E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia* Science & Technology Asia
Vol.30 No.2 April - June 2025 Page: [38-46]

Original research article

Functions Whose Images are Terms of a Weakly Fixed Variable

Chortip Siwapornanan, Thodsaporn Kumduang, Prapairat Junlouchai*

Faculty of Science and Technology, Rajamangala University of Technology Rattanakosin, Nakhon Pathom 73170, Thailand

Received 4 June 2024; Received in revised form 2 April 2025
Accepted 13 April 2025; Available online 18 June 2025

ABSTRACT

In this paper, the set of terms of a weakly fixed variable of type τ which is a generalization of terms of a fixed variable is introduced. Applying the generalized superposition operation, the superassociative algebra of terms of a weakly fixed variable is formed. Binary associative systems induced by such operations are obtained. We also discuss properties of functions whose images are terms of a weakly fixed variable called wfv-generalized hypersubstitions.

Keywords: Function; Semigroup; Superposition; Term

1. Introduction and Preliminaries

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called variables. The set $X_n = \{x_1, x_2, \ldots, x_n\}$ denotes the set of n elements, i.e., $\{x_1, \ldots, x_n\}$. Let $(f_i)_{i\in I}$ be an n_i-ary operation symbol, where $n_i \geq 1$ is a natural number. The type $\tau = (n_i)_{i\in I}$ is a sequence of n-ary operation symbols f_i. Formally, an n-ary term of type τ is defined as follows: (1) the variable $x_j \in X_n$ is an n-ary term of type τ and (2) $f_i(t_1, \ldots, t_{n_i})$ is also an n-ary term of type τ if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ previously. The set of all n-ary terms of type τ is denoted by $W_\tau(X_n)$. Over alphabet X, we write $W_\tau(X)$ instead of $W_\tau(X_n)$. Moreover, $var(t)$ stands for the set of all variables in a term t. Recent progress in the study of terms can be seen in [1–5]. Besides, terms under which the longest path form the root to each vertex is equal called completely expanded terms were mentioned in [6].

Terms used in the theory of strong hyperidentities and solid varieties, see [7, 8], can be computed by the generalized superposition $S^n : W_\tau(X)^{n+1} \to W_\tau(X)$, where n is a fixed positive integer, which is defined by the following steps: for $t \in W_\tau(X)$,

*Corresponding author: prapairat.jun@rmutr.ac.th
doi: 10.14456/scitechasia.2025.24

(1) if t is a variable $x_j, 1 \leq j \leq n$, then $S^n(x_j, t_1, \ldots, t_n) = t_j$,

(2) if t is a variable x_j from $X \setminus X_n$, then $S^n(x_j, t_1, \ldots, t_n) = x_j$,

(3) if $t = f_i(s_1, \ldots, s_{n_i})$ for any $s_1, \ldots, s_{n_i} \in W_\tau(X)$, then $S^n(t, t_1, \ldots, t_n)$ $= f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$ where $S^n(s_i, t_1, \ldots, t_n)$ for all $1 \leq i \leq n$ are defined.

It was proved in [9] that the operation S^n is superassociative, which means that it satisfies the equation $S^n(S^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$ $= S^n(a, S^n(b_1, d_1, \ldots, d_n), \ldots, S^n(b_n, d_1, \ldots, d_n))$. For more details on superassociativity, see [10].

In 2020, Wattanatripop and Changphas [11] observed that there are many equations having only one variable, for instance $x + x = x$, an idempotent law. For this reason, a specific class of terms called a term of a fixed variable of type τ was inductively defined as follows:

(1) every $x_i \in X_n$ is an n-ary term of a fixed variable of type τ,

(2) if $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable of type τ and if $var(t_j) = var(t_k)$ for all $1 \leq j < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable of type τ.

Let $W_\tau^{fv}(X_n)$ be a set of all n-ary terms of a fixed variable of type τ. Furthermore, we write $W_\tau^{fv}(X) = (W_\tau^{fv}(X_n))_{n \geq 1}$. Let us see the following examples: for a binary operation symbol Δ of type (2), $x_1, x_2, x_3, x_7, \Delta(x_1, x_1), \Delta(\Delta(x_3, x_3), x_3)$ are elements in $W_{(2)}^{fv}(X)$, but $\Delta(x_2, x_4)$ and $\Delta(x_1, \Delta(x_1, x_2))$ are not. For more details and backgrounds on terms of a fixed variable, the reader is referred to [12–14].

