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Alternative Method for the Estimation of Parameters for the Normal Inverse Gaussian Distribution

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ABSTRACT

This article presents a study on the parameter estimation of the Normal Inverse Gaussian distribution, which is a specialized instance of the generalized hyperbolic distribution that is extensively utilized in the analysis of financial time series. Conventionally, the maximum likelihood method and the method of moment are used to estimate the parameters; however, these methods have a restriction on the feasible domain of possible skewness and excess kurtosis values. Therefore, we propose an alternative parameter estimation method for the Normal Inverse Gaussian distribution based on the Metropolis Hasting exponential maximum likelihood method. Moreover, the performance of this method will be compared with the maximum likelihood estimator, the epsilon maximum likelihood estimator, the exponential maximum likelihood estimator, using both simulated and real-world datasets. For simulation, we use the smallest root mean square error and provide descriptive statistics, including means and standard deviations to evaluate the performance of the model. For real data application, the selection of the model is guided by a goodness-of-fit test using the Anderson-Darling test statistics criterion. Furthermore, the model selection should demonstrate the smallest AD value alongside the highest p-value.

Keywords: Financial Time Series; Generalized hyperbolic; Maximum likelihood; Metropolis Hasting; Normal Inverse Gaussian distribution

1. Introduction

The Generalized Hyperbolic (GH) distribution was presented by Barndorff-Nielsen [1] in 1977. It is derived by mixing Normal variance-mean mixtures distribution with generalized inverse Gaussian

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(GIG) mixing distribution. The GH distribution has advantaged in capturing stylized empirical facts in financial assets. We are interested in the special case of GH distribution, namely the Normal Inverse Gaussian (NIG) distribution. It was introduced by Barndorff-Nielsen [2], obtained by mixing the Inverse Gaussian (IG) distribution with the Normal variance-mean mixtures distribution. Many authors have successfully fitted the NIG distribution in financial time series. For example, in 2006, Trejo et al. [3] examined the performance of the NIG distribution using stock data from the American and Mexican markets. In 2017, Shen et al. [4] illustrated the ability of the NIG distribution to fit gold and other precious metals and in 2018, Núñez et al. [5] showed the efficacy of the NIG distribution to fit the indexes of the BRIC economies.

Some authors studied on the parameter estimation of the NIG distribution. For instance, in 2002, Dimitris Karlis [6] introduced the application of the Expectation-Maximization (EM) algorithm to implement the Maximum Likelihood Estimator (MLE) method for estimating parameters of the NIG distribution. The focus of this study pertained to the calculation of parameters for the NIG distribution with a specific application to the general index of the Athens Stock Exchange. In 2011, Figueroa-López et al. [7] used the method of moment estimator (MOM) to estimate parameters of NIG and Variance Gamma (VG) distribution for high-frequency financial data. In 2014, Ghysels and Wang [8] showed the first four moments estimators of the NIG distribution, the VG distribution, and the generalized skewed t (GST) distribution as well as demonstrated a feasible domain of the MOM method for the NIG, the VG, and the GST distributions. Next, Yoon and Song [9] considered the feasible domain problem and proposed the epsilon estimation for NIG parameters in 2016. Recently, in 2020, Yoon et al. [10] compared the performance of parameter estimation methods for the NIG distribution, considering the problem related to the feasible domain. They used the MLE method, the MOM method, the epsilon MLE (ε-MLE) method by Yoon and Song in 2016 [9], and the exponential MLE (exp-MLE) method by Kim [11]. In 2021, Dhull and Kumar [12] introduced the first-order autoregressive (AR1) model with the NIG innovation by using the EM algorithm. They used the NIG autoregressive (NIGAR(1)) model to fit the historical financial data associated with Google equity. In this research, we aim to propose an alternative parameter estimation method for the NIG distribution involving the Metropolis Hasting algorithm and the exponential maximum likelihood (MH-exp-MLE) method. We check the performance of the alternative parameter estimation method by comparing the alternative parameter estimation among the MLE method, the ε-MLE method, and the exp-MLE method. Furthermore, we evaluate the goodness of fit of the NIG distribution using the Anderson-Darling (AD) test statistics. The selection of an appropriate model is based on the criterion that the chosen model should exhibit the smallest AD value and the highest p-value. The subsequent sections of the article are organized as follows. In section 2, the NIG distribution is briefly reviewed. In Section 3 describes the estimation methods utilized in this study. The derivation of the alternative parameter estimation method based on MH-exp-MLE is presented in Section 4. In Section 5, the simulations of all parameter estimation methods are shown and discussed. In Section 6, the application results using the bitcoin data are presented and explained. In section 7, we conclude this study. Last section discusses the study.

2. The Normal Inverse Gaussian distribution

The NIG distribution is a special case of the GH distribution which we focus on the $\lambda = -1/2$ [1]. The NIG distribution is a semi-heavy-tailed distribution which can be obtained as a normal mean–variance mixture with Inverse Gaussian as mixing distributions. Since the NIG distribution is the normal mean-variance mixture, it can be showed that the marginal distribution of X in the pair (X, Z), where the condition probability X|Z is given in Eq. (2.1)):

$$X|Z = z \sim N(\mu + \beta z, z), \tag{2.1}$$

where μ is mean. Additionally, the variable Z is the IG(γ, δ) distribution with parameters$\gamma$and δ. The IG distribution is a special case of the GIG distribution when replaced $\lambda = -1/2$ [2]. The probability density function (pdf) of IG distribution can be written in Eq. (2.2):

$$f(z) = \frac{\delta}{\sqrt{2\pi}} \exp(\delta\gamma) z^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\left(\frac{\delta^2}{z} + \gamma^2 z\right)\right). \tag{2.2}$$

