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An Investigation of Extended Proinov Contraction under Non-Triangular Metric Structure and the Corresponding Applications

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ABSTRACT

This article demonstrates several fixed point theorems of extended Proinov-type, on the existence and uniqueness of fixed points within the context of non-triangular metric spaces. Additionally, we provide two applications of our primary result, in solving existence and uniqueness of a solution for a non-homogeneous linear parabolic partial differential equation and a stochastic integral equation.

Keywords: Non-triangular Metric Space (NTMS); Proinov Fixed Point Theorem; Initial Value Problem (IVP); Parabolic Partial Differential Equation; Stochastic Integral Equation

1. Introduction

One of the most essential aspects of fixed-point theory is its ability to solve a wide range of operator equations. In 1922, Banach produced the famous contraction condition:

$$d(Tv, Tw) \le kd(v, w) \text{ for all } v, w \in X,$$

(here d is complete metric on X, $k \in [0, 1)$ and T is a self-mapping on X) to prove the existence and uniqueness of fixed-point for T. This fixed-point acted as a solution to an operator equation. Initially, this finding was overlooked as a minor aspect of a broader project. However, it sparked the growth of a robust field of study that is currently prospering.

To contextualize the presented data and better understand the study's goals, we will address these aspects in both ways. Fixed-point theory requires a combination

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of a generic contraction condition and an abstract metric space for optimal results. To contextualize the data and better understand the study objectives, we will discuss both ways.

On the other hand, classical metric spaces marked a crucial breakthrough in the realm of mathematics. However, these spaces soon became old-fashioned for modeling natural phenomena. Many generalizations of the idea of metric were given in an increasingly abstract manner (see for Example: [1, 2] etc.) In this field of study, it is critical to recognize the challenges of experimenting with events that cannot be accurately seen or situations in which randomness plays a significant role. For such scenarios, the concept of non-triangular metric (NTM) was introduced by Khojasteh and Khandani [3] in 2020. Due to their generality and particular features, we are going to focus our research on a class of spaces within this type of metric structure. The topological properties of this space have been introduced by Aniruddha Deshmukh and Dhananjay Gopal [4] for the following reason: It has useful qualities for working in the realm of fixed-point theory. Classical metric spaces are a particular type of NTM spaces, thus we are going to tackle them separately. We are going to provide our main result for NTM spaces, since the former allow for an estimation of weaker restrictions.

On the other hand, the contractivity is the second most important component of any exploration in this field of study. Various scholars have proposed significant modifications to Banach's theorem, proposing increasingly general contractivity standards. The contributions that must be cited are as follows: [5–11], etc. In 2020 Proinov [12] unified many contractions. Erdal Karapınar and Juan Martinez-Moreno et al. [13] recently extended Proinov's [12] outstanding result. In recent years some generalized fixed-point results like (see for instance: Quasi-contractions [14], (A, S)-contraction [15], (ψ, ϕ)-contraction [16] etc.) have been established in the context of NTMS. So, inspired by this type of work, we extend Proinov contraction result in the setting of NTMS. Use these results to solve other problems: for instance, non-homogeneous linear parabolic partial differential equations and stochastic mixed-type nonlinear integral equations.

The article has been organized as follows. In Section 2, we provide the required background information to better understand the development and ideas. In Section 3, we propose existence and uniqueness fixed-point theorems for a modern class of contractions under the non-triangular metric spaces. In Section 4, our claims can be supported by applications and examples.

2. Preliminaries

This part collects the concepts and results that will be used throughout the study. It has been separated into three subsections. The first one is dedicated to introducing the basic concepts of non-triangular metric spaces. The last two subsections recalls the basics about Proinov and Extended Proinov fixed-point theorems.

2.1 Non-triangular metric space

Here, we recall only those definitions relating to non-triangular metric spaces that will be crucial in our upcoming work. For a thorough discussion, we refer the reader to [3, 4].

Definition 2.1 (Non-Triangular Metric Space)**.** Let us consider X is a non-empty set. A real valued function d is said to be a NTM on X, if d is satisfies following:

(a_1) $\mathrm{d}(\mathrm{v}, \mathrm{v}) = 0$ for every $\mathrm{v} \in \mathrm{X}$;

(a_2) $\mathrm{d}(\mathrm{v}, \mathrm{w}) = \mathrm{d}(\mathrm{w}, \mathrm{v})$ for every $\mathrm{v}, \mathrm{w} \in \mathrm{X}$;

(a_3) for every v, w and $\{\mathrm{v}_n\} \subset \mathrm{X}$ such that $\lim_{n\to\infty} \mathrm{d}(\mathrm{v}_n, \mathrm{v}) = 0$ and $\lim_{n\to\infty} \mathrm{d}(\mathrm{v}_n, \mathrm{w}) = 0$ then $\mathrm{v} = \mathrm{w}$.

Then (X, d) is a NTMS.

Example 2.2. Consider $X = [0, \infty)$ and define $\mathrm{d} : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ given by

$$\mathrm{d}(\mathrm{v}, \mathrm{w}) = \begin{cases} \frac{(\mathrm{v}+\mathrm{w})^2}{(\mathrm{v}+\mathrm{w})^2+1}, & \text{if } 0 \neq \mathrm{v} \neq \mathrm{w} \neq 0 \\ \frac{\mathrm{v}}{2}, & \text{if } \mathrm{w} = 0 \\ \frac{\mathrm{w}}{2}, & \text{if } \mathrm{v} = 0 \\ 0, & \text{if } \mathrm{v} = \mathrm{w}. \end{cases}$$

Conditions $(\mathfrak{a}_1)$ and $(\mathfrak{a}_2)$ are obvious. We have to verify condition $(\mathfrak{a}_3)$. Therefore, we take $\mathrm{v}, \mathrm{w} \in \mathrm{X}$ and $\{\mathrm{v}_n\} \in \mathrm{X}$ such that $\mathrm{d}(\mathrm{v}_n, \mathrm{v}) \to 0$, $\mathrm{d}(\mathrm{v}_n, \mathrm{w}) \to 0, n \to \infty$. Then implies that $\lim_{n\to\infty} \frac{(\mathrm{v}_n+\mathrm{v})^2}{(\mathrm{v}_n+\mathrm{v})^2+1} = \lim_{n\to\infty} \frac{(\mathrm{v}_n+\mathrm{w})^2}{(\mathrm{v}_n+\mathrm{w})^2+1} = 0$ and these hold if and only if $\lim_{n\to\infty} \mathrm{v}_n = -\mathrm{v} = -\mathrm{w}$ in $\mathbb{R}$ and so $\mathrm{v} = \mathrm{w}$. Hence, condition $(\mathfrak{a}_3)$ is verified. So, (X, d) is a NTMS.

