho}(b)\},\\ \overline{\nu}(q) &\preceq \text{rmax}\{(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(q)\}\\ &\preceq \text{rmax}\{\overline{\nu}(t), \overline{\nu}(b)\}.\end{aligned}$$

For the converse, let $q \in S$. Then $q = th = eb$ for some $t, h, e, b \in S$. Thus

$$\begin{aligned}\overline{\alpha}(q) &\succeq \text{rmin}\big\{\overline{\alpha}(t), \overline{\alpha}(b)\big\}\\ &= \text{rmin}\Big\{\underset{q=th}{\text{rsup}}\, \text{rmin}\{\overline{\alpha}(t), \overline{\kappa}_S(h)\},\\ &\qquad \underset{q=eb}{\text{rsup}}\, \text{rmin}\{\overline{\kappa}_S(e), \overline{\alpha}(b)\}\Big\}\\ &= \text{rmin}\big\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\big\},\end{aligned}$$

and

$$\begin{aligned}\overline{\varrho}(q) &\preceq \text{rmax}\big\{\overline{\varrho}(t), \overline{\varrho}(h)\big\},\\ &= \text{rmax}\Big\{\underset{q=th}{\text{rinf}}\, \text{rmax}\{\overline{\varrho}(t), \overline{\kappa}'_S(h)\},\\ &\qquad \underset{q=eb}{\text{rinf}}\, \text{rmax}\{\overline{\kappa}'_S(e), \overline{\varrho}(b)\}\Big\}\\ &= \text{rmax}\big\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\big\}.\end{aligned}$$

Similarly,

$$\overline{\nu}(q) \preceq \text{rmax}\big\{(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(q)\big\}.$$

Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-quasi-ideal of S. □

Lemma 2.30. *Let $\mathcal{Q} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus $\mathcal{Q}$ is an IvPF-quasi-ideal of S if and only if*

$$(\mathcal{Q} \,\bar{\circ}_p\, \mathcal{S}) \cap (\mathcal{S} \,\bar{\circ}_p\, \mathcal{Q}) \subseteq \mathcal{Q}.$$

Proof. First, we recall

$$\begin{aligned}\Big(\mathcal{Q} \,\bar{\circ}_p\, \mathcal{S}\Big) \cap \Big(\mathcal{S} \,\bar{\circ}_p\, \mathcal{Q}\Big) &= \Big((\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S) \cap (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha}),\\ &\quad (\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho}),\\ &\quad (\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})\Big).\end{aligned}$$

Next, we assume that $\mathcal{Q}$ is an IvPF-quasi-ideal of S. Let q be an element in S. Since $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-quasi-ideal of S,

$$\begin{aligned}&\big((\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S) \cap (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})\big)(q)\\ &\quad= \mathrm{rmin}\big\{(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S)(q), (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(q)\big\}\\ &\quad\preceq \overline{\alpha}(q),\\ &\big((\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})\big)(q)\\ &\quad= \mathrm{rmax}\big\{(\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(q)\big\}\\ &\quad\succeq \overline{\varrho}(q),\\ &\big((\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})\big)(q)\\ &\quad= \mathrm{rmax}\big\{(\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(q)\big\}\\ &\quad\succeq \overline{\nu}(q).\end{aligned}$$

Then we conclude that

$$\Big(\mathcal{Q} \,\bar{\circ}_p\, \mathcal{S}\Big) \cap \Big(\mathcal{S} \,\bar{\circ}_p\, \mathcal{Q}\Big) \subseteq \mathcal{Q}.$$

To prove the converse, we assume that

$$\Big(\mathcal{Q} \,\bar{\circ}_p\, \mathcal{S}\Big) \cap \Big(\mathcal{S} \,\bar{\circ}_p\, \mathcal{Q}\Big) \subseteq \mathcal{Q}.$$