The mean and the variance of the IG distribution are $\mathrm{E}(Z) = \delta/\gamma$ and $\mathrm{Var}(Z) = \delta/\gamma^3$. Therefore, the result of mixing between two distributions is the NIG distribution with four parameters α, β, δ, and μ denoted as NIG$(\alpha, \beta, \delta, \mu)$, where $\alpha = \sqrt{\gamma^2 + \beta^2}$. The pdf of the NIG distribution can be written in Eq. (2.3):

$$(x) = \frac{\alpha}{\pi} \exp\left(\delta\left(\sqrt{\alpha^2 - \beta^2}\right) - \beta\mu\right), \tag{2.3}$$

where $\phi(x) = 1 + \left[\frac{(x-\mu)}{\delta}\right]^2$ and $K_1(x)$ is the modified Bessel function of the third order with index 1 [6]. We used the property of the modified Bessel function to calculate the modified Bessel function of the third order which is

$$K_{-n}(x) = K_n(x), \text{ and}$$

$$K_2(x) = K_0(x) + (2/x) K_1(x).$$

We need to compute$K_0(x)$ and $K_1(x)$. It was introduced by Abramowirz and Stegum [13]. Furthermore, $0 < |\beta| < \alpha, \alpha > 0, \beta > 0, \mu \in \mathbb{R}$. As seen from the pdf of NIG distribution in Eq. (2.3), the shape of NIG density is very flexible indeed owing to its definition through the utilization of four parameters, making it possible to adjust many kinds of shape and with many decay rates of the tail. Moreover, the four parameters of the NIG distribution have an interesting characterization as follows: α parameter is a steepness of the density and reflects the tail behavior, in sense that the steepness of density increases monotonically with increasing α. Thus, we can describe the tail behavior that a small values of α implies heavy tails, while larger values of α imply lighter tails. Next, the β parameter is a skewness. The $\beta < 0$ is insinuated that a density skew to the left, but $\beta > 0$ is insinuated that a density skew to the right and the $\beta = 0$ is insinuated a symmetric around μ. The final two parameters, the μ and the δ are location and scale, respectively. The moment generating function of the NIG distribution can be demonstrated in Eq. (2.4) in term of α, β, δ and μ by

$$M_X(t) = \exp(t\mu) \exp\left(\delta\sqrt{\alpha^2 - \beta^2}\right) \exp\left(-\delta\sqrt{\alpha^2 - (\beta + t)^2}\right). \tag{2.4}$$

From Eq. (2.3), the pdf of the NIG distribution is rather complicated. From Eq. (2.4), we can write the mean, the variance, the skewness (S), and the kurtosis (K) of the

NIG distribution. It is easily derived the four moments of the NIG distribution if we let $\gamma = \sqrt{\alpha^2 - \beta^2}$ in the following Eqs. (2.5)-(2.8).

$$\mathrm{E}(X) = \mu + \frac{\delta\beta}{\gamma}, \quad (2.5)$$

$$\mathrm{Var}(X) = \frac{\delta\alpha^2}{\gamma^3}, \quad (2.6)$$

$$S = \gamma_1 = \frac{3\beta}{\alpha\sqrt{\gamma\delta}}, \quad (2.7)$$

$$K = \gamma_2 = \frac{3\left(1 + \frac{4\beta^2}{\alpha^2}\right)}{\delta\gamma}. \quad (2.8)$$

3. Estimation methods

3.1 Method of Moment

The moment of the NIG distribution which is given in Eq. (2.5). We calculate estimators of NIG parameters $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\mu}$, from the MOM [14], respectively. The parameters $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\mu}$ can be shown from Eq. (3.1) to Eq. (3.5):

$$\hat{\gamma} = \frac{3}{s\sqrt{3\overline{\gamma}_2 - 5\overline{\gamma}_1^{\,2}}}, \quad (3.1)$$

$$\hat{\beta} = \frac{\overline{\gamma}_1 s^2 \hat{\gamma}^2}{3}, \quad (3.2)$$

$$\hat{\delta} = \frac{s^2\hat{\gamma}^3}{\hat{\beta}^2 + \hat{\gamma}^2}, \quad (3.3)$$

$$\hat{\mu} = \overline{x} - \frac{\hat{\beta}\hat{\delta}}{\hat{\gamma}} \quad (3.4)$$

$$\hat{\alpha} = \sqrt{\gamma^2 + \beta^2}. \quad (3.5)$$

where $\bar{x}, s^2$ are the sample mean and variance, respectively. We denote

$$\overline{\gamma}_1 = \mu_3/\mu_2^{\,3/2}, \text{ and}$$
$$\overline{\gamma}_2 = \mu_4/\mu_2^{\,2} - 3,$$

while $\mu_k = n^{-1}\sum_{i=1}^{n}(x_i - \overline{x})^k$, i.e. the sample skewness and the kurtosis, respectively. The parameter estimation $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\mu}$ can be seen in the Eq. (3.1) to Eq. (3.5), the parameters do not exist if $3\overline{\gamma}_2 - 5\overline{\gamma}_1^{\,2} < 0$. We show the MOM method because it can be used for setting the initial value in parameter estimation process.