We remark that the condition $var(t_1) = var(t_2) = \cdots = var(t_{n_i})$ is strong, which means that there is one variable occuring in a term of a fixed variable. This leads us to seek for a weaker one. Thus, as a continuation of the papers [11–14], the main purposes of this work are to introduce a generalization of terms of a fixed variable of arbitrary type under which in some position the sets of variables are identical and to apply the generalized superposition to this set. In Section 2, we also define binary operations arising from such operations and study some algebraic properties. The set of generalized hypersubstititons whose ranges are restricted to the set of terms of a weakly fixed variable is given in Section 3.

2. Construction of Terms of a Weakly Fixed Variable

In this section, we begin with the definition of terms of a weakly fixed variable of arbitrary type and some examples.

Definition 2.1. A term of a weakly fixed variable of type τ is inductively defined by

(1) every variable x_i in X is a term of a weakly fixed variable of type τ,

(2) if $t_1, \ldots, t_{n_i}$ are terms of a weakly fixed variable type τ and $var(t_j) = var(t_k)$ for some $1 \leq j < k \leq n_i$, then $f(t_1, \ldots, t_{n_i})$ is a term of a weakly fixed variable of type τ,

(3) the set $W_\tau^{wfv}(X)$ of all terms of a weakly fixed variable of type τ is the smallest set which is closed under finite application of (2).

Example 2.2. Consider type $\tau = (3, 2)$ with ternary operation symbol f_1 and binary operation symbol f_2 on infinite set of variable X. Then some examples of elements in $W^{wfv}_{(3,2)}(X)$ are:

$$x_1, x_2, x_7, f_1(x_1, x_1, x_2), f_1(x_1, x_5, x_5), f_2(x_6, x_6), f_2(f_2(x_4, x_4), x_4), f_1(f_2(x_1, x_1), x_3, x_1), f_2(f_1(x_8, x_8, x_9), f_1(x_8, x_8, x_9).$$

In this list, we see that x_1, x_2, x_7 and $f_2(x_6, x_6)$ also belong to the sets $W^{wfv}_{(3,2)}(X)$ and $W^{fv}_{(3,2)}(X)$, but the others only contains in the $W^{wfv}_{(3,2)}(X)$.

Example 2.3. Let us consider type $\tau = (5, 4, 3)$ with 5-ary operation symbol f_1, quaternary operation symbol f_2 and ternary operation symbol f_3 on infinite set of variable X. Then some elements in $W^{wfv}_{(5,4,3)}(X)$ are:

$$x_1, x_2, f_1(x_1, x_2, x_1, x_3, x_4), f_2(x_3, x_5, x_7, x_5), f_3(x_9, x_8, x_8), f_1(x_1, f_3(x_2, x_1, x_1), f_3(x_1, x_2, x_1), x_3, x_2).$$

On the other hand, the following do not contain in $W^{wfv}_{(5,4,3)}(X)$:

$$f_1(x_1, x_2, x_3, x_4, x_5),$$
$$f_3(f_2(x_7, x_7, x_1, x_2), f_3(x_1, x_2, x_3), f_3(x_1, x_7, x_1)).$$

Remark 2.4. *The relationship between $W^{fv}_\tau(X)$ and $W^{wfv}_\tau(X)$ is described as follows:*

(1) if $\tau = (1, 1, \ldots)$ or $\tau = (2, 2, \ldots)$, then $W^{wfv}_\tau(X) = W^{fv}_\tau(X)$,

(2) if $\tau = (n_i)_{i\in I}$ where $n_i > 2$, then $W^{fv}_\tau(X) \subset W^{wfv}_\tau(X)$.

To ensure that the set $W^{wfv}_\tau(X)$ is closed under an application of the operation S^n, we prove the following lemma.

Lemma 2.5. *For a positive integer n, if $s, t_1, \ldots, t_n$ are terms of a weakly fixed variable of type τ, then $S^n(s, t_1, \ldots, t_n)$ is a term of a weakly fixed variable of type τ.*

Proof. We give a proof by induction on the complexity of a term s. If $s = x_i \in X_n$, then $S^n(s, t_1, \ldots, t_n) = t_i \in W^{wfv}_\tau(X)$. If $s = x_i \in X \setminus X_n$, then $S^n(s, t_1, \ldots, t_n) = s \in W^{wfv}_\tau(X)$. If $s = f_i(s_1, \ldots, s_{n_i})$ and without loss of generality, we assume that $var(s_l) = var(s_k)$ for some $l, k \in \{1, \ldots, n_i\}$ and $l \neq k$, we prove that $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) \in W^{wfv}_\tau(X)$. Because $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)$ $= f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$, we need to show that the term $f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$ belongs to $W^{wfv}_\tau(X)$. This means that

(1) $S^n(s_j, t_1, \ldots, t_n) \in W^{wfv}_\tau(X)$ for all $1 \leq j \leq n_i$,

(2) $var(S^n(s_l, t_1, \ldots, t_n)) = var(S^n(s_k, t_1, \ldots, t_n))$ for some $1 \leq l < k \leq n_i$.