Remark 2.3. *Consider a NTMS (X, d) and $v, w \in X$, then $d(v, w) = 0$ implies $v = w$ by condition $(\mathfrak{a}_3)$.*

Definition 2.4 (Convergence)**.** Let (X, d) be a NTMS and a sequence $\{\mathrm{v}_n\} \in \mathrm{X}$. We say that $\{\mathrm{v}_n\}_{n\in\mathbb{N}}$ converges to $\mathrm{v} \in \mathrm{X}$, if $\lim_{n\to\infty} \mathrm{d}(\mathrm{v}_n, \mathrm{v}) = 0$.

Remark 2.5. *Notice that $(\mathfrak{a}_3)$ in definition 2.1 ensures that the limit of a convergent sequence is unique.*

Definition 2.6 (Cauchy Sequence)**.** Let (X, d) be a NTMS and a sequence $\{\mathrm{v}_n\} \in \mathrm{X}$. We say that $\{\mathrm{v}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if $\lim_{n\to\infty} \sup\{\mathrm{d}(\mathrm{v}_n, \mathrm{v}_m) | m \geq n\} = 0$.

Definition 2.7 (Completeness)**.** A NTMS (X, d) is considered complete if every Cauchy sequence in X converges to a point $\mathrm{v} \in \mathrm{X}$.

Definition 2.8 (Property C)**.** Let (X, d) be a NTMS; then d is said to satisfy property C, if for, any sequence $\{\mathrm{v}_n\}$ with $\mathrm{d}(\mathrm{v}_n, \mathrm{v}) \to 0$, we have $\mathrm{d}(\mathrm{v}_n, \mathrm{w}) \to \mathrm{d}(\mathrm{v}, \mathrm{w})$ for every $\mathrm{w} \in \mathrm{X}$.

Definition 2.9 (Fixed Point)**.** Let X be a non-empty set and T be a self-mapping on X. A point $v \in X$ is called fixed point if $T(v) = v$. A set of fixed points is denoted by Fix(T).

2.2 Proinov contractions in metric space

In this subsection, we review the fundamentals of Proinov Contractions in classical metric spaces. Our primary references for these topics are [12, 13]. Proinov recently announced several results that unified previously known results of fixed point theory.

Theorem 2.10. *Let us consider a complete metric space (X, d) and T be a self map on X such that*

$$\psi(d(Tv, Tw)) \leq \phi(d(v, w)) \tag{2.1}$$

for all $v, w \in X, d(Tv, Tw) > 0$, where the real valued functions ψ, ϕ defined on $(0, \infty)$ satisfy the certain conditions:

$(\mathfrak{b}_1)$ *ψ is non decreasing;*

$(\mathfrak{b}_2)$ *$\phi(s) < \psi(s)$ for any $s > 0$;*

$(\mathfrak{b}_3)$ *$\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ for any $e > 0$.*

Then T has a unique fixed-point $v_0 \in X$ and the iterative sequence $\{T^{\ell} u\}$ converges to v_0 for each $v \in X$.
A self-map T on X is considered a Proinov

contraction if T contains pair of functions ψ, ϕ satisfied above conditions $(\mathfrak{b}_1) - (\mathfrak{b}_3)$, ensured the contractivity condition Eq. (2.1).

2.3 Extended proinov contraction in metric space

An extended Proinov Contraction is the most recent generalization of many contractions. In this subsection we give basic definitions and concepts which are needed for our work, For further information, refer to [13].

Let $\mathfrak{X}$ denote the family of pairs of real valued functions ψ, ϕ defined on $(0, \infty)$, which satisfied the properties:

- $(\mathfrak{X}_1)$ since $\{t_\ell\} \subset [0, \infty)$ with $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$, then $\{t_\ell\} \to 0$;
- $(\mathfrak{X}_2)$ since $\{t_\ell\} \subset [0, \infty), \{s_\ell\} \subset [0, \infty)$ are converging to the exact same limit $e \geq 0$ with $t_\ell > e$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, so $e = 0$;
- $(\mathfrak{X}_3)$ since $\{t_\ell\}, \{s_\ell\}$ are two sequences of non-negative real numbers with $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 0$.

Lemma 2.11. *Let $\psi, \phi : (0, \infty) \to \mathbb{R}$ be two functions that satisfy the certain conditions:*

- *(i) ψ is non-decreasing;*
- *(ii) $\phi(s) < \psi(s)$ for any $s > 0$;*
- *(iii) $\limsup_{s \to e^+} \phi(s) < \lim_{s \to e^+} \psi(s)$ for any $e > 0$.*

Then $(\psi, \phi) \in \mathfrak{X}$.

Theorem 2.12. *Let us consider a complete metric space (X, d) and a self-map T on X for which there exists $(\psi, \phi) \in \mathfrak{X}$ such that*

$$\psi(d(Tv, Tw)) \leq \phi(d(v, w)) \tag{2.2}$$

for all $v, w \in X, d(Tv, Tw) > 0$. Then every Picard sequence $\{T^\ell v\}$ converges to a fixed-point of T.

Theorem 2.13. *Under assumptions of the theorem 2.12, consider that $(\psi, \phi) \in \mathfrak{X}$ satisfies the certain property:*

- $(\mathfrak{X}_9)$*: there is a subset $S \subseteq X$ such that $Fix(T) \subseteq S$ and $\psi(d(v, w)) > \phi(d(v, w))$ for all $v \neq w \in S$.*

Then T has a unique fixed-point $v_0 \in X$ and every Piccard sequence $\{T^\ell u\}$ converges to v_0 for each $v \in X$.