3.2 Maximum likelihood

The MLE method is employed for the determination of statistical model parameters by finding the parameter values that optimize the likelihood function. When confronted with a random sample of size n drawn from the NIG$(\alpha, \beta, \delta, \mu)$ distribution, the likelihood function can be expressed in Eq. (3.6):

$$L(\alpha,\beta,\delta,\mu) = \left(\frac{\alpha}{\pi}\right)^n \times \frac{e^{n\delta\gamma} \cdot e^{-n\beta\mu} \cdot e^{\sum_{i=1}^{n}\beta x_i}}{\prod_{i=1}^{n}\sqrt{\delta^2 + (x_i - \mu)^2}} \times \prod_{i=1}^{n} K_1\left(\alpha\delta\sqrt{\delta^2 + (x_i - \mu)^2}\right). \quad (3.6)$$

To solve the MLE method, it suffices to calculate the likelihood function and maximize it concerning the parameter of interest. The easy way for fixing this problem, we used the natural logarithm (ln) of the likelihood function, called log-likelihood function. It can be shown in Eq. (3.7).

$$\ln L(\alpha,\beta,\delta,\mu) = n\ln(\alpha) - n\ln(\pi) + n(\delta\gamma - \beta\mu) + \beta\sum_{i=1}^{n} x_i - \frac{1}{2}\sum_{i=1}^{n}\ln\left(\delta^2 + (x_i - \mu)^2\right)$$

$$+\sum_{i=1}^{n} \ln\left(K_1\left(\alpha\delta\sqrt{\delta^2+(x_i-\mu)^2}\right)\right). \tag{3.7}$$

Due to the intricacies associated with the Bessel function appearing in the derivative of the log-likelihood function, direct computation of the maximum likelihood becomes highly complex and challenging for estimating parameters. Consequently, implemented the Expectation-Maximization (EM) algorithm as an alternative method to derive parameter estimators for the NIG distribution [6]. The EM algorithm iterates between two steps, namely E-step and M-step.

E-step: In this step, let $\theta = (\alpha, \beta, \delta, \mu)$ be the parameter vector for computing the expectation of the log-likelihood of complete data with respect to the conditional distribution. Upon obtaining the parameter values at the kth iteration, denoted as $\theta^{(k)}$ representing a current parameter value, the pseudo-values s_i and w_i are computed in Eq. (3.8) and Eq. (3.9), respectively.

$$\begin{aligned} s_i =& \mathrm{E}\left(z_i|x_i,\theta^{(k)}\right) \\ =& \frac{\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}}{\alpha^{(k)}} \\ & \frac{K_0\left(\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}\right)}{K_1\left(\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}\right)}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} w_i =& \mathrm{E}\left({z_i}^{-1}|x_i,\theta^{(k)}\right) \\ =& \frac{\alpha^{(k)}}{\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}} \\ & \frac{K_{-2}\left(\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}\right)}{K_{-1}\left(\delta^{(k)}\theta^{(k)}(x_i)^{\frac{1}{2}}\right)}, \end{aligned} \tag{3.9}$$

for $i = 1, 2, 3, \ldots, k$,

and $\phi^{(k)}(x) = 1 + \left[\frac{\left(x-\mu^{(k)}\right)}{\delta^{(k)}}\right]^2$.

M-step: Update the parameters using the pseudo-values that calculated during E-step. Calculate

$$tM = \sum_{i=1}^{n} \frac{s_i}{n}, \text{ and}$$

$$\hat{\Lambda} = n\left(\sum_{i=1}^{n} w_i - \hat{M}^{-1}\right)^{-1},$$

then update the following terms from Eqs. (3.10)-(3.14):

$$\delta^{(k+1)} = \hat{\Lambda}^{\frac{1}{2}}, \tag{3.10}$$

$$\gamma^{(k+1)} = \frac{\delta^{(k+1)}}{\hat{M}}, \tag{3.11}$$

$$\beta^{(k+1)} = \frac{\sum_{i=1}^{n} x_i w_i - \bar{x}\sum_{i=1}^{n} w_i}{n - \bar{s}\sum_{i=1}^{n} w_i}, \tag{3.12}$$

$$\mu^{(k+1)} = \bar{x} - \beta^{(k+1)}\bar{s}, \tag{3.13}$$

$$\alpha^{(k+1)} = \left[\left(\gamma^{(k+1)}\right)^2 + \left(\beta^{(k+1)}\right)^2\right]^{\frac{1}{2}}, \tag{3.14}$$

where $\bar{s} = \sum_{i=1}^{n} \frac{s}{n}$.

3.3 Epsilon maximum likelihood

Referring to Eq. (2.7) and Eq. (2.8), we can simplify those equations to reveal the following relationship in Eq. (3.15):

$$3K - 5S^2 = \frac{9\sqrt{\alpha^2-\beta^2}}{\alpha^2\delta}. \tag{3.15}$$

Consequently, it is obvious that $3K - 5S^2 > 0$. However, during attempts to fit the NIG distribution to real data and compute sample moments, instances may occur where $3K - 5S^2 < 0$. In such scenarios, issues may happen in parameter estimation using the NIG distribution, as highlighted by Yoon and Song [9]. According to Yoon and Song, conventional the MLE method becomes unstable, and the MOM method cannot be computed when $3K - 5S^2 < 0$. To address this challenge, Yoon and Song [9] set up a valueε to substitute for $3K - 5S^2$ when $3K - 5S^2 < \varepsilon$. This approach is referred to as the ε-estimation method, and we will denote the MLE with $3\hat{K} - 5\hat{S}^2$ replaced by $\max\{3\hat{K} - 5\hat{S}^2, \varepsilon\}$ as ε-MLE. The specific value for ε is chosen by Yoon and Song [9] through the computation of ε-MLE using 1,000 simulated observations across a range of parameters. Subsequently, this process is iterated 1,000 times to calculate the root mean square error (RMSE) ofε-MLE, and the value ofε is selected to minimize the overall RMSE. The RMSE is defined in Eq. (3.16) :

$$RMSE\left(\hat{\theta}\right) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(\widehat{\theta}_m - \theta\right)^2}, \tag{3.16}$$

where $\widehat{\theta}_m$ is the estimated parameter value for m^{th} replication and θ is the true parametre value.