To show that the condition (1) holds, let $j \in \{1, \ldots, n_i\}$. If s_j is a variable from X_n, then $S^n(s_j, t_1, \ldots, t_n)$ belongs to $W^{wfv}_\tau(X)$. If s_j is a variable from $X \setminus X_n$, then $S^n(S_j, t_1, \ldots, t_n)$ equals s_j, which contains in $W^{wfv}_\tau(X)$. Let $s_j = f_i(s'_1, \ldots, s'_{n_i})$ and $S^n(s'_p, t_1, \ldots, t_n) \in W^{wfv}_\tau(X)$ for all $p = 1, \ldots, n_i$. Without loss of generality, suppose that $var(s'_l) = var(s'_k)$ for some $1 \leq l < k \leq n_i$. Then we obtain $S^n(f_i(s'_1, \ldots, s'_{n_i}), t_1, \ldots, t_n)$ $= f_i(S^n(s'_1, t_1, \ldots, t_n), \ldots, S^n(s'_n, t_1, \ldots, t_n))$ belongs to $W^{wfv}_\tau(X)$ because $var(s'_l) = var(s'_k)$. Moreover, $var(S^n(s'_l, t_1, \ldots, t_n)) = var(S^n(s'_k, t_1, \ldots, t_n))$. Thus, the condition (1) holds. Next, we show that (2) holds. Since we know that there exist $l, k \in \{1, \ldots, n_i\}$ and $l \neq k$ such that $var(s_l) = var(s_k)$, by the definition of S^n, we obtain $var(S^n(s_l, t_1, \ldots, t_n)) = var(S^n(s_k, t_1, \ldots, t_n))$. This completes the proof. □

As a consequence, by Lemma 2.5, we have the folllowing results.

Theorem 2.6. $(W_\tau^{wfv}(X), S^n, (x_i)_{i\geq 1})$ *is a subalgebra of* $(W_\tau(X), S^n, (x_i)_{i\geq 1})$.

Proof. It follows immediately from Lemma 2.5. □

The fact that the generalized clone axioms are also valid in the algebra $(W_\tau^{wfv}(X), S^n, (x_i)_{i\geq 1})$ is now proposed.

Theorem 2.7. $(W_\tau^{wfv}(X), S^n, (x_i)_{i\geq 1})$ *satisfies the following axioms:*

(1) $S^n(S^n(s, t_1, \ldots, t_n), u_1, \ldots, u_n) = S^n(s, S^n(t_1, u_1, \ldots, u_n), \ldots, S^n(t_n, u_1, \ldots, u_n))$,

(2) $S^n(x_i, t_1, \ldots, t_n) = t_i$ *for* $1 \leq i \leq n$,

(3) $S^n(x_i, t_1, \ldots, t_n) = x_i$ *for* $i > n$,

(4) $S^n(t, x_1, \ldots, x_n) = t$.

Proof. It follows from Theorem 2.6. □

Other properties of the algebra $(W_\tau(X), S^n, (x_i)_{i\geq 1})$, we refer to [9].

From Theorem 2.7, we say that the algebra $(W_\tau^{wfv}(X), S^n, (x_i)_{i\geq 1})$ is a unitary superassociative algebra where each variable x_i in X acts as a scalar.

In the paper [15], consider s and t in $W_\tau(X)$. The operations $+^G$ and $\cdot^r$, where $1 \leq r \leq n$, are defined on $W_\tau(X)$ as the following:

(1) $+^G : W_\tau(X)^2 \to W_\tau(X)$ is defined by $s +^G t = S^n(s, t, \ldots, t)$,

(2) $\cdot^r : W_\tau(X)^2 \to W_\tau(X)$ is defined by $s \cdot^r t = S^n(s, x_1, x_2, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n)$.

Then we have the following results.

Theorem 2.8. *For a type* $\tau_n = (n_i)_{i\in I}$ *where* $n_i = n$, *the following statement holds:*

(1) $W_{\tau_n}^{wfv}(X)$ *is a subsemigroup of* $W_{\tau_n}(X)$ *with respect to the operation* $+^G$,

(2) $W_{\tau_n}^{wfv}(X)$ *is a subsemigroup of* $W_{\tau_n}(X)$ *with respect to the operation* $\cdot^r$.