Given the preceding outcome, we'll describe a self-map T on X as an extended Proinov contraction with pair of real valued functions ψ, ϕ defined on $(0, \infty)$, satisfy the axioms $(i) - (iii)$ with the contraction condition Eq. (2.2) holds.

3. Main Result

Influenced by the Proinov theorem[12], we'll investigate a modern class of contractions within the context of NTM spaces, that can be characterized using certain auxiliary functions.

Let us consider $\mathfrak{X}$ is the family of pairs of functions of real valued (ψ, ϕ) defined on $(0, \infty)$ satisfied the certain properties:

- $(\mathfrak{X}_1)$ Since $\{t_\ell\} \subset [0, \infty)$ with $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 0$;
- $(\mathfrak{X}_2)$ since $\{t_\ell\} \subset [0, \infty)$ and $\{s_\ell\} \subset [0, \infty)$ are converging to $e \geq 0$ with $t_\ell > e$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, so $e = 0$;
- $(\mathfrak{X}_3)$ since $\{t_\ell\} \subset [0, \infty), \{s_\ell\} \subset [0, \infty)$ with $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 0$.

Proposition 3.1 ([17]). *Consider $\{v_\ell\}$ is a Picard sequence in a NTMS (X, d) with $\{d(v_\ell, v_{\ell+1})\} \to 0$. Since $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 < \ell_2$ and $v_{\ell_1} = v_{\ell_2}$, then there is $\ell_0 \in \mathbb{N}$ and $v_0 \in X$ such that $v_\ell = v_0$ for all $\ell \geq \ell_0$ (that is, $\{v_\ell\}$ is constant from a term on wards). In such a case, v_0 will be a fixed-point of operator for which $\{v_\ell\}$ is a Picard sequence.*

Lemma 3.2 ([17]). *If the mapping $\{v_\ell\}$ is a sequence under a NTMS (X, d) with $\{d(v_\ell, v_{\ell+1})\} \to 0$ as $\ell \to \infty$. since the sequence $\{v_\ell\}$ is not Cauchy, then there exists $e > 0$ and two partial sub-sequences $\{v_{p(\ell)}\}$ and $\{v_{q(\ell)}\}$ of $\{v_\ell\}_{\ell\in\mathbb{N}}$ such that*

$$p(\ell) < q(\ell) < p(\ell + 1)$$
$$\text{and} \quad e < d(v_{p(\ell)+1}, v_{q(\ell)+1}) \text{ for every } \ell \in \mathbb{N},$$
$$\lim_{\ell\to\infty} d(v_{p(\ell)}, v_{q(\ell)}) = \lim_{\ell\to\infty} d(v_{p(\ell)+1}, v_{q(\ell)})$$
$$= \lim_{\ell\to\infty} d(v_{p(\ell)}, v_{q(\ell)+1})$$
$$= \lim_{\ell\to\infty} d(v_{p(\ell)+1}, v_{q(\ell)+1}) = e.$$

The purpose of this portion is to provide proof that the pair (ψ, ϕ) of real valued functions, along with Proinov contraction, belongs to $\mathfrak{X}$, which is basically the claim of this statement:

Lemma 3.3 ([17]). *If the mapping $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfies the certain conditions:*

(i) *ψ is non-decreasing;*

(ii) *$\phi(s) < \psi(s)$ for any $s > 0$;*

(iii) *$\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ for any $e > 0$.*

Then (ψ, ϕ) in $\mathfrak{X}$.

The main result of this section will be presented next.

Theorem 3.4. *Let (X, d) be a complete NTMS with the property C and T is a self-map on X for which there exists pair of functions $(\psi, \phi) \in \mathfrak{X}$ such that*

$$\psi(d(Tv, Tw)) \leq \phi(d(v, w)), \tag{3.1}$$

for all $v, w \in X$ with $d(Tv, Tw) > 0$. Then every iterative Picard sequence $\{T^\ell v\}$ converges to a fixed point of T.

Proof. Consider an arbitrary $v \in X$ and let us define $v_1 = v$ and $v_{\ell+1} = Tv_\ell$ for every $\ell \in \mathbb{N}$. If there's $\ell_0 \in \mathbb{N}$ such that $v_{\ell_0} = v_{\ell_0+1}$, then v_{ℓ_0} is a fixed point of T. In this case, $\{d(v_\ell, v_{\ell+1})\}_{\ell\geq\ell_0} = 0$. On the other hand, suppose that $v_\ell \neq v_{\ell+1}$ for every $\ell \in \mathbb{N}$. Then every v_ℓ is not a fixed-point of T and also

$$d(v_\ell, v_{\ell+1}) > 0 \quad \text{and} \quad d(Tv_\ell, Tv_{\ell+1}) > 0, \quad \text{for every } \ell \in \mathbb{N}.$$

Applying the condition Eq. (3.1), we obtain that, for every $\ell \in \mathbb{N}$,

$$\psi(d(v_{\ell+1}, v_{\ell+2})) = \psi(d(Tv_\ell, Tv_{\ell+1})) \leq \phi(d(v_\ell, v_{\ell+1})).$$

If we define $s_\ell = d(v_\ell, v_{\ell+1})$ for every $\ell \in \mathbb{N}$, the previous inequality means that the sequence $\{s_\ell\}$ satisfies $\psi(s_{\ell+1}) \leq \phi(s_\ell)$ for every $\ell \in \mathbb{N}$. Under condition $\mathfrak{X}_1$, we deduce that $\{d(v_\ell, v_{\ell+1})\} = \{s_\ell\} \to 0$. If there are $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 < \ell_2$ and $v_{\ell_1} = v_{\ell_2}$, then proposition 3.1 ensures that there is $\ell_0 \in \mathbb{N}$ and $v_0 \in X$ such that $v_\ell = v_0$ for every $\ell \geq \ell_0$. In such a case, v_0 is a fixed-point of T, and the existence of fixed points is assured. Next, suppose that $v_{\ell_1} \neq v_{\ell_2}$ for every $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \neq \ell_2$, that is, $\{v_\ell\}_{\ell\in\mathbb{N}}$ is an infinite sequence. In particular, $d(Tv_{\ell_1}, Tv_{\ell_2}) = d(v_{\ell_1+1}, v_{\ell_2+1}) > 0$ for all $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \neq \ell_2$. To show that $\{v_\ell\}_{\ell\in\mathbb{N}}$ is a