Let q be an element of S. Thus

$$\begin{aligned}\overline{\alpha}(q) &\succeq \big((\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S) \cap (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})\big)(q)\\ &= \mathrm{rmin}\big\{(\overline{\alpha} \,\bar{\circ}\, \overline{\kappa}_S)(q), (\overline{\kappa}_S \,\bar{\circ}\, \overline{\alpha})(q)\big\},\\ \overline{\varrho}(q) &\preceq \big((\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})\big)(q)\\ &= \mathrm{rmax}\big\{(\overline{\varrho} \,\bar{\bullet}\, \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\varrho})(q)\big\},\\ \overline{\nu}(q) &\preceq \big((\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S) \cup (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})\big)(q)\\ &= \mathrm{rmax}\big\{(\overline{\nu} \,\bar{\bullet}\, \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \,\bar{\bullet}\, \overline{\nu})(q)\big\}.\end{aligned}$$

Therefore, $\mathcal{Q}$ is an IvPF-quasi-ideal of S. □

Theorem 2.31. *A nonempty subset Q is a quasi-ideal of S if and only if $(\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q)$ is an IvPF-quasi-ideal of S.*

Proof. Suppose that Q is a quasi-ideal of S. Then $QS \cap SQ \subseteq Q$. By Propositions 1.13 and 1.10, we have that

$$\begin{aligned}&\Big((\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q) \,\bar{\circ}_p\, (\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S)\Big)\\ &\qquad\cap \Big((\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S) \,\bar{\circ}_p\, (\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q)\Big)\\ &= (\overline{\kappa}_{QS}, \overline{\kappa}'_{QS}, \overline{\kappa}'_{QS}) \cap (\overline{\kappa}_{SQ}, \overline{\kappa}'_{SQ}, \overline{\kappa}'_{SQ})\\ &= (\overline{\kappa}_{QS\cap SQ}, \overline{\kappa}'_{QS\cap SQ}, \overline{\kappa}'_{QS\cap SQ})\\ &\subseteq (\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q).\end{aligned}$$

Therefore, $(\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q)$ is an IvPF-quasi-ideal of S.

To prove the converse, assume that $(\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q)$ is an IvPF-quasi-ideal of S. Hence,

$$\begin{aligned}&(\overline{\kappa}_{QS\cap SQ}, \overline{\kappa}'_{QS\cap SQ}, \overline{\kappa}'_{QS\cap SQ})\\ &\quad= (\overline{\kappa}_{QS}, \overline{\kappa}'_{QS}, \overline{\kappa}'_{QS}) \cap (\overline{\kappa}_{SQ}, \overline{\kappa}'_{SQ}, \overline{\kappa}'_{SQ})\\ &\quad= \Big((\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q) \,\bar{\circ}_p\, (\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S)\Big)\\ &\qquad\quad \cap \Big((\overline{\kappa}_S, \overline{\kappa}'_S, \overline{\kappa}'_S) \,\bar{\circ}_p\, (\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q)\Big)\\ &\quad\subseteq (\overline{\kappa}_Q, \overline{\kappa}'_Q, \overline{\kappa}'_Q).\end{aligned}$$

Thus $QS \cap SQ \subseteq Q$. Therefore, Q is a quasi-ideal of S. □

Theorem 2.32. *If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are IvPF-quasi-ideals of S, then $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is also an IvPF-quasi-ideal of S.*

Proof. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two IvPF-quasi-ideal of S. Then by Lemma 2.30 $(\mathcal{Q}_1 \,\bar{\circ}_p\, \mathcal{S}) \cap (\mathcal{S} \,\bar{\circ}_p\, \mathcal{Q}_1) \subseteq \mathcal{Q}_1$ and

$(\mathcal{Q}_2 \mathbin{\bar{\circ}_p} \mathcal{S}) \cap (\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{Q}_2) \subseteq \mathcal{Q}_2$. By Proposition 1.12,

$$\begin{aligned}
&\Big((\mathcal{Q}_1 \cap \mathcal{Q}_2) \mathbin{\bar{\circ}_p} \mathcal{S}\Big) \cap \Big(\mathcal{S} \mathbin{\bar{\circ}_p} (\mathcal{Q}_1 \cap \mathcal{Q}_2)\Big) \\
&= \Big((\mathcal{Q}_1 \mathbin{\bar{\circ}_p} \mathcal{S}) \cap (\mathcal{Q}_2 \mathbin{\bar{\circ}_p} \mathcal{S})\Big) \\
&\quad \cap \Big((\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{Q}_1) \cap (\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{Q}_2)\Big) \\
&= (\mathcal{Q}_1 \mathbin{\bar{\circ}_p} \mathcal{S}) \cap (\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{Q}_1) \\
&\quad \cap (\mathcal{Q}_2 \mathbin{\bar{\circ}_p} \mathcal{S}) \cap (\mathcal{S} \mathbin{\bar{\circ}_p} \mathcal{Q}_2) \\
&\subseteq \mathcal{Q}_1 \cap \mathcal{Q}_2.
\end{aligned}$$