3.4 Exponential maximum likelihood

Exponential-estimation method is introduced as an innovative alternative to the MLE method, with a specific focus on mitigating the limitations associated with parameter estimation for the NIG distribution. The primary objectives of Exponential-estimation encompass achieving a more stable distribution of estimated parameters and capitalizing on the strengths inherent in the MLE method, particularly within contexts involving extreme distributions. We define the condition for the estimators of the NIG parameters, denoted as

$$-\alpha < \beta < \alpha, \ \alpha > 0, \text{ and } \delta > 0,$$

we write

$$\alpha = \sqrt{\beta^2 + e^u} \text{ and } \delta = e^w.$$

Suppose u, β, w, μ are a new parameterization. We can express the pdf of the exponential maximum likelihood method in Eq. (3.17):

$$f(x) = e^w\sqrt{\beta^2 + e^u} \frac{e^{e^w + \frac{u}{2} + \beta(x-\mu)}}{\pi\sqrt{e^{2w + (x-\mu)^2}}} K_1\left(\sqrt{\beta^2 + e^u}\sqrt{e^{2w} + (x-\mu)^2}\right), \tag{3.17}$$

where $u = \log(\alpha^2 - \beta^2)$ and $w = \log(\delta)$. We will refer the MLE under this parameterization as exp-MLE, which was used in, for example, the study research of in Kim [11].

4. The Metropolis-Hasting with exponential maximum likelihood

The Metropolis Hasting (MH) method extensively employed in the analysis of numerous intricate probability distributions. This approach, introduced by Metropolis et al. [15], is a notable member of the Markov Chain Monte Carlo (MCMC) algorithms. In this study, we apply the MH method with the exp-MLE method, namely Metropolis-Hasting exponential maximum likelihood (MH-exp-MLE) method, to estimate parameters of the NIG distribution. Moreover, the MH algorithm has two distributions: the target distribution (π) and the proposal distribution $q(u^*,\ \beta^*,\ w^*,\ \mu^*|u,\ \beta,w,\mu)$. For the former distribution, we can show the equation of the target distribution in Eq. (4.1):

$$\ln L(u,\beta,w,\ \mu) = n\ln\left(\sqrt{\beta^2+e^w}\right) - nln(\pi) + n(\gamma e^w - \beta\mu) + \beta\sum_{i=1}^{n} x_i - \frac{1}{2}\sum_{i=1}^{n}\ln\left(e^{2w}+(x_i-\mu)^2\right) + \sum_{i=1}^{n}\ln\left(K_1\left(e^w\sqrt{\beta^2+e^u}\sqrt{\delta^2+(x_i-\mu)^2}\right)\right). \tag{4.1}$$

For the latter distribution, we discussed in this article that the proposal distribution is symmetric satisfying. It can be shown in Eq. (4.2).

$$q(u^*,\ \beta^*,\ w^*,\ \mu^*|u,\ \beta,w,\mu) = q(u,\ \beta,w,\mu|u^*,\ \beta^*,\ w^*,\ \mu^*). \tag{4.2}$$

Therefore, let $(u^*,\ \beta^*,\ w^*,\ \mu^*)$ be the candidate value that is created by the proposal distribution. The acceptance of $(u^*,\ \beta^*,\ w^*,\ \mu^*)$ depended on the probability $\min\{1,r\}$ when

$$r = \frac{L(u^*,\beta^*,w^*,\mu^*|\tilde{x})\,\pi(u^*,\beta^*,w^*,\mu^*)}{L(u,\beta,w,\mu|\tilde{x})\,\pi(u,\beta,w,\mu)}.$$

Let $(u^{(t-1)},\ \beta^{(t-1)},w^{(t-1)},\mu^{(t-1)})$ be a current state of Markov chain when t is an iteration $t = 1,\ 2,\ 3,\ \ldots,\ n$ and let the initial value be $u^{(0)},\beta^{(0)},w^{(0)},\mu^{(0)}$. The procedural steps of the MH algorithm can be delineated as follows:

1. Initial value: $u^{(0)},\beta^{(0)},w^{(0)},\mu^{(0)}$.

2. Repeat for $t = 1,\ 2,\ 3,\ \ldots,\ n$.

2.1 Generate $\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ from a multivariate normal distribution characterized by a zero-vector mean and a covariance matrix denoted by

$$\begin{bmatrix} \sigma_u^2 & 0 & 0 & 0 \\ 0 & \sigma_\beta^2 & 0 & 0 \\ 0 & 0 & \sigma_w^2 & 0 \\ 0 & 0 & 0 & \sigma_\mu^2 \end{bmatrix}.$$

Furthermore, let

$$\begin{aligned} u^* &= u^{(t-1)} + \varepsilon_1, \\ \beta^* &= \beta^{(t-1)} + \varepsilon_2, \\ w^* &= w^{(t-1)} + \varepsilon_3, \text{ and} \\ \mu^* &= \mu^{(t-1)} + \varepsilon_4. \end{aligned}$$

2.2 Calculate

$$p = \min\{1,r\}$$

where

$$r = \frac{L(u^*,\beta^*,w^*,\mu^*|\tilde{x})\,\pi(u^*,\beta^*,w^*,\mu^*)}{L(u,\beta,w,\mu|\tilde{x})\,\pi(u,\beta,w,\mu)}.$$

2.3 Generate random variable V from a uniform distribution $v \sim U(0,1)$.

2.4 If $v \le p$ then

$$u^{(t)},\ \beta^{(t)},w^{(t)},\mu^{(t)} = u^{(*)},\ \beta^{(*)},w^{(*)},\mu^{(*)},$$

respectively with probability p. Otherwise, $v > p$ then

$$u^{(t)},\ \beta^{(t)},w^{(t)},\mu^{(t)} = u^{(t-1)},\ \beta^{(t-1)},w^{(t-1)},\mu^{(t-1)},$$

respectively with probability $p-1$.