Cauchy sequence, consider it is not . In this case, Lemma 3.2 say that there exists $e > 0$ and two partial sub-sequences $\{v_{p(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{v_{q(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{v_\ell\}_{\ell\in\mathbb{N}}$ such that

$$p(\ell) < q(\ell) < p(\ell+1),$$

and

$$e < d(v_{p(\ell)+1}, v_{q(\ell)+1}) \quad \text{for all } \ell \in \mathbb{N}, \tag{3.2}$$

$$\lim_{\ell\to\infty} d(v_{p(\ell)}, v_{q(\ell)}) = \lim_{\ell\to\infty} d(v_{p(\ell)+1}, v_{q(\ell)})$$
$$= \lim_{\ell\to\infty} d(v_{p(\ell)}, v_{q(\ell)+1})$$
$$= \lim_{r\to\infty} d(v_{p(\ell)+1}, v_{q(\ell)+1}) = e. \tag{3.3}$$

Using the condition Eq. (3.1) we get, for all $\ell \in \mathbb{N}$,

$$\psi(d(v_{p(\ell)+1}, v_{q(\ell)+1})) = \psi(d(Tv_{p(\ell)}, Tv_{q(\ell)}))$$
$$\leq \phi(d(v_{p(\ell)}, v_{q(\ell)})). \tag{3.4}$$

If we define $t_\ell = d(v_{p(\ell)+1}, v_{q(\ell)+1})$ and $s_\ell = d(v_{p(\ell)}, v_{q(\ell)})$ for all $\ell \in \mathbb{N}$, then Eqs. (3.2)-(3.4) guarantees that

$$t_\ell > e \quad \text{and} \quad \psi(t_\ell) \leq \phi(s_\ell), \tag{3.5}$$

for all $\ell \in \mathbb{N}$. However, $e > 0$ contradicts the property $(\mathfrak{X}_2)$. This contradiction comes from the assumption that $\{v_\ell\}_{\ell\in\mathbb{N}}$ is not a d-Cauchy sequence, which demonstrates that actually $\{v_\ell\}_{\ell\in\mathbb{N}}$ is a d-Cauchy sequence. As (X, d) is a complete non triangular metric space, there is $v_0 \in X$ such that $\{v_\ell\}_{\ell\in\mathbb{N}}$ d-converges to v_0. As the sequence $\{v_\ell\}_{\ell\in\mathbb{N}}$ is infinite, there is $\ell_0 \in \mathbb{N}$ such that $v_\ell \neq v_0$ and $Tv_\ell \neq Tv_0$ for every $\ell \geq \ell_0$. The contractivity condition Eq. (3.1) leads to

$$\psi(d(v_{\ell+1}, Tv_0)) = \psi(d(Tv_\ell, Tv_0))$$
$$\leq \phi(d(v_\ell, v_0)). \tag{3.6}$$

It follows from property $(\mathfrak{X}_3)$ that $\{d(v_{\ell+1}, Tv_0)\} \to 0$, so $Tv_0 = v_0$. This completes the proof. □

To deduce the uniqueness of the fixed-point, it is necessary to add an additional condition.

Theorem 3.5. *Assumption of theorem 3.4, suppose that the pair of real valued functions $(\psi, \phi) \in \mathfrak{X}$ satisfies the certain properties:*

$(\mathfrak{X}_4)$*: $S \subset X$ with $Fix(T) \subseteq S$, $\psi(d(v, w)) > \phi(d(v, w))$ for every $v \neq w \in S$.*

Then T admit a unique fixed-point $v_0 \in X$ and the iterative Picard sequence $\{T^\ell v\}$ converges to v_0 for each $v \in X$.

Proof. For the uniqueness of the fixed-point of T, consider $v_1 \neq v_2 \in X$ such that $v_1 = T(v_1) \neq T(v_2) = v_2$. Therefore $d(Tv_1, Tv_2) = d(v_1, v_2) > 0$. By the condition Eq. (3.1), we have

$$\psi(d(v_1, v_2)) = \psi(d(Tv_1, Tv_2)) \leq \phi(d(v_1, v_2)),$$

which contradicts the assumption $(\mathfrak{X}_4)$. Hence we have a unique fixed-point. □

The following example supports our main result:

Example 3.6. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ with the non-triangular metric defined as

$$d(\frac{1}{n}, \frac{1}{m}) = \begin{cases} |\frac{1}{n} - \frac{1}{m}|, & \text{if } |m-n| \neq 1 \\ 1, & \text{if } |m-n| = 1 \end{cases}$$

and $d(\frac{1}{n}, 0) = \frac{1}{3n}$, $d(0, 0) = 0$.

To show d is not metric: Take $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{5}{12}$, by definition of d, $d(\frac{1}{2}, \frac{1}{3}) = 1, d(\frac{1}{2}, \frac{5}{12}) = \frac{1}{12}, d(\frac{5}{12}, \frac{1}{3}) = \frac{1}{12}$. Clearly, seen that d does not satisfy triangle inequality. Therefore d is not metric.