Therefore, $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is an IvPF-quasi-ideal of S. □

Theorem 2.33. *Let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPFS of S. Thus $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-quasi-ideal of S if and only if for each $I_1, I_2, I_3 \in CI[0,1]$, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a quasi-ideal of S.*

Proof. First of all, we suppose that $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-quasi-ideal of S. Let $I_1, I_2, I_3 \in CI[0,1]$. Let $q \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} S \cap S(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then $q = th = eb$ for some $h, e \in S$ and $t, b \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. We get

$$\begin{aligned}
\overline{\alpha}(t) \succeq I_1, \overline{\varrho}(t) \preceq I_2, \overline{\nu}(t) \preceq I_3, \\
\overline{\alpha}(b) \succeq I_1, \overline{\varrho}(b) \preceq I_2, \overline{\nu}(b) \preceq I_3.
\end{aligned}$$

Thus

$$\begin{aligned}
(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q) &\succeq \mathrm{rmin}\{\overline{\alpha}(t), \overline{\kappa}_S(h)\} \\
&= \mathrm{rmin}\{\overline{\alpha}(t), \mathbf{1}\} = \overline{\alpha}(t),
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q) &\succeq \mathrm{rmin}\{\overline{\kappa}_S(e), \overline{\alpha}(b)\} \\
&= \mathrm{rmin}\{\mathbf{1}, \overline{\alpha}(b)\} = \overline{\alpha}(b).
\end{aligned}$$

Then

$$\begin{aligned}
\overline{\alpha}(q) &\succeq \mathrm{rmin}\big\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\big\} \\
&\succeq \mathrm{rmin}\big\{\overline{\alpha}(t), \overline{\alpha}(b)\big\} \succeq I_1,
\end{aligned}$$

also,

$$\begin{aligned}
(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q) &\preceq \mathrm{rmax}\{\overline{\varrho}(t), \overline{\kappa}'_S(h)\} \\
&= \mathrm{rmax}\{\overline{\varrho}(t), \mathbf{0}\} = \overline{\varrho}(t),
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q) &\preceq \mathrm{rmax}\{\overline{\kappa}'_S(e), \overline{\varrho}(b)\} \\
&= \mathrm{rmax}\{\mathbf{0}, \overline{\varrho}(b)\} = \overline{\varrho}(b).
\end{aligned}$$

Then

$$\begin{aligned}
\overline{\varrho}(q) &\preceq \mathrm{rmax}\big\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\big\} \\
&\preceq \mathrm{rmax}\big\{\overline{\varrho}(t), \overline{\varrho}(b)\big\} \preceq I_2.
\end{aligned}$$

Similarly, we get $\overline{\nu}(q) \preceq I_3$.
Hence, $q \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$.
Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a quasi-ideal of S.

Conversely, suppose that for each $I_1, I_2, I_3 \in CI[0,1]$, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a quasi-ideal of S. Let $q \in S$
Case 1: $q \notin S^2$. Then

$$\begin{aligned}
\overline{\alpha}(q) &\succeq \mathbf{0} = \mathrm{rmin}\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\}, \\
\overline{\varrho}(q) &\preceq \mathbf{1} = \mathrm{rmax}\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\}, \\
\overline{\nu}(q) &\preceq \mathbf{1} = \mathrm{rmax}\{(\overline{\nu} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\nu})(q)\}.
\end{aligned}$$