Proof. We show that $s +^G t \in W_{\tau_n}^{wfv}(X)$. To do this, we consider a few cases. If s and t are variables from X, it is clear that $s +^G t \in W_{\tau_n}^{wfv}(X)$. If $s = x_k$ for $1 \leq k \leq n$ and $t = f_i(t_1, \ldots, t_{n_i})$, then $s +^G t = x_k +^G f_i(t_1, \ldots, t_{n_i}) = S^n(x_k, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i})) = f_i(t_1, \ldots, t_{n_i})$. Thus $s +^G t \in W_{\tau_n}^{wfv}(X)$. If $s = x_k$ for $n < k$ and $t = f_i(t_1, \ldots, t_{n_i})$, then $s +^G t = x_k +^G f_i(t_1, \ldots, t_{n_i}) = S^n(x_k, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i})) = x_k$. Thus $s +^G t \in W_{\tau_n}^{wfv}(X)$. If $s = f_i(s_1, \ldots, s_{n_i})$ and t is variable in X, and we assume that $var(s_k) = var(s_m)$ for some $1 \leq k < m \leq n_i$. Then $s +^G t = f_i(s_1, \ldots, s_{n_i}) +^G x_l = f_i(S^n(s_1, x_l, \ldots, x_l), \ldots, S^n(s_{n_i}, x_l, \ldots, x_l))$. Because $var(s_k) = var(s_m)$ we have that $var(S^n(s_k, x_l, \ldots, x_l)) = var(S^n(s_m, x_l, \ldots, x_l))$. Hence $s +^G t \in W_{\tau_n}^{wfv}(X)$. If $s = f_i(s_1, \ldots, s_{n_i})$ and $t = f_i(t_1, \ldots, t_{n_i})$. Suppose that $var(s_{k_1}) = var(s_{m_1})$ for some $1 \leq k_1 < m_1 \leq n_i$ and $var(t_{k_2}) = var(t_{m_2})$ for some $1 \leq k_2 < m_2 \leq n_i$. Then $s +^G t$

$$\begin{aligned} &= f_i(s_1, \ldots, s_{n_i}) +^G f_i(t_1, \ldots, t_{n_i}) \\ &= S^n(f_i(s_1, \ldots, s_{n_i}), f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i})) \\ &= f_i(S^n(s_1, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i})), \ldots, S^n(s_{n_i}, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i}))). \end{aligned}$$

Since $var(s_{k_1}) = var(s_{m_1})$, we get that $var(S^n(s_{k_1}, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i}))$ $= var(S^n(s_{m_1}, f_i(t_1, \ldots, t_{n_i}), \ldots, f_i(t_1, \ldots, t_{n_i}))$. Therefore $s +^G t \in W^{wfv}_{\tau_n}(X)$.

To prove that (2) holds. It is clear that $s \cdot^r t \in W^{wfv}_{\tau_n}(X)$ for $s, t \in X$. If $s = x_k$ for $1 \le k \le n$ and $t = f_i(t_1, \ldots, t_{n_i})$, then $s \cdot^r t = S^n(x_k, x_1, \ldots, x_{r-1}, f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n)$ $= x_k$ for $k \ne r$ and $s \cdot^r t = f_i(t_1, \ldots, t_{n_i})$ for $k = r$. So $s \cdot^r t \in W^{wfv}_{\tau_n}(X)$. If $s = x_k$ for $n < k$ and $t = f_i(t_1, \ldots, t_{n_i})$, then it is clearly $s \cdot^r t = x_k \in W^{wfv}_{\tau_n}(X)$. If $s = f_i(s_1, \ldots, s_{n_i})$ and t is variables in X and we assume that $var(s_k) = var(s_m)$ for some $1 \le k < m \le n_i$. Then $s \cdot^r t =$ $S^n(f_i(s_1, \ldots, s_{n_i}), x_l, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n)$ $= f_i(S^n(s_1, x_l, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n),$ $\ldots, S^n(s_{n_i}, x_l, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n))$. Because of $var(s_k) = var(s_m)$, we get that $var(S^n(s_k, x_l, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n)) =$ $var(S^n(s_m, x_l, \ldots, x_{r-1}, t, x_{r+1}, \ldots, x_n))$, thus $s \cdot^r t = x_k \in W^{wfv}_{\tau_n}(X)$. Let us consider $s = f_i(s_1, \ldots, s_{n_i})$ and $t = f_i(t_1, \ldots, t_{n_i})$ and assume that $var(s_{k_1}) = var(s_{m_1})$ for some $1 \le k_1 < m_1 \le n_i$ and $var(t_{k_2}) = var(t_{m_2})$ for some $1 \le k_2 < m_2 \le n_i$. Then $s \cdot^r t = S^n(f_i(s_1, \ldots, s_{n_i}), x_1, \ldots, x_{r-1},$ $f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n) =$ $f_i(S^n(s_1, x_1, \ldots, x_{r-1}, f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n),$ $\ldots, S^n(s_{n_i}, x_1, \ldots, x_{r-1}, f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n))$. Since $var(s_{k_1}) = var(s_{m_1})$, then we have $var(S^n(s_{k_1}, x_1, \ldots, x_{r-1}, f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n))$ $= var(S^n(s_{m_1}, x_1, \ldots, x_{r-1}, f_i(t_1, \ldots, t_{n_i}), x_{r+1}, \ldots, x_n))$. Therefore, $s \cdot^r t \in W^{wfv}_{\tau_n}(X)$. □