We define self-map T on X as $T(x) = \frac{x}{2}$ and $\phi, \psi : (0, \infty) \to \mathbb{R}$

$$\phi(s) = \begin{cases} \frac{2s}{3}, & \text{if } s \in (0, \frac{1}{3}], \\ \frac{2}{3}, & \text{if } s > \frac{1}{3}; \end{cases}$$

and

$$\psi(s) = \begin{cases} s, & \text{if } s \in (0, \frac{1}{3}], \\ 2 + \frac{1}{\pi}\cos\left(\frac{3}{1-3s}\right), & \text{if } s > \frac{1}{3}. \end{cases}$$

Let us consider $x_0 = 0$ and $y_0 = 1$ then $e_0 = d(x_0, y_0) = \frac{1}{3}$, however $\lim_{s \to e_0^+} \psi(s)$ does not exist. Because

$$\lim_{s \to e_0^+} \psi(s) = \lim_{s \to \frac{1}{3}^+} \left[2 + \frac{1}{\pi} \cos\left(\frac{3}{1-3s} \right) \right].$$

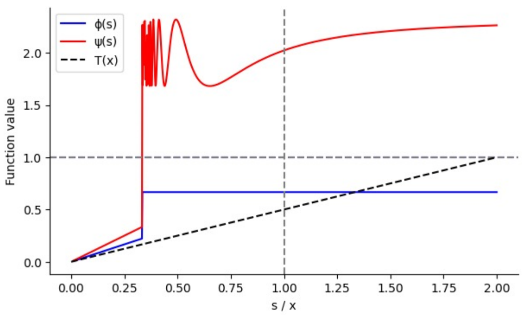

Fig. 1. Graph of $\psi(s), \phi(s)$, and $T(x)$.

So, the conditions (ii) and (iii) of Lemma 3.3 are not satisfied. Therefore, Lemma 3.3 cannot be applied, But we claim that (ϕ, ψ) belongs to $\mathfrak{X}$, such that $T(x) = [0, 0.5], \phi(0, \infty) = (0, \frac{2}{3}]$ and $\psi((0, \infty)) = [0, 1] \cup [1.5, 2.5]$.

($\mathfrak{X}_1$) Let $\{t_\ell\}$ be a sequence of non-negative real numbers such that $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for every $\ell \in \mathbb{N}$. Then $\psi(t_{\ell+1}) \leq \phi(t_\ell) \leq \frac{2}{3}$, So $\psi(t_{\ell+1})$ belongs to $(0, \frac{2}{3})$ for every $\ell \in \mathbb{N}$. This means $t_{\ell+1} \in (0, \frac{2}{3}]$ for every $\ell \in \mathbb{N}$. Therefore this $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ that is $t_{\ell+1} \leq \frac{2}{3} t_\ell$ for every $\ell \in \mathbb{N}$ and this ensures that $\{t_l\} \to 0$.

($\mathfrak{X}_2$) Let $\{t_\ell\} \subset [0, \infty), \{s_\ell\} \subset [0, \infty)$ be converging to the exact same limit $e \geq 0$ with the property satisfying $t_\ell > e, \psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Then $\psi(t_\ell) \leq \phi(s_\ell) \leq \frac{2}{3}$; hence $\psi(t_\ell) \in (0, \frac{2}{3})$ for all $\ell \in \mathbb{N}$ that is $e < t_\ell \leq \frac{2}{3}$ for every ℓ belongs to $\mathbb{N}$. Since $\{t_\ell\}, \{s_\ell\}$ are converging to $e \geq 0$, there exists $\ell_0 \in \mathbb{N}$, such that t_ℓ, s_ℓ belongs to $(0, \frac{2}{3}]$ for every $\ell \leq \ell_0$.
In specific $\psi(t_\ell \leq \phi(s_\ell))$ gives to $t_\ell \leq \frac{2}{3} s_\ell$ for every $\ell \geq \ell_0$, since $\ell \to \infty$, so we obtain $e \leq \frac{2}{3} e$ and as $e \geq 0$ implies $e = 0$.

($\mathfrak{X}_3$) Let $\{t_\ell\}, \{s_\ell\}$ be two sequences of non-negative real numbers with $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for every $\ell \in \mathbb{N}$. Since $\psi(t_\ell) \leq \phi(s_\ell) \leq \frac{2}{3}$. Then $\psi(t_\ell) \in (0, \frac{2}{3}] \implies t_\ell \in (0, \frac{2}{3}]$ for every $\ell \in \mathbb{N}$.
Moreover,as $\{s_\ell\} \to 0$, so, there is $\ell_0 \in \mathbb{N}$ such that $t_l, s_\ell \in (0, \frac{2}{3}]$ for every $\ell \geq \ell_0$.
Specifically, $\psi(t_\ell) \leq \phi(s_\ell)$ leads ℓ_0, $t_\ell \leq \frac{2}{3} s_\ell$ for every $\ell \geq \ell_0$. as $\{s_\ell\} \to 0 \implies \{t_\ell\} \to 0$. Hence $(\psi, \phi) \in \chi$.

Now, we can easily check the condition

$$\psi(d(Tv, Tw)) \leq \phi(d(v, w)),$$

for all $v, w \in X$ with $d(Tv, Tw) > 0$.

Hence, by using theorems 3.4-3.5, T admits a unique fixed-point.

Corollary 3.7. *Let (X, d) be complete NTMS with Property C and T be self map on X such that*

$$d(Tv, Tw) \leq \mu d(v, w) \text{ for all } v, w \in X$$

and $\mu \in [0, 1)$. Then T admit a unique fixed point $v_0 \in X$ and the iterative Picard sequence $\{T^\ell v\}$ converges to v_0 for each $v \in X$.

Proof. By substituting $\psi(s) = s$ and $\phi(s) = \mu s$ into theorems 3.4-3.5, a complete proof is obtained. □

Example 3.8. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ with the non-triangular metric defined as

$$d(\frac{1}{n}, \frac{1}{m}) = \begin{cases} |\frac{1}{n} - \frac{1}{m}|, & \text{if } |m-n| \neq 1 \\ 1, & \text{if } |m-n| = 1 \end{cases}$$

and $d(\frac{1}{n}, 0) = \frac{1}{3n}, d(0,0) = 0$.
In this function d is a non-triangular metric but not a metric because d does not satisfied triangle inequality.
We define self-map T on X as $T(x) = \frac{x}{2}$ and $\phi, \psi : (0, \infty) \to \mathbb{R}$

$$\psi(s) = s \text{ for all } s \in [0, \infty)$$

and

$$\phi(s) = \frac{3\psi(s)}{5}.$$

We claim that (ϕ, ψ) belongs to $\mathfrak{X}$, so we chek

($\mathfrak{X}_1$) Let $\{t_\ell\}$ be a sequence of non-negative real numbers such that $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$. that is, $\psi(t_{l+1}) \leq \frac{3\psi(t_l)}{5}$ for all $\ell \in \mathbb{N}$. Therefore $\psi(t_\ell) \to o$ as $\{t_\ell\} \to 0$.