Case 2: $q \in S^2$. Then $q = tb$ for some $t, b \in S$. Let

$$\begin{aligned}
I_1 &= \mathrm{rmin}\{\overline{\alpha}(t), \overline{\alpha}(b)\}, \\
I_2 &= \mathrm{rmax}\{\overline{\varrho}(t), \overline{\varrho}(b)\}, \\
I_3 &= \mathrm{rmax}\{\overline{\nu}(t), \overline{\nu}(b)\}.
\end{aligned}$$

Then $I_1, I_2, I_3 \in CI[0,1]$. By assumption, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a quasi-ideal of S. It is clear that $t, b \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$q = tb \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}} S \cap S (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}.$$

Since $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$ is a quasi-ideal of S, we obtain that $q \in (\overline{\alpha}, \overline{\varrho}, \overline{\nu})_{\{I_1,I_2,I_3\}}$. Then

$$\begin{aligned}
\overline{\alpha}(q) \succeq I_1 &= \mathrm{rmin}\{\overline{\alpha}(t), \overline{\alpha}(b)\} \\
&= \mathrm{rmin}\{(\overline{\alpha} \mathbin{\bar{\circ}} \overline{\kappa}_S)(q), (\overline{\kappa}_S \mathbin{\bar{\circ}} \overline{\alpha})(q)\}, \\
\overline{\varrho}(q) \preceq I_2 &= \mathrm{rmax}\{\overline{\varrho}(t), \overline{\varrho}(b)\} \\
&= \mathrm{rmax}\{(\overline{\varrho} \mathbin{\bar{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\bar{\bullet}} \overline{\varrho})(q)\},
\end{aligned}$$

$$\begin{aligned}\overline{\nu}(q) \preceq I_3 &= \text{rmax}\{\overline{\nu}(t), \overline{\nu}(b)\} \\ &= \text{rmax}\{(\overline{\nu} \mathbin{\overline{\bullet}} \overline{\kappa}'_S)(q), (\overline{\kappa}'_S \mathbin{\overline{\bullet}} \overline{\nu})(q)\}.\end{aligned}$$

Therefore, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-quasi-ideal of S. □

Finally, we investigate some relationships between each type of the interval-valued picture fuzzy ideal.

Theorem 2.34. *If $\mathcal{R}$ is an IvPF-right ideal and $\mathcal{L}$ is an IvPF-left ideal of S, then $\mathcal{R} \cap \mathcal{L}$ is a IvPF-quasi-ideal of S.*

Proof. Let $\mathcal{R}$ be an IvPF-right ideal of S, and $\mathcal{L}$ be an IvPF-left ideal of S. Then

$$\begin{aligned}&\Big((\mathcal{R}\cap\mathcal{L}) \mathbin{\overline{\circ}_p} \mathcal{S}\Big) \cap \Big(\mathcal{S} \mathbin{\overline{\circ}_p} (\mathcal{R} \cap \mathcal{L})\Big) \\ &\quad= \Big((\mathcal{R} \mathbin{\overline{\circ}_p} \mathcal{S}) \cap (\mathcal{L} \mathbin{\overline{\circ}_p} \mathcal{S})\Big) \\ &\qquad\cap \Big((\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{R}) \cap (\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{L})\Big) \\ &\quad\subseteq \mathcal{R} \cap (\mathcal{L} \mathbin{\overline{\circ}_p} \mathcal{S}) \cap (\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{R}) \cap \mathcal{L} \\ &\quad\subseteq \mathcal{R} \cap \mathcal{L}.\end{aligned}$$

Consequently, $\mathcal{R} \cap \mathcal{L}$ is an IvPF-quasi-ideal of S. □

Theorem 2.35. *Let $\mathcal{K}$ be an IvPFS and $\mathcal{B}$ be an IvPF-bi-ideal of S. Then $\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}$ and $\mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{K}$ are both IvPF-bi-ideals of S.*

Proof. We see that

$$\begin{aligned}(\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}) \mathbin{\overline{\circ}_p} (\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}) &= \mathcal{K} \mathbin{\overline{\circ}_p} (\mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}) \\ &\subseteq \mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}.\end{aligned}$$

Then $\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}$ is an IvPF-subsemigroup of S. Next, we have

$$\begin{aligned}(\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}) \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} (\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}) &\subseteq \mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B} \mathbin{\overline{\circ}_p} (\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{S}) \mathbin{\overline{\circ}_p} \mathcal{B} \\ &\subseteq \mathcal{K} \mathbin{\overline{\circ}_p} (\mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{B}) \\ &\subseteq \mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}.\end{aligned}$$

Then $\mathcal{K} \mathbin{\overline{\circ}_p} \mathcal{B}$ is an IvPF-bi-ideal of S.