The choice of the covariance matrix is often tuned during the exploration process. You might start with an initial guess for the covariance and then adapt it based on the acceptance rate of proposed points. If the acceptance rate is too low, you may need to increase the exploration by increasing the variances in covariance and vice versa

5. Simulation

In this research, the R-studio program was employed for analysis. For parts of simulation, we discuss and compare the performance of the MLE,ε-MLE, exp-MLE, and MH-exp-MLE by using the RMSE, mean and standard deviation. To obtain the mean and standard deviation, we can compute from Eq. (5.1) to Eq. (5.2)

$$\text{mean}\left(\hat{\theta}\right) = \frac{1}{M}\sum_{m=1}^{M}\widehat{\theta}_m, \qquad (5.1)$$

and

$$\text{standard deviation}\left(\hat{\theta}\right) = \sqrt{\frac{\sum_{m=1}^{M}\left(\widehat{\theta}_m - \bar{\hat{\theta}}\right)^2}{n-1}}, \qquad (5.2)$$

respectively.

We showed the steps for choosing the ε value in this study as follows:

1. Select dataset.
2. Compute the daily log return of dataset.
3. Calculate $3K - 5S^2$:
   - 3.1 Calculate $3K - 5S^2$ for the first 200 consecutive days from the daily log return data.
   - 3.2 Drop the first day and add a new day.
   - 3.3 Calculate $3K - 5S^2$ for the next 200 consecutive days of the daily log return data.
   - 3.4 Repeat steps (3.2) to (3.3) until the end of the daily log return data.
4. dentify three distinct sets from step (3) where $3K - 5S^2$ closed to 0.
5. Record the parameter values of the three distinct sets from step (4).
6. Set the sample size n and the parameter vector $\theta = (\alpha, \beta, \delta, \mu)$ in each set of parameter values from step (5).
7. Generate a random sample from the NIG distribution with various parameters from each set of parameter values and sample size in step (6) and check the goodness of fit test for the NIG distribution by using the Anderson-Darling test at 0.05 significant level.
8. Estimate the parameters in step (7) using the ε-MLE method by varying ε values from 0.005 to 0.12 by 0.005.
9. Repeat steps (7) to (8) until we have 1,000 estimated parameter vectors.
10. Calculate the RMSE of parameter vectors in each set of ε values from step (9).
11. Plot graph between RMSE and all ε values for each parameter.
12. Select the ε value.

To determine the ε value, we adhered to the approach outlined by Yoon et al. [10]. We discussed to use the daily log return between January 1, 2001 and 31 December, 2021. Next, we first computed $3K - 5S^2$ with 200 consecutive from the daily log return of the dataset. Subsequently, we omitted the first day and added a new day at the end of the dataset. Following this computation, we identified three distinct sets, as illustrated in Table 1.

Table 1 displays the selected parameter values in three distinct sets: Firstly, $3K-5S^2$. is 0.0841 in Set 1. Next, $3K-5S^2$ is 0.1046 in Set2. Finally, $3K - 5S^2$ is 0.3800 in Set 3.

For each set of parameter values, we applied the ε-MLE method to estimate parameter $\alpha, \beta, \delta, \mu$ with ε varying from 0.005 to 0.12 by 0.005 using 1,000 simulated observations. The procedure is repeated 1,000 times The details of each data sets are shown in Table 1.

Table 1. Parameter values in the three distinct sets.

	Set 1	Set 2	Set 3
α	8.3179	21.0295	4.3862
β	-1.3563	2.0793	-1.0337
δ	0.0081	0.0457	0.0224
μ	0.0006	-0.0112	-0.0046
$\mathrm{E}(X)$	-0.0007	-0.0067	-0.0101
$\mathrm{Var}(X)$	0.0011	0.0022	0.0056
S	-5.4603	1.3914	-4.7900
K	49.7195	3.2616	38.3668

The results of all data sets follow from the procedure of Yoon et al. [10]. Fig. 1 shows the RMSEs of ε-MLE method by changing from 0.005 to 0.012 by 0.005 for the three data sets in Table 1.

In Fig. 1, the RMSEs for α and the RMSE for β exhibit a sharp initial decline followed by a gradual increase in Set 2 around $\varepsilon = 0.03$ and 0.04. Conversely, for δ and μ, the RMSEs display a distinct pattern compared to α and β, remaining relatively stable except in Set 2. Consequently, we have chosen to utilize $\varepsilon = 0.035$ in our subsequent studies.