($\mathfrak{X}_2$) Let $\{t_\ell\} \subset [0, \infty), \{s_\ell\} \subset [0, \infty)$ are converging to the exact same limit $e \geq 0$ with the property satisfying $t_\ell > e, \psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Le us consider that $e > 0$, with both functions ψ and ϕ are continuous, Then

$$\begin{aligned} 0 < e = \psi(e) &= \lim_{s \to e} \psi(s) = \lim_{s \to e^+} \psi(s) \\ &= \lim_{\ell \to \infty} \psi(t_\ell) \leq \lim_{\ell \to \infty} \phi(s_\ell) = \lim_{s \to e^+} \phi(s) \\ &= \lim_{s \to e} \psi(s) = \phi(e) = \frac{3\psi(e)}{5} = \frac{3e}{5} \end{aligned}$$

which is impossible. As a consequence, $e = 0$. Similarly we can check condition $\mathfrak{X}_8$.
Now, we can easily check the condition

$$d(Tv, Tw) \leq \mu d(v, w),$$

for all $v, w \in X$ and $\mu \in [0, 1)$.

Hence, by using the corollary 3.7 , T admits a unique fixed-point.

4. Applications

This section focuses on the applications of our main results, which will be applicable in solving various real-world problems. It has been separated into two subsections. The first section demonstrates how to apply our main result to solve non-homogeneous linear parabolic partial differential equations with a supporting numerical example. The last one focuses on applying our main result to solve stochastic integral equations.

4.1 Applications to non-homogeneous linear parabolic partial differential equations

Inspired by[11, 18, 19], this section uses our theorems to prove the existence and uniqueness of a solution to a non-homogeneous linear parabolic partial differential equation with a given initial condition. We consider the following problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + J(x, t, u(x,t), u_x(x,t)), \\ \qquad \text{for } -\infty < x < \infty, 0 < t < L; \\ u(x, 0) = \phi(x) \geq 0, \\ \qquad \text{for } -\infty < x < \infty; \end{cases} \tag{4.1}$$

Where J is continuous, ϕ is a continuously differentiable, and ϕ and ϕ' are bounded. A solution of problem (4.1) is a function $u = u(x, t)$ defined on $\mathbb{R} \times I$, where $I = [0, L]$ assume to satisfies the following conditions:

(i) $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I), \{C(\mathbb{R} \times I)\}$ is the space of all continuous real-valued functions.

(ii) u and u_x are bounded in $\mathbb{R} \times I$;

(iii) $u_t(x,t) = u_{xx}(x,t) + J(x,t,u(x,t),u_x(x,t))$, for all $(x,t) \in \mathbb{R} \times I$;

(iv) $u(x,0) = \phi(x)$ for all $x \in \mathbb{R}$.

We observed the initial value problem (4.1) is equivalent to the following integral equation:

$$u(x,t) = \int_{-\infty}^{\infty} \rho(x-\delta,t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} \rho(x-\delta,t-\xi)\Lambda d\delta d\xi \quad (4.2)$$

for all $x \in \mathbb{R}$ and $0 < t < L$, where

$$\rho(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{\frac{-x^2}{4t}}$$

and $\Lambda = H(\delta,\xi,u(\delta,\xi),u_x(\delta,\xi))$. Thus the given problem (4.1) admits a solution if and only if the corresponding integral Eq. (4.2) has a solution.
Consider

$$X = \{u(x,t) : u, u_x \in C(\mathbb{R} \times I), \|u\| < \infty\},$$

where

$$\|u\| = \sup_{x\in\mathbb{R},t\in I} |u(x,t)|^2 + \sup |u_x(x,t)|^2.$$

Obviously, the function $d : X \times X \to \mathbb{R}^+$ defined by

$$d(u,v) = \|u-v\|$$

is a complete NTM on X and has property C. But note that the above defined distance function d is not a classical metric because d does not hold triangle inequality.

Theorem 4.1. *Let us Consider IVP (4.1) and if the following conditions:*

(i) for $c > 0$ with $|s| < c$ and $|p| < c$, the function $J(x,t,s,p)$ is uniformly Holder continuous in x and t for every compact subsets of $\mathbb{R}$

(ii) J is bounded for every s and p.

(iii) There exists $u_0 \in X$ such that $d(u_0, T(u_0)) > 0$ where the self map $T : X \to X$ is defined by

$$(Tu)(x,t) = \int_{-\infty}^{\infty} \rho(x-\delta,t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} \rho(x-\delta,t-\xi)\Lambda d\delta d\xi,$$

for all $(x,t) \in \mathbb{R} \times I$ where $\Lambda = H(\delta,\xi,u(\delta,\xi),u_x(\delta,\xi))$.

(iv) There exist a constant $D_J \leq (L^2 + (2\pi)^{-1}L) \leq 1$ such that

$$0 \leq [J(x,t,s_2,p_2) - J(x,t,s_1,p_1)] \leq D_J[(s_2-s_1)^2 + (p_2-p_1)^2]$$

for all $(s_1,s_2),(p_1,p_2) \in \mathbb{R}\times\mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$

holds. Then, the given IVP (4.1) admit a solution.