Example 2.9. Let A be the set $\{x_2, x_5, f(x_4, x_5, x_4), g(x_7, x_7, x_7)\}$, which is a subset of $W^{wfv}_{(3,3)}(X)$. The results of operations $+^G$ and $\cdot^3$, are shown in the Tables 1-2.

3. Functions Whose Images are Terms of a Weakly Fixed Variable

This section starts with recalling the definition of generalized hypersubstitutions. Recall that the generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$, which does not necessarily preserve the arity, and the set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. By the symbol σ_t we mean a generalized hypersubstitution which takes each operation symbol to a term t.

It may be seen that we can not apply the usual composition of funtions to the set $Hyp_G(\tau)$. So it is possible to set some preparations. Actually, the extension $\widehat{\sigma}$ of each σ on $Hyp_G(\tau)$ is defined by

(1) $\widehat{\sigma}[x_i] = x_i \in X$,

(2) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})]$ $= S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and $\widehat{\sigma}[t_j], 1 \le j \le n_i$, are already defined.

Thus the binary operation $\circ_G : Hyp_G(\tau)^2 \to Hyp_G(\tau)$ is defined by $\sigma_1 \circ_G \sigma_2 = \widehat{\sigma_1} \circ \sigma_2$. Let σ_{id} be a hypersubstitution mapping which maps each n_i-ary operation symbol f_i to the term $f_i(x_i, \ldots, x_{n_i})$. The fact that $(Hyp_G(\tau), \circ_G, \sigma_{id})$ forms a monoid was proved in [7, 9, 16].

Another binary composition on the set $Hyp_G(\tau_n)$ is denoted by $+^G$. By the definition, for all $i \in I$ and $\alpha, \beta \in Hyp_G(\tau_n)$, $(\alpha +^G \beta)(f_i) = S^n(\alpha(f_i), \beta(f_i), \ldots, \beta(f_i))$. Because of the fact that a generalized superposition S^n is superassociative, thus $+^G$ is associative. Consequently, we have a semigroup $(Hyp_G(\tau_n), +^G)$.

In this section, we restrict our study from the set $W_\tau(X)$ to the set $W^{wfv}_\tau(X)$ of all terms of a weakly fixed variable. It is

Table 1. Example of the computational process of elements in $A \subset W^{wfv}_{(3,3)}(X)$ under $+^G$.

$+^G$	x_2	x_5	$f(x_4,x_5,x_4)$	$g(x_7,x_7,x_7)$
x_2	x_2	x_5	$f(x_4,x_5,x_4)$	$g(x_7,x_7,x_7)$
x_5	x_5	x_5	x_5	x_5
$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$
$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$

Table 2. Example of the computational process of elements in $A \subset W^{wfv}_{(3,3)}(X)$ under $\cdot^3$.

$\cdot^3$	x_2	x_5	$f(x_4,x_5,x_4)$	$g(x_7,x_7,x_7)$
x_2	x_2	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_5
$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$	$f(x_4,x_5,x_4)$
$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$	$g(x_7,x_7,x_7)$

known that an identity $xyx \approx x$ is an identity which uses to classify any semigroup to a variety of regular semigroups. In this case we see that both sides of this equation are terms of a weakly fixed variable. In order to study strong hyperidentities and strong hypervarieties of regular semigroups, it is necessary to define a mapping that changes any operation symbol to a term of a weakly fixed variable in the first step and apply term operations on algebra $\mathcal{A}$ in the second one.

Definition 3.1. A generalized hypersubstitution of type τ is called a generalized hypersubstitution of weakly fixed variable (for short, wfv-generalized hypersubstitution) of type τ if $\sigma(f_i) \in W^{wfv}_{\tau}(X)$. The set of all wfv-hypersubstitutions of type τ is denoted by $Hyp^{wfv}_G(\tau)$, i.e., $Hyp^{wfv}_G(\tau) = \{\sigma : \{f_i \mid i \in I\} \to W^{wfv}_{\tau}(X)\}$.