Likewise, we can observe that $\mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{K}$ is an IvPF-bi-ideal of S □

Theorem 2.36. *If Q is an IvPF-quasi-ideal of S, then Q is an IvPF-bi-ideal of S.*

Proof. Let Q be an IvPF-quasi-ideal of S. Since

$Q \mathbin{\overline{\circ}_p} Q \subseteq \mathcal{S} \mathbin{\overline{\circ}_p} Q$ and $Q \mathbin{\overline{\circ}_p} Q \subseteq Q \mathbin{\overline{\circ}_p} \mathcal{S}$,

we have

$$Q \mathbin{\overline{\circ}_p} Q \subseteq \big(Q \mathbin{\overline{\circ}_p} \mathcal{S}\big) \cap \big(\mathcal{S} \mathbin{\overline{\circ}_p} Q\big) \subseteq Q.$$

Therefore, Q is an IvPF-subsemigroup of S. Since

$$Q \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} Q \subseteq Q \mathbin{\overline{\circ}_p} \mathcal{S} \text{ and } Q \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} Q \subseteq \mathcal{S} \mathbin{\overline{\circ}_p} Q,$$

we have

$$Q \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} Q \subseteq \big(Q \mathbin{\overline{\circ}_p} \mathcal{S}\big) \cap \big(\mathcal{S} \mathbin{\overline{\circ}_p} Q\big) \subseteq Q.$$

Hence, Q is an IvPF-bi-ideal of S. □

Corollary 2.37. *If Q_1 or Q_2 is an IvPF-quasi-ideal of S, then $Q_1 \mathbin{\overline{\circ}_p} Q_2$ is an IvPF-bi-ideal of S.*

Proof. The result is following Theorems 2.35 and 2.36. □

Theorem 2.38. *Let S be a regular semigroup. Then $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-ideal of S if and only if $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-interior ideal of S.*

Proof. Let $\mathcal{I} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPF-ideal of S. Then

$$\mathcal{I} \mathbin{\overline{\circ}_p} \mathcal{S} \subseteq \mathcal{I} \text{ and } \mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{I} \subseteq \mathcal{I}, \text{ so}$$

$$\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{I} \mathbin{\overline{\circ}_p} \mathcal{S} \subseteq \mathcal{I}.$$

Hence, $\mathcal{I}$ is an IvPF-interior ideal of S.

In another way, let $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPF-interior ideal of S. Let x and y be elements in S. Since S is regular, we have $a, b \in S$ where $x = xax$ and $y = yby$. Then

$$\begin{aligned}
\overline{\alpha}(xy) &= \overline{\alpha}(xy(by)) \succeq \overline{\alpha}(y),\\
\overline{\varrho}(xy) &= \overline{\varrho}(xy(by)) \preceq \overline{\varrho}(y),\\
\overline{\nu}(xy) &= \overline{\nu}(xy(by)) \preceq \overline{\nu}(y),\\
\overline{\alpha}(xy) &= \overline{\alpha}((xa)xy) \succeq \overline{\alpha}(x),\\
\overline{\varrho}(xy) &= \overline{\varrho}((xa)xy) \preceq \overline{\varrho}(y),\\
\overline{\nu}(xy) &= \overline{\nu}((xa)xy) \preceq \overline{\nu}(y).
\end{aligned}$$

Hence, $(\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ is an IvPF-ideal of S. □

Theorem 2.39. *For any IvPF-right ideal $\mathcal{R}$ and IvPF-left ideal $\mathcal{L}$ of S, then*

S is regular if and only if $\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} = \mathcal{R} \cap \mathcal{L}$.