Proof. Consider $u, v \in X$ such that $d(u,v) \geq 0$. Then by the definition of T and (ii), we have

$$\begin{aligned}
&|(Tv)(x,t) - (Tu(x,t))|^2 \\
&\leq \left(\int_0^t \int_0^\infty \rho(x-\delta,t-\xi)\Theta d\delta d\xi\right)^2 \\
&\leq \left(\int_0^t \int_0^\infty \rho(x-\delta,t-\xi)D_J\Phi d\delta d\xi\right)^2 \\
&\leq D_J L^2 d(u,v) \qquad (4.3)
\end{aligned}$$

where $\Theta = |J(\delta,\xi,v(\delta,\xi),v_x(\delta,\xi)) - J(\delta,\xi,u(\delta,\xi),u_x(\delta,\xi))|$ and $\Phi = |(v(\delta,\xi)-u(\delta,\xi))^2 + (v_x(\delta,\xi)-u_x(\delta,\xi))^2|$
Similarly, we obtain

$$\begin{aligned}
&|(Tv_x)(x,t) - (Tu_x)(x,t)|^2 \\
&\leq D_J d(u,v)\left(\int_0^t \int_0^\infty |\rho(x-\delta,t-\xi)|d\delta d\xi\right)^2
\end{aligned}$$

$$\leq ((2\pi)^{-1}\mathrm{L})\mathrm{D_J}\mathrm{d}(\mathrm{u},\mathrm{v}). \quad (4.4)$$

Therefore, from Eqs. (4.3)-(4.4), we have

$$|(\mathrm{Tv})(\mathrm{x},\mathrm{t}) - (\mathrm{Tu}(\mathrm{x},\mathrm{t}))|^2 + |(\mathrm{Tv_x})(\mathrm{x},\mathrm{t}) - (\mathrm{Tu_x})(\mathrm{x},\mathrm{t})|^2 = \mathrm{d}(\mathrm{Tu},\mathrm{Tv}) \leq ((\mathrm{L}^2 + (2\pi)^{-1}\mathrm{L})\mathrm{D_J})\mathrm{d}(\mathrm{u},\mathrm{v})$$

Now we define $\psi(\mathrm{u}) = \mathrm{u}$ and $\phi(\mathrm{u}) = \mathrm{Ku}$, where $\mathrm{K} = ((\mathrm{L}^2 + (2\pi)^{-1}\mathrm{L})\mathrm{D_J}) < 1$. Then T, ϕ and ψ follow all the conditions of theorem 3.4 on X. Consequently, we deduce the existence of $\mathrm{u}^* \in \mathrm{X}$ and hence u^* is a solution of problem (4.1). □

Example 4.2.

$$\begin{cases} \mathrm{u_t}(\mathrm{x},\mathrm{t}) = \mathrm{u_{xx}}(\mathrm{x},\mathrm{t}) + \mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{x},\mathrm{t}),\mathrm{u_x}(\mathrm{x},\mathrm{t})), \\ \text{for} -\infty < \mathrm{x} < \infty, 0 < \mathrm{t} < \frac{1}{2}; \\ \mathrm{u}(\mathrm{x},0) = \phi(\mathrm{x}) \geq 0, \\ \text{for} -\infty < \mathrm{x} < \infty; \end{cases} \quad (4.5)$$

where $\mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{x},\mathrm{t}),\mathrm{u_x}(\mathrm{x},\mathrm{t})) = \frac{\mathrm{u}(\mathrm{x},\mathrm{t})}{6} = \frac{e^{-\mathrm{t}}\sin(\mathrm{x})}{6}$ is continuous, $\mathrm{u}(\mathrm{x},0) = \phi(\mathrm{x}) = \sin(\mathrm{x})$ is continually differentiable and $\phi = \sin(\mathrm{x})$ and $\phi' = \cos(\mathrm{x})$ are bounded. Now we verified condition(iv) of theorem 4.1 for $\mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{s},\mathrm{p}) = \frac{\mathrm{s}}{6}$, So we have $\mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{s_1},\mathrm{p}_1) = \frac{\mathrm{s_1}}{6}$, $\mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{s_2},\mathrm{p}_2) = \frac{\mathrm{s_2}}{6}$ and $\mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{s_2},\mathrm{p}_2) - \mathrm{J}(\mathrm{x},\mathrm{t},\mathrm{s_1},\mathrm{p}_1) = \frac{\mathrm{s_2}}{6} - \frac{\mathrm{s_1}}{6} \leq \mathrm{D_J}(\frac{1}{6})(\mathrm{s_2} - \mathrm{s_1})^2$ and $(\mathrm{L}^2 + (2\pi)^{-1}\mathrm{L}) = \frac{1}{4} + \frac{1}{4\pi} = 0.339 > \frac{1}{6} = 0.167$, Therefore, Eq. (4.5) satisfies all the condition of theorem 4.1. Hence, Eq. (4.5) has a unique solution namely $\mathrm{u}(\mathrm{x},\mathrm{t}) = e^{-\mathrm{t}}\sin(\mathrm{x})$.

4.2 Application for stochastic integral equations

Inspired by [20, 21], we investigate the existence of unique solution for the stochastic integral equation of mixed type. More precisely, let us consider the stochastic integral equation of the form:

$$\mathrm{u}(\mathrm{t},\mathrm{x}) = \mathfrak{f}(\mathrm{t},\mathrm{x}) + \int_0^{\mathrm{t}} \alpha(\mathrm{t},\mathrm{r},\mathrm{x})\mathfrak{g}(\mathrm{r},\mathrm{u}(\mathrm{r},\mathrm{x}))\mathrm{dr} + \int_0^{\infty} \beta(\mathrm{t},\mathrm{r},\mathrm{x})\mathfrak{h}(\mathrm{r},\mathrm{u}(\mathrm{r},\mathrm{x}))\mathrm{dr}, \mathrm{t} \geq 0 \quad (4.6)$$

Theorem 4.3. *Considering the following:*

(i) Given probability one

$$\sup_{t\geq 0} \int_0^t |\alpha(t,r,x)|dr \leq A_0(x) < \infty \quad (4.7)$$

$$\sup_{t\geq 0} \int_0^t |\beta(t,r,x)|dr \leq B_0(x) < \infty; \quad (4.8)$$

(ii) $\mathfrak{g}(t,0) \equiv \mathfrak{h}(t,0) \equiv 0$;

(iii) for every given $\epsilon > 0$, $\exists$ *a* δ_1 *such that* $|\mathfrak{g}(t,u) - \mathfrak{g}(t,u)| \leq \epsilon|u-v|$, *for every* $|u|,|v| \leq \delta_1$, *for all* t;

(iv) for every given $\eta > 0$, $\exists$ *a* δ_2 *such that* $|\mathfrak{h}(t,u) - \mathfrak{h}(t,v)| \leq \eta|u-v|$, *for every* $|u|,|v| \leq \delta_2$, *for all* t.