Example 3.2. We try to compute in terms of a weakly fixed variable $Hyp_G(3) = \{\sigma : \{f\} \to W^{wfv}_{(3)}(X)\}$ as follows:

(1) if $\sigma(f) = f(x_1,x_1,x_2)$, then
$$\widehat{\sigma}[f(x_3,x_1,x_3)] = S^3(f(x_1,x_1,x_2),x_3,x_1,x_3) = f(x_3,x_3,x_1),$$

(2) if $\sigma(f) = f(f(x_7,x_6,x_7),x_7,f(x_6,x_7,x_6))$, then $\widehat{\sigma}[f(x_4,x_5,x_4)]$
$$= S^3(f(f(x_7,x_6,x_7),x_7,f(x_6,x_7,x_6)),x_4,x_5,x_4) = f(f(x_7,x_6,x_7),x_7,f(x_6,x_7,x_6)),$$

(3) if $\sigma(f) = f(x_1,f(x_2,x_5,x_2),f(x_5,x_5,x_2))$, then $\widehat{\sigma}[f(x_3,x_7,x_3)]$
$$= S^3(f(x_1,f(x_2,x_5,x_2),f(x_5,x_5,x_2)),x_3,x_7,x_3) = f(x_3,f(x_7,x_5,x_7),f(x_5,x_5,x_7)).$$

To guarantee that we can apply the extension of each generalized hypersubstititon to the set $W^{wfv}_{\tau}(X)$, the following lemma is essential.

Lemma 3.3. *For any $\sigma \in Hyp^{wfv}_G(\tau)$, the extension $\widehat{\sigma}$ of σ maps a term of a weakly fixed variable to a term of a weakly fixed variable.*

Proof. Let σ be a mapping in $Hyp^{wfv}_G(\tau)$ and $t \in W^{wfv}_{\tau}(X)$. We show that $\widehat{\sigma}[t] \in W^{wfv}_{\tau}(X)$. For this, we give a proof on complexity of a term t. Clearly,

$\widehat{\sigma}[t] \in W_{\tau}^{wfv}(X)$ if t is a variable from X. Suppose that $t = f_i(s_1, \ldots, s_{n_i})$ belongs to $W_{\tau}^{wfv}(X)$. Without loss of generality, we assume that $var(s_l) = var(s_k)$ for fixed integers $1 \leq l < k \leq n_i$. Inductively, we also assume that each $\widehat{\sigma}[s_j]$ belongs to $W_{\tau}^{wfv}(X)$ for $j = 1, \ldots, n_i$. Since we assume that $var(s_l) = var(s_k)$, we have $var(\widehat{\sigma}[s_l]) = var(\widehat{\sigma}[s_k])$. Because $\sigma(f_i) \in W_{\tau}^{wfv}(X)$, it follows that $S^{n_i}(\sigma(f_i), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}]) \in W_{\tau}^{wfv}(X)$. Therefore, $\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})]$ belongs to the set $W_{\tau}^{wfv}(X)$. □

According to Lemma 3.3, we prove the following:

Theorem 3.4. *The extension $\widehat{\sigma}$ of each wfv-generalized hypersubstitution σ of type τ is an endomorphism of the algebra $(W_{\tau}^{wfv}(X), S^n, (x_i)_{i\geq 1})$.*

Proof. Let $s, t_1, \ldots, t_n$ be terms of a weakly fixed variable of type τ. To show that the equation $\widehat{\sigma}[S^n(s, t_1, \ldots, t_n)] = S^n(\widehat{\sigma}[s], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ holds, we give a proof on a structure of s. If s is a variable x_j in X_n, then $\widehat{\sigma}[S^n(x_j, t_1, \ldots, t_n)] = \widehat{\sigma}[t_j] = S^n(\widehat{\sigma}[x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$. If s is a variable x_j in $X \setminus X_n$, then $\widehat{\sigma}[S^n(x_j, t_1, \ldots, t_n)] = \widehat{\sigma}[x_j] = x_j = S^n(\widehat{\sigma}[x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$. Suppose that $s = f_i(s_1, \ldots, s_{n_i})$ and inductively assume that the theorem is satisfied for $s_1, \ldots, s_{n_i}$. Without loss of generality, we may assume that $var(s_j) = var(s_k)$ for some $1 \leq j < k \leq n_i$. Then by Theorem 2.7, we have $\widehat{\sigma}[S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)]$

$$\begin{aligned}
&= \widehat{\sigma}[f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))] \\
&= S^{n_i}(\sigma(f_i), \widehat{\sigma}[S^{n_i}(s_1, t_1, \ldots, t_n)], \ldots, \\
&\quad \widehat{\sigma}[S^{n_i}(s_{n_i}, t_1, \ldots, t_n)]) \\
&= S^{n_i}(\sigma(f_i), S^{n_i}(\widehat{\sigma}[s_1], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]), \ldots, \\
&\quad S^{n_i}(\widehat{\sigma}[s_{n_i}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])) \\
&= S^{n_i}\left(S^{n_i}(\sigma(f_i), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\right) \\
&= S^{n_i}\left(\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\right).
\end{aligned}$$

This completes the proof. □

From Theorem 3.4, we say that the mapping $\widehat{\sigma}$ preserves the operation S^n.