Proof. Assume that S is regular. Let $\mathcal{R} = (\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ be an IvPF-right ideal of S, and $\mathcal{L} = (\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ be an IvPF-left ideal of S. Thus, we have

$$\begin{aligned}
\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} &\subseteq \mathcal{R} \,\overline{\circ}_p\, \mathcal{S} \subseteq \mathcal{R},\\
\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} &\subseteq \mathcal{S} \,\overline{\circ}_p\, \mathcal{L} \subseteq \mathcal{L}.
\end{aligned}$$

Hence,

$$\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}.$$

Let $x \in S$. Since S is regular, $x = xsx$ for some $s \in S$. Thus,

$$\begin{aligned}
(\overline{\alpha}_1 \,\overline{\circ}\, \overline{\alpha}_2)(x) &= \operatorname*{rsup}_{x=ab} \operatorname{rmin}\{\overline{\alpha}_1(a), \overline{\alpha}_2(b)\}\\
&\succeq \operatorname{rmin}\{\overline{\alpha}_1(xs), \overline{\alpha}_2(x)\}\\
&\succeq \operatorname{rmin}\{\overline{\alpha}_1(x), \overline{\alpha}_2(x)\},\\
(\overline{\varrho}_1 \,\overline{\bullet}\, \overline{\varrho}_2)(x) &= \operatorname*{rinf}_{x=ab} \operatorname{rmax}\{\overline{\varrho}_1(a), \overline{\varrho}_2(b)\}\\
&\preceq \operatorname{rmax}\{\overline{\varrho}_1(xs), \overline{\varrho}_2(x)\}\\
&\preceq \operatorname{rmax}\{\overline{\varrho}_1(x), \overline{\varrho}_2(x)\},\\
(\overline{\nu}_1 \,\overline{\bullet}\, \overline{\nu}_2)(x) &= \operatorname*{rinf}_{x=ab} \operatorname{rmax}\{\overline{\nu}_1(a), \overline{\nu}_2(b)\}\\
&\preceq \operatorname{rmax}\{\overline{\nu}_1(xs), \overline{\nu}_2(x)\}\\
&\preceq \operatorname{rmax}\{\overline{\nu}_1(x), \overline{\nu}_2(x)\}.
\end{aligned}$$

Therefore, $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \,\overline{\circ}_p\, \mathcal{L}$.
Consequently, $\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} = \mathcal{R} \cap \mathcal{L}$.

To prove the converse, we assume that $\mathcal{R}$ is an IvPF-right ideal and $\mathcal{L}$ an IvPF-left ideal of S such that

$$\mathcal{R} \,\overline{\circ}_p\, \mathcal{L} = \mathcal{R} \cap \mathcal{L}.$$

Let R be a right ideal and L be a left ideal of S. Then $(\overline{\kappa}_R, \overline{\kappa}'_R, \overline{\kappa}'_R)$ is an IvPF-right ideal and $(\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)$ is an IvPF-left ideal of S. Since $RL \subseteq RS \subseteq R$ and $RL \subseteq SL \subseteq L$, we get $RL \subseteq R \cap L$. Let $x \in R \cap L$. By Proposition 1.13 and assumption,

$$\begin{aligned}
&(\overline{\kappa}_{RL}, \overline{\kappa}'_{RL}, \overline{\kappa}'_{RL})\\
&\quad= (\overline{\kappa}_R, \overline{\kappa}'_R, \overline{\kappa}'_R) \,\overline{\circ}_p\, (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)\\
&\quad= (\overline{\kappa}_R, \overline{\kappa}'_R, \overline{\kappa}'_R) \cap (\overline{\kappa}_L, \overline{\kappa}'_L, \overline{\kappa}'_L)\\
&\quad= (\overline{\kappa}_{R\cap L}, \overline{\kappa}'_{R\cap L}, \overline{\kappa}'_{R\cap L}).
\end{aligned}$$

So that $\overline{\kappa}_{RL}(x) = \overline{\kappa}_{R\cap L}(x) = \mathbf{1}$, and $\overline{\kappa}'_{RL}(x) = \overline{\kappa}'_{R\cap L}(x) = \mathbf{0}$. Then $x \in RL$. Therefore, $R \cap L \subseteq RL$. Accordingly, $R \cap L = RL$. Hence, S is regular. □

Theorem 2.40. *In a regular semigroup, an IvPF-bi-ideal and an IvPF-quasi-ideal are coincide.*

Proof. By Theorem 2.36, every IvPF-quasi-ideal of S is an IvPF-bi-ideal of S.