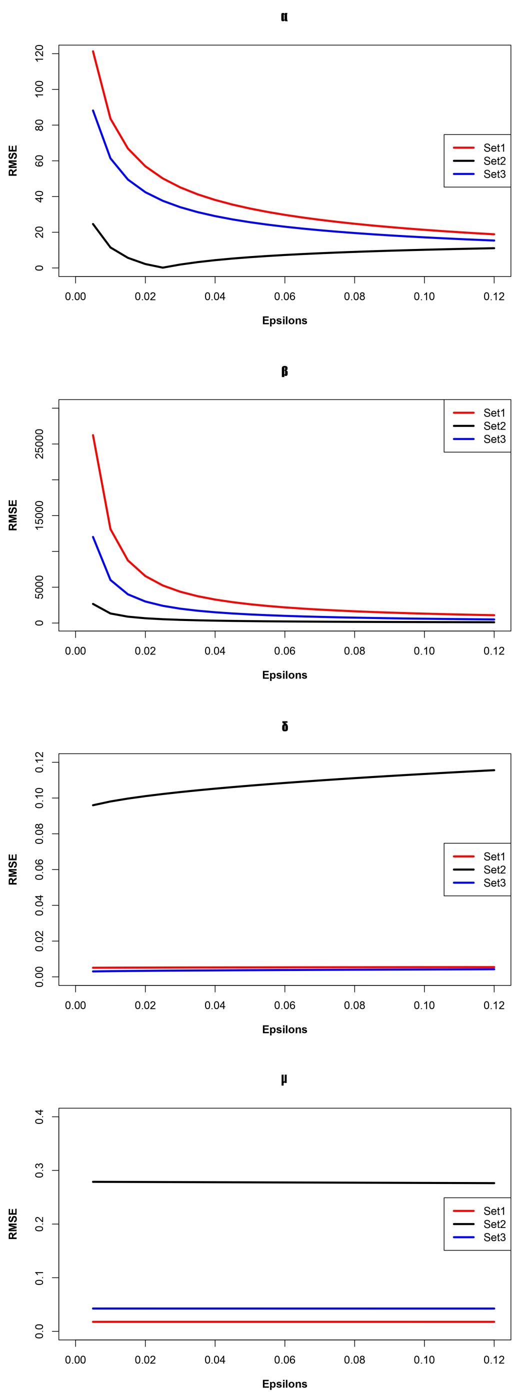

Fig. 1. The RMSEs of the ε-MLE by changing from 0.005 to 0.012 by 0.005 for the three data sets in Table 1.

We demonstrated the steps of parameter estimation for simulation studies as follows:

1. Follow the steps (1) to (3) from the choosing the ε value steps.
2. Identify two scenarios from step (1)
   2.1 Scenario 1: $3K - 5S^2$ close to 0.
   2.2 Scenario 2: $3K - 5S^2 > 0$.
3. Record the parameter values of the two scenarios from step (2).
4. Set the sample size n and the parameter vector $\theta = (\alpha, \beta, \delta, \mu)$ in each scenario from step (3).
5. Generate a random sample from the NIG distribution with various parameters of each scenario in step (4) and sample size and check the goodness of fit test for the NIG distribution by using the Anderson-Darling test at 0.05 significant level.
6. Set initial value into two cases:
   6.1 Case 1: start with $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$.
   6.2 Case 2: start with ε-MOM.
7. Estimate parameters in step (5) using the MLE, the ε-MLE, the exp-MLE, and the exp-MH-MLE methods with the two initial value cases in step (6).
8. Repeat steps (5) to (7) until we have 1,000 estimated parameters vectors.
9. Calculate the RMSE in step (8).
10. Select the appropriate parameter estimation method with the smallest RMSE value.

We explored two sets of parameter estimation denoted as Scenario 1 and Scenario 2, respectively. In Scenario 1, we suppose that $3K - 5S^2$ is 0.0628 which approximates to 0. In the contrast, Scenario 2 involved a much larger parameter value of $3K - 5S^2$ that is 15.7818.

Table 2 provides the parameter values and the first four moments for both scenarios.

Table 2. Parameter values are used in simulations: $3K - 5S^2 = 0.0628$ in Scenario 1 and $3K - 5S^2 = 15.7818$ in Scenario 2.

	Scenario1	Scenario2
α	24.5794	18.0046
β	-2.2436	0.6146
δ	0.0546	0.0285
μ	0.0047	0.0026
E(X)	-0.0003	0.0036
Var(X)	0.0022	0.0016
S	-1.1743	0.6066
K	2.3192	5.8738

In each scenario, we generate 1,000 observations to compute estimates, repeating this process 1,000 times for robustness. Additionally, we consider the impact of the initial value in the MLE by considering two sets. In Case 1, the maximization procedure starts at $\alpha = 1$, $\beta = 0, \delta = 1$ and $\mu = 0$. Conversely, in Case 2, we initiate the procedure at ε-MOM because the MOM is commonly used as the initial values for numerical optimization.

For the ε-MOM, we referred to the MOM method in the Eq. (3.1). Let us examine in term of $3K - 5S^2$ which must be positive. In the contrast, we sometimes observe that $3K - 5S^2$ is negative. Therefore, we replaced $3K - 5S^2$ with $\max\{3K - 5S^2, \varepsilon\}$ that we call ε -MOM. We set up the $\varepsilon = 0.035$ in all scenarios.