Using Lemma 3.3 and Theorem 3.4, we prove:

Theorem 3.5. *The following statement holds:*

(1) $Hyp_G^{wfv}(\tau)$ is a subsemigroup of $(Hyp_G(\tau), \circ^G)$,

(2) $Hyp_G^{wfv}(\tau_n)$ is a subsemigroup of $(Hyp_G(\tau_n), +_h^G)$.

Proof. Let σ_1 and σ_2 be elements in $Hyp_G^{wfv}(\tau)$. We show that $\sigma_1 \circ^G \sigma_2 \in Hyp_G^{wfv}(\tau)$, i.e., for any f_i, we show that $(\sigma_1 \circ^G \sigma_2)(f_i) \in W_{\tau}^{wfv}(X)$. Consider $(\sigma_1 \circ^G \sigma_2)(f_i) = (\widehat{\sigma_1} \circ \sigma_2)(f_i) = \widehat{\sigma_1}[\sigma_2(f_i)]$. Because $\sigma_2(f_i) \in W_{\tau}^{wfv}(X)$ then by Lemma 3.3 we get that $\widehat{\sigma_1}[\sigma_2(f_i)] \in W_{\tau}^{wfv}(X)$. Next, we show that $\sigma_1 +_h^G \sigma_2 \in Hyp_G^{wfv}(\tau_n)$, i.e., for any f_i, we show that $(\sigma_1 +_h^G \sigma_2)(f_i) \in W_{\tau_n}^{wfv}(X)$. Consider $(\sigma_1 +_h^G \sigma_2)(f_i) = S^{n_i}(\sigma_1(f_i), \sigma_2(f_i), \ldots, \sigma_2(f_i))$. Because $\sigma_2(f_i) \in W_{\tau_n}^{wfv}(X)$ then by Theorem 3.4 we get that $(\sigma_1 +_h^G \sigma_2)(f_i) \in W_{\tau_n}^{wfv}(X)$. Hence, $\sigma_1 +_h^G \sigma_2 \in Hyp_G^{wfv}(\tau_n)$. □

Consider a type τ_n instead of an arbitrary type τ on the set $Hyp_G^{wfv}(\tau)$, we have the set $Hyp_G^{wfv}(\tau_n)$ of all wfv-generalized hypersubstitution of type τ_n.

Our next goal is to show that the set of terms of a weakly fixed variable of type τ_n under the binary operation $+^G$ can be isomorphically represented by a function σ on the set $Hyp_G^{wfv}(\tau_n)$.

Theorem 3.6. *The semigroup $(W_{\tau_n}^{wfv}(X), +^G)$ is embeddable into $(Hyp_G^{wfv}(\tau_n), +_h^G)$.*

Table 3. Example of some elements in $Hyp_G^{wfv}(3,3)$ under $+_h^G$.

$+_h^G$	σ_{x_2}	σ_{x_5}	$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{g(x_7,x_7,x_7)}$
σ_{x_2}	σ_{x_2}	σ_{x_5}	$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{g(x_7,x_7,x_7)}$
σ_{x_5}	σ_{x_5}	σ_{x_5}	σ_{x_5}	σ_{x_5}
$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{f(x_4,x_5,x_4)}$	$\sigma_{f(x_4,x_5,x_4)}$
$\sigma_{g(x_7,x_7,x_7)}$	$\sigma_{g(x_7,x_7,x_7)}$	$\sigma_{g(x_7,x_7,x_7)}$	$\sigma_{g(x_7,x_7,x_7)}$	$\sigma_{g(x_7,x_7,x_7)}$

Proof. For each term t of a weakly fixed variable of type τ_n, a mapping μ that maps from $W_{\tau_n}^{wfv}(X)$ to $Hyp_G^{wfv}(\tau_n)$ can be defined by $\mu(t) = \sigma_t$ where σ_t we mean a wfv-generalized hypersubstitution of type τ_n which maps every n-ary operation symbol to t, i.e., $\sigma_t(f_i) = t$ for all $i \in I$. Obviously, μ is an injection. Infact, from $\mu(t_1) = \mu(t_2)$, we have $\sigma_{t_1} = \sigma_{t_2}$ and thus $t_1 = t_2$. To show that μ preserves operations of those two sets, let s and t be elements in $W_{\tau_n}^{wfv}(X)$. For this, we first need to ensure that the equation $\sigma_{s+^G t} = \sigma_s +_h^G \sigma_t$ holds. Let f_i be an n-ary operation symbol. Then we obtain that $(\sigma_{s+^G t})(f_i) = s +^G t = S^n(s,t,\ldots,t) = S^n(\sigma_s(f_i),\sigma_t(f_i),\ldots,\sigma_t(f_i)) = (\sigma_s +_h^G \sigma_t)(f_i)$. As a result, we have $\mu(s +^G t) = \sigma_{s+^G t} = \sigma_s +_h^G \sigma_t = \mu(s) +_h^G \mu(t)$, which means that a mapping μ is a homomorphism. □