Then, there exists a $\mathfrak{Z}(variable) \in \mathbb{Z}^+$, *with probability such that if* $0 < \rho(x) < \mathfrak{Z}(x)$, *then there exists a* $\delta(x) > 0$ *such that, for* $\mathfrak{f}(t,x)$ *with* $\|\mathfrak{f}(t,x)\| \leq \delta(x)$, $\exists$ *a unique solution* $u(t,x)$ *of Eq. (4.6) on* $[0,\infty)$ *with* $\|u(t,x)\| \leq \rho(x)$.

Proof. In virtue of the considerations, $\exists$ set $\mathrm{S_0}$ with $\mathrm{P}(\mathrm{S_0}) = 0$ such that for every $\mathrm{x} \in \mathrm{S} - \mathrm{S_0}$, expressions Eqs. (4.7)-(4.8) are satisfied.
Choose and fix $\mathrm{x_0} \in \mathrm{S} - \mathrm{S_0}$.
Fix $\eta > 0$ such that $\eta B_0 < 1$. From (ii) and (iv) we have a $\delta_2 > 0$ such that $|\mathfrak{h}(\mathrm{t},\mathrm{u})| \leq \eta|\mathrm{u}|, |\mathrm{u}| \leq \delta_2$. Consider $\gamma = \frac{(1-\eta\mathrm{B_0})}{2\mathrm{A}}$, and $\delta_1 > 0 \implies |\mathfrak{g}(\mathrm{t},\mathrm{u}) - \mathfrak{g}(\mathrm{t},\mathrm{v})| \leq \gamma|\mathrm{u}-\mathrm{v}|$, whenever $|\mathrm{u}|,|\mathrm{v}| \leq \delta_1$, uniformly in $\mathrm{t} \geq 0$. Let $\mathfrak{Z}(\mathrm{x_0}) = \min(\delta_1,\delta_2)$. For a given $0 < \rho < \mathfrak{Z}(\mathrm{x_0})$, we defined the set $\mathrm{X}_\rho = \{\mathrm{u} \in C(\mathbb{R} \times \mathrm{S}) : \|\mathrm{u}\| < \rho = \rho(\mathrm{v_0})\}$, where $\|\mathrm{u}\| = \sup_{\mathrm{t}\geq 0} |\mathrm{u}(\mathrm{t})|^2$. Obviously, a function $\mathrm{d} : \mathrm{X}_\rho \times \mathrm{X}_\rho \to [0,\infty)$ given by $\mathrm{d}(\mathrm{u},\mathrm{v}) = \|\mathrm{u}-\mathrm{v}\| = |\mathrm{u}-\mathrm{v}|^2$ is a CNTM on X_ρ and has

property C. For $u \in X_\rho$ define an operator T on X_ρ, such that:

$$(Tu)(t, x_0) = \mathfrak{f}(t, x_0) + \int_0^t \alpha(t, r, x_0)\mathfrak{g}(r, u(r, x_0))dr + \int_0^\infty \beta(t, r, u(r, x_0))\mathfrak{h}(r, u(r, x_0))dr, t \geq 0,$$

for every $x_0 \in S - S_0$.

For $u \in X_\rho$, From the conditions (i)-(iv) of the theorem 4.3, we get

$$\|Tu\| \leq \delta(x_0) + \gamma A_0\rho + \eta B_0\rho < \rho|u|^2$$

provided that $\delta(x_0) < \frac{(1-\eta B_0)\rho}{2}$. Let $u, v \in X_\rho$. Then we have

$$\|Tu - Tv\| \leq (\gamma A_0 + \eta B_0)|u - v|^2 = \frac{1 - \eta B_0}{2}\|u - v\|.$$

Now define $\psi(u) = u$, $\phi(u) = Ku$, where $K = \frac{1-\eta B_0}{2} < 1$. Consequently T, ϕ and ψ follow all the conditions of theorem 3.4 on X_ρ, then there exists a unique solution $u(t, x_0)$ on $[0, \infty)$ for Eq. (4.6). Additionally, this solution is continuous in t for each $x_0 \in S - S_0$, and $\|u\| < \rho(x_0)$. Since the above argument is true for every $x_0 \in S - S_0, P(S_0) = 0$, hence there exists a unique solution(in general) $u(t, x)$ for Eq. (4.6). □

5. Conclusions

Inspired by the extensive literature on fixed-point theorems and it's applications, in this article, we have investigated the recently remarkable result given by Erdal Karapınar and Juan Martinez-Moreno et al. [13] is known as extended Proinov Contraction in non triangular metric structure, which is a weaker form of unified result of many contractions is given by Proinov [12].

However, a comprehensive examination of the solution mechanism of operator equation issues shows that the method of application of fixed point theorem to operator equations consists of certain key steps:
(i) since integrals are easier to deal with than non-homogeneous linear parabolic partial differentials, the given operator equation is first converted into an equivalent equation using a theory of non-homogeneous linear parabolic partial differential equations and integral calculus, and then the obtained integral equation is written as a corresponding equation in a suitable metric space.
(ii) Depending on the nature of stochastic calculus included in an operator equation, a fixed point theorem on an appropriate metric space is used to show that the resulting equivalent operator equation has a solution, which implies that the operator equation exists.

In this article, we proved the applicability of fixed point theorems within the context of NTMS in solving (i) A non-homogeneous linear parabolic partial differential equation with the given conditions (ii) Stochastic integral equation.

The technique in theorem 3.4 can be used to prove the existence and uniqueness of solutions to a variety of mathematical models (differential, Integral, variational inequalities). This can also be used in other areas such as steady-state temperature distribution, chemical reactions, neutron transport theory, economic theory, game theory, optimal control theory, etc.

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