Table 3. RMSE of estimations in Scenario 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	α	β	δ	μ
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$				
MLE	**19.0940**	2.2398	0.3409	0.0047
ε-MLE	10.4691	298.4482	0.1812	0.3949
exp-MLE	22.1614	2.2409	0.2037	0.0047
MH-exp-MLE	5.8320	1.9376	0.0081	0.0034
Case 2: Initial values are ε-MOM				
MLE	**6.4009**	1.0523	0.0118	0.0021
ε-MLE	19.0909	333.1603	0.1055	0.1499
exp-MLE	19.7468	1.9377	0.0235	0.0030
MH-exp-MLE	5.8911	1.9367	0.0082	0.0034

Table 4. Means and standard deviations of estimators in Scenario 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	α	β	δ	μ
True value	24.5794	-2.2436	0.0546	0.0047
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$				
MLE	34.1358 (6.0661)	-3.3399 (2.5469)	0.0736 (0.0113)	0.0069 (0.0049)
ε-MLE	15.4982 (-14.3329)	-323.9366 (-358.5014)	0.1641 (-0.1459)	0.2452 (-0.1905)
exp-MLE	4.7301 (0.1416)	-0.3059 (0.2267)	0.0311 (0.0012)	0.0017 (0.0015)
MH-exp-MLE	18.6505 (2.1514)	-0.3061 (0.2289)	0.0466 (0.0039)	0.0013 (0.0013)
Case 2: Initial values are ε-MOM				
MLE	25.0735 (3.3074)	-2.3433 (1.7490)	0.0555 (0.0057)	0.0049 (0.0034)
ε-MLE	0.2948 (0.5055)	-323.9336 (358.5014)	0.1641 (0.1459)	0.2452 (0.1905)
exp-MLE	15.4982 (14.3329)	-0.3059 (0.2267)	0.0311 (0.0012)	0.0017 0.0015
MH-exp-MLE	18.5924 (2.1401)	-0.3070 (0.2298)	0.0465 (0.0040)	0.0013 (0.0013)

Table 5. RMSE of estimations in Scenario 2: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	α	β	δ	μ
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$				
MLE	(12.5879)	0.6132	0.3627	0.0012
ε-MLE	16.9941	0.6140	0.8812	0.00119
exp-MLE	15.6627	0.6136	0.2257	0.0012
MH-exp-MLE	4.3758	0.4718	0.0029	0.0008
Case 2: Initial values are ε-MOM				
MLE	**1.4162**	0.9978	0.0012	0.0011
ε-MLE	18.0290	178.9601	0.0946	0.0679
exp-MLE	12.9341	0.4737	0.0065	0.0008
MH-exp-MLE	4.3073	0.4713	0.0029	0.0008

Tables 3 and 4 illustrate the impact of initial values on estimation. Notably, the MLE demonstrates a significant improvement from Case 1 to Case 2, as evident from the bold-faced numbers in Table 3. In contrast, the MH-exp-MLE is less affected by initial values compared with the MLE; nevertheless, the means of MH-exp-MLE in Table 4 converge closer to the true values for all parameters. Overall, the best performance for fitting the Scenario 1 is the MH-exp-MLE method.

Tables 5 and 6 are the RMSEs and the means and the standard deviations of the parameter estimation method in Scenario 2, respectively. In this scenario, the RMSEs are much smaller that those with the Scenario 1, since $3K - 5S^2$ is much larger than 0 in this scenario. Note that, the RMSEs in Table 5 and the means and standard deviation in Table 6 of the MLE are very large in Case 1. This indicates that the MLE can often provide etremely large value. The performance of the MLE, ε-MLE, exp-MLE, and MH-exp-MLE are similar. However, the MH-exp-MLE still shows the best performance for all parameters to fit the data in the Scenario 2.

6. Application

In a real data application, we estimated the parameters of the NIG distribution by using the MLE, the ε-MLE, exp-MLE, and MH-exp-MLE methods. These methods were specifically applied to analyze the daily log-return of the bitcoin over a peroid spaning from January 1, 2018 to December 31, 2018. The total 365 observations were recorded. The price of bitcoin data is illustrated in Fig.2.

For the analysis, we divided the Bitcoin data into two distinct parts. In the first part, we initialized the parameters with $\alpha = 1, \beta = 0, \delta = 1$ and $\mu = 0$. In the second part, an alternative set of initial values was established using the ε-MOM method. We fitted the NIG distribution, which has been recommended as a suitable model for such financial data. Additionally, we conducted AD test statistic to assess the appropriateness of the NIG distribution in representing the characteristics of the bitcoin

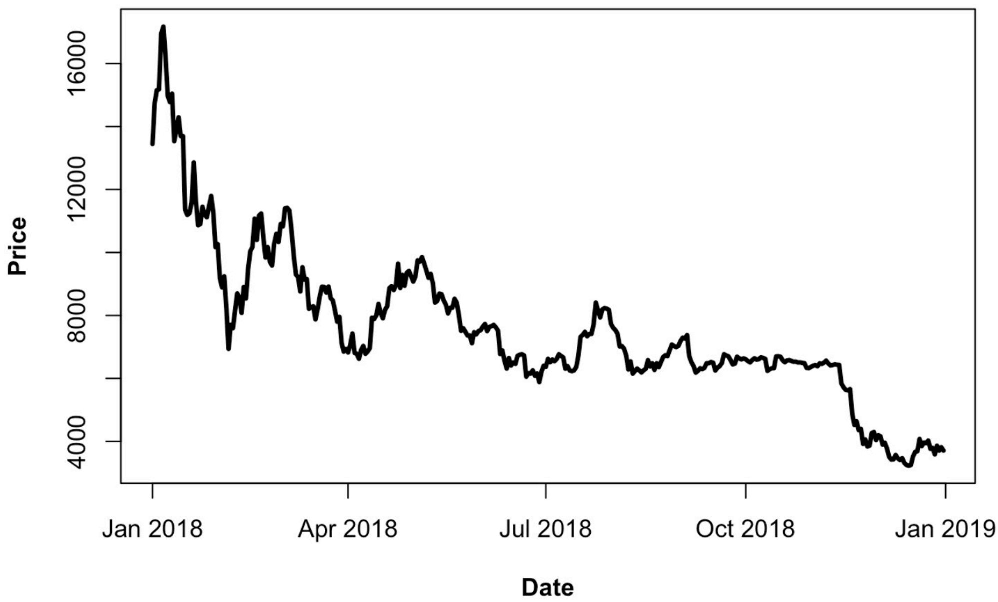

Fig. 2. The price of the bitcoin data.

dataset. The AD test statistic, along with corresponding p-values, served as criteria for model fitting and facilitated the comparison of the efficacy of each estimation method. Both of cases of parameter values for each estimation method are presented in Table 7.