From Example 2.9 and Theorem 3.6, we can write Table 3 which shows a characteristic of each function on $(Hyp_G^{wfv}(\tau_n), +_h^G)$ of some type.

It appears that Table 3 is similar to Table 1, but the characteristic of each element has been renamed by wfv-generalized hypersubstititons. In this matter, we can compute the results of functions in $Hyp_\tau^{wfv}(X)$ by the composition $\circ^G$ and return the answer to the set $W_\tau^{wfv}(X)$ under the binary operation $+^G$.

4. Conclusion

A term that extends a term of a fixed variable of type τ is presented. Some concrete examples are given. We also apply the generalized superposition to the set $W_\tau^{wfv}(X)$ and demonstrate the process of computation. Two semigroups under the operation $+^G$ and $\cdot^r$ for $r \in \{1,\ldots,n\}$ are proved. The semigroups of functions called wfv-generalized hypersubstitutions under two binary operations say $\circ^G$ and $+_h^G$ and their properties are discussed. To continue the work, we suggest the reader to extend the study from the set $W_\tau^{wfv}(X)$ to the power set $P(W_\tau^{wfv}(X))$ and determine the conditions under which such power set is closed under the non-deterministic operation.

Acknowledgement

This work was supported by Rajamangala University of Technology Rattanakosin, Thailand. The authors are grateful to the anonymous referees for their valuable comments and suggestions.

References

[1] Denecke K, Hounnon H. Partial Menger algebras of terms. Asian-Eur. J. Math. 2021; 14(6): Art. ID 2150092.

[2] Kitpratyakul P, Pibaljommee B. On substructures of semigroups of inductive terms. AIMS Mathematics. 2022; 7(6): 9835-9845.

[3] Kumduang T, Sriwongsa S. Superassociative structures of terms and formulas defined by transformations preserving a partition. Commun. Algebra. 2023; 51(48): 3203-3220.

[4] Phusanga D, Koppitz J. The semigroup of linear terms. Asian-Eur. J. Math. 2020; 13(1): Art. ID 2050005.

[5] Lipparini P. Exact-m-majority terms. Mathematica Slovaca. 2024; 74(2): 293-298.

[6] Wattanatripop K, Kumduang T. The partial clone of completely expanded terms. Asian-Eur. J. Math. 2024; doi.org/10.1142/S1793557124500633.

[7] Leeratanavalee S. Outermost-Strongly Solid Variety of Commutative Semigroups. Thai Journal of Mathematics. 2016; 14(2): 305–313.

[8] Denecke K. Partial clones of terms: an algebraic approach to trees, formulas and languages. Eliva Press, Chisinau; 2024.

[9] Puninagool W, Leeratanavalee S. Green's relations on $Hyp_G(2)$. An. □tiin□. Univ. Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica. 2012; 20(1): 249-264.

[10] Dudek WA, Trokhimenko VS. Menger Algebras of associative and self-distributive n-ary operations. Quasigroups Relat. Syst. 2018; 26: 45-52.

[11] Wattanatripop K, Changphas T. Clones of terms of a fixed variable. Mathematics. 2020; 8: Art. ID 260.

[12] Kumduang T, Wattanatripop K, Changphas T. Tree languages with fixed variables and their algebraic structures. International Journal of Mathematics and Computer Science. 2021; 16: 1683-1696.

[13] Phuapong S, Chansuriya N, Kumduang T. Algebras of generalized tree languages with fixed variables. Algebra and Discrete Mathematics. 2023; 36: 202-216.

[14] Phuapong S, Pookpienlert C. Fixed variables generalized hypersubstitutions. International Journal of Mathematics and Computer Science. 2021; 16: 133-142.

[15] Kumduang T, Leeratanavalee S. Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems. Symmetry. 2021; 13: 558.

[16] Kunama P, Leeratanavalee S. Green's relations on submonoids of generalized hypersubstitutions of type (n). Discuss. Math. Gen. Algebra Appl. 2021; 41(2): 239-248.