Let $\mathcal{B} = (\overline{\alpha}, \overline{\varrho}, \overline{\nu})$ be an IvPF-bi-ideal of a regular semigroup S. Then

$$\big(\mathcal{B} \,\overline{\circ}_p\, \mathcal{S}\big) \,\overline{\circ}_p\, \mathcal{S} \subseteq \mathcal{B} \,\overline{\circ}_p\, \mathcal{S}$$

and

$$\mathcal{S} \,\overline{\circ}_p\, \big(\mathcal{S} \,\overline{\circ}_p\, \mathcal{B}\big) \subseteq \mathcal{S} \,\overline{\circ}_p\, \mathcal{B}.$$

Thus $\mathcal{B} \,\overline{\circ}_p\, \mathcal{S}$ is an IvPF-right ideal of S and $\mathcal{S} \,\overline{\circ}_p\, \mathcal{B}$ is an IvPF-left ideal of S. By Theorem 2.39,

$$(\mathcal{B} \,\overline{\circ}_p\, \mathcal{S}) \cap (\mathcal{S} \,\overline{\circ}_p\, \mathcal{B})$$

$$= (\mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{S}) \mathbin{\overline{\circ}_p} (\mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{B})$$
$$\subseteq \mathcal{B} \mathbin{\overline{\circ}_p} \mathcal{S} \mathbin{\overline{\circ}_p} \mathcal{B}$$
$$\subseteq \mathcal{B}.$$

Consequently, $\mathcal{B}$ is an IvPF-quasi-ideal of S. □

Example 2.41. Consider the semigroup $S = \{a, b, c\}$ with a multiplication table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

and define the IvPFSs $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ of S by

$$\begin{aligned}(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) &= \Big\{(a, [1.0, 1.0], [0.0, 0.0], [0.0, 0.0]),\\ &\quad (b, [0.0, 0.0], [1.0, 1.0], [1.0, 1.0]),\\ &\quad (c, [0.0, 0.0], [1.0, 1.0], [1.0, 1.0])\Big\},\end{aligned}$$

$$\begin{aligned}(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2) &= \Big\{(a, [1.0, 1.0], [0.0, 0.0], [0.0, 0.0]),\\ &\quad (b, [1.0, 1.0], [0.0, 0.0], [0.0, 0.0]),\\ &\quad (c, [0.0, 0.0], [1.0, 1.0], [1.0, 1.0])\Big\}.\end{aligned}$$

Then $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1)$ and $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ are IvPF-right ideal and IvPF-left ideal of S, respectively. Since S is a regular semigroup, we have $(\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) \mathbin{\overline{\circ}_p} (\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2) = (\overline{\alpha}_1, \overline{\varrho}_1, \overline{\nu}_1) \cap (\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ by Theorem 2.39.

Moreover, $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ is an IvPF-bi ideal of S. By Theorem 2.40, $(\overline{\alpha}_2, \overline{\varrho}_2, \overline{\nu}_2)$ is also an IvPF-quasi ideal of S.

3. Conclusion

An IvPF-subsemigroup, an IvPF-left ideal[right ideal, ideal, bi-ideal, interior ideal, quasi-ideal] of semigroups were defined. Some properties were investigated. Let A be a nonempty subset of S. Then A is a subsemigroup (a left ideal, a right ideal, an ideal, a bi-ideal, an interior ideal, a quasi-ideal) of S if and only if $(\overline{\kappa}_A, \overline{\kappa}'_A, \overline{\kappa}'_A)$ is an IvPF-subsemigroup (an IvPF-left ideal[right ideal, ideal, bi-ideal, interior ideal, quasi-ideal]) of S. These are the relationships between each ideal of semigroups and its interval-valued picture fuzzification. Every IvPF-quasi-ideal of S is an IvPF-bi-ideal of S. An IvPF-ideal and an IvPF-interior ideal of a regular semigroup S are identical. Moreover, in a regular semigroup, an IvPF-bi-ideal of S and an IvPF-quasi-ideal are coincident.

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