The AD test statistic and p-value were presented in Table 8. The method of estimating in this way shows that the method with the smallest AD value and highest p-value performed the best [16]. In Table 8, the bold-faced p-values are found to be smaller than 0.05, accompanied by higher AD values, for the ε-MLE in both Case 1 and Case 2. This discrepancy with the acknowledge in [16] indicates that the ε-MLE method is not well-suited for estimating Bitcoin data in this study. Conversely, the MH-exp-MLE exhibits the most favorable performance, as evidenced by the smallest AD value and the highest p-value when compared to other methods. Consequently, the MH-exp-MLE is deemed suitable for accurately estimating parameter values in Bitcoin data.

7. Conclusion

The MH-exp-MLE method, combined with the NIG distribution, is a highly effective approach for modelling and fitting Bitcoin data. This study examines the period from January 1, 2018, to December 31, 2018, providing valuable insights into the behavior of the cryptocurrency during this time. The choice of the NIG distribution is noteworthy due to its semi-heavy-tail nature and the flexibility it offers with its four parameters. In financial modelling, using distributions that accurately capture the complex dynamics of asset prices is crucial, and the NIG distribution has proven effective for this purpose. However, a challenge in this process is directly deriving the like-

Table 6. Means and standard deviations of estimators in Scenario 2: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	α	β	δ	μ
True value	18.0046	0.6146	0.0285	0.0026
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$				
MLE	5.4062	0.0013	0.3912	0.0035
(0.0053)	(0.0036)	(0.0007)	(0.0011)	
ε-MLE	1.0000	0.0006	0.9097	0.0032
	(0.0000)	(0.0016)	(0.0012)	(0.0012)
exp-MLE	1.1410	1.1851	1.1537	1.1325
	(2.2864)	(2.4755)	(2.4616)	(2.4072)
MH-exp-MLE	13.6560	0.1575	0.0258	0.0030
	(1.8586)	(0.2915)	(0.0019)	(0.0009)
Case 2: Initial values are ε-MOM				
MLE	18.8113	0.6746	0.0294	0.0026
	(1.5765)	(1.2445)	(0.0012)	(0.0014)
ε-MLE	15.4982	-323.9336	0.1641	0.2452
	(14.3329)	(358.5014)	(0.1459)	(0.1905)
exp-MLE	5.0600	0.1519	0.0220	0.0030
	(0.2150)	(0.2795)	(0.0008)	(0.0010)
MH-exp-MLE	13.7067	0.1580	0.0258	0.0030
	(1.8301)	(0.2916)	(0.0019)	(0.0009)

lihood function within the MH-exp-MLE method due to the Bessel function, making the equation complex to solve. To overcome this challenge, we resort to using the natural logarithm, a common practice in statistical modeling, enabling a more manageable and efficient computation of the likelihood function within the exp-MLE method framework. Furthermore, in applying this method to real data, the outcomes stand out as the most fitting and appropriate for the given Bitcoin data. This is evident in both initial cases, as it yields the smallest AD value and the highest p-value.

8. Discussion

An inherent difficulty faced when utilizing Maximum Likelihood Estimation (MLE) for estimating parameters in the Normal-Inverse Gaussian (NIG) distribution is the vulnerability of estimates to instability. This instability becomes particularly pronounced when working with limited data or encountering extreme observa-

Table 7. The value of parameter estimation of the NIG distribution in different initial value: $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	α	β	δ	μ
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$				
MLE	23.3752	4.6788	0.0415	0.0782
ε-MLE	35.1148	4.6789	0.0415	0.0782
exp-MLE	4.6205	1.0319	0.0222	0.0815
MH-exp-MLE	15.9283	1.0141	0.0314	0.0823
Case 2: Initial values are ε-MOM				
MLE	15.2142	3.4215	0.0297	0.0798
ε-MLE	46.0007	3.4215	0.0297	0.0798
exp-MLE	4.6205	1.0319	0.0222	0.0815
MH-exp-MLE	16.3284	1.1608	0.0318	0.0821

Table 8. The AD test statistics and p-value of the goodness of fit test for the NIG distribution in different initial value: $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$ in Case 1 and ε-MOM in Case 2.

	AD value	p**-value**
Case 1: Initial values are $\alpha = 1, \beta = 0, \delta = 1, \mu = 0$		
MLE	1.1121	0.3034
ε-MLE	3.0771	**0.0250**
exp-MLE	0.9240	0.3997
MH-exp-MLE	0.8956	0.4169
Case 2: Initial values are ε-MOM		
MLE	0.9617	0.3780
ε-MLE	17.8740	**0.0000**
exp-MLE	0.9240	0.3997
MH-exp-MLE	0.8878	0.4217

tions. To address this issue, MH-exp-MLE introduces an innovative estimation framework aimed at bolstering stability. Diverging from traditional MLE techniques, MH-exp-MLE incorporates adjustments to foster more robust parameter estimations, even in challenging scenarios where standard methods falter.

However, this approach has limitations, especially in high-dimensional spaces, where exploring the distribution becomes more arduous. In such cases, the acceptance rate of proposed adjustments may diminish, resulting in poor mixing and protracted convergence times. To overcome this, we propose exploring alternative parameter estimation methods, such as those employing the Gibbs algorithm, in future endeavors.

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