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Original research article

Seating Derangements without Horizontal Displacement

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ABSTRACT

This paper investigates the permutations of seating arrangements where individuals can only move to adjacent seats, without any horizontal displacement. We consider a grid consisting of m rows and n columns, with each seat occupied by a single person. The allowed movements are restricted to vertical or diagonal shifts to neighboring seats. We establish recurrence relations to determine the number of possible seating derangements for given values of m and n. Solutions to these recurrence relations are provided. Additionally, we extend our analysis to larger grids of size $2m \times n$, subject to the same movement constraints.

Keywords: Derangement; Permutation; Rearrangement; Recurrence relation; Seating derangement

1. Introduction

A seating derangement involves rearranging individuals within a structured grid such that no one occupies their original position. These problems are fundamental to combinatorial mathematics, focusing on constrained permutations and movement restrictions.

Seating derangement problems have practical applications in classroom management, optimization, and discrete mathematics. By examining constraints such as adjacency rules and grid limitations, researchers aim to determine the feasibility of complete rearrangements and enumerate distinct solutions. One of the fundamental questions in this domain is whether a given arrangement of individuals can transition to new positions following specific movement constraints, such as only being allowed to move to adjacent seats in designated directions. The study of these constraints leads to fascinating mathematical insights, often revealing connections to well-known numerical sequences such as the Fibonacci sequence. Researchers have explored special cases of seating derangements, including those where movement is restricted to

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horizontal, vertical, or diagonal shifts, leading to formulas that describe the total number of valid derangements under these conditions.

Several studies have contributed to the understanding of seating derangements and their mathematical properties. Honsberger [2] introduced an initial version of the problem: Consider a classroom with five rows of five desks per row, where each desk is occupied by a student. The teacher issues a directive requiring each student to move to an adjacent seat, with movement permitted in one of four directions: forward, backward, leftward, or rightward. However, certain students, particularly those seated at the edges or corners of the classroom, have limited movement options. The key question posed in this problem is whether it is always possible to execute such a seating rearrangement successfully without leaving any student without a valid seat.

A more generalized version of this problem was later presented by Kennedy and Cooper [3], considering a classroom consisting of m rows and n desks per row, with every seat occupied. Each student must move to an adjacent seat, similar to the constraints in the original problem. The primary focus of this variation is to determine the total number of distinct ways in which this directive can be executed, given the constraints imposed by the seating arrangement and the availability of adjacent seats.

Subsequent research by Otake et al. [4] further analyzed the structure of seating derangements in classrooms with two rows and n desks per row. They established a mathematical relationship between these derangements and Fibonacci numbers. Specifically, they demonstrated that the number of possible seating arrangements in a 2-row, $n-$column classroom follows the squared Fibonacci sequence, given by $F^2_{(n+1)}$, where F_n represents the nth Fibonacci number. This result provided a fundamental connection between combinatorial seat rearrangement problems and classical number theory.

Recent research has further explored the mathematical foundation of seating derangements. Eriksen et al. [1] examined cases of restricted derangements where additional constraints were imposed on movement, influencing the total number of valid seating arrangements. Later, in 2018, Sirivoravit and Leerawat [5] studied seating derangements in a $2 \times n$ classroom. They defined a seating derangement as any rearrangement of individuals among the $2n$ seats, arranged in two rows and n columns, such that each seat is occupied by a single person, and each person moves to a neighboring seat either horizontally, vertically, or diagonally. They also derived a formula for calculating the number of $2 \times n$ seating derangements, further advancing the mathematical study of constrained seating arrangements.

The $m \times n$ seating derangement problem is an extension of the classical seating derangement problem. In this variant, we consider a rectangular grid of seats with m rows and n columns, where all seats are initially occupied by a single person. Each person is assigned a specific seat, but due to a derangement, no one remains in their assigned seat. The goal is to analyze the possible rearrangements under defined movement constraints and determine the number of valid seating permutations. This paper focuses on a specific type of seating derangement where a person moving to a new seat cannot move to any horizontally adjacent seat. The objective is to determine the number of ways people can rearrange them-

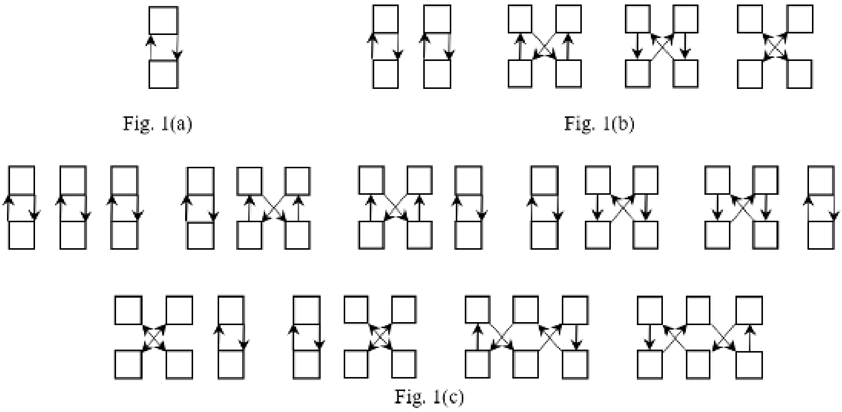

Fig. 1. the symbol "□" represents a seat, and "→" indicates the movement of a person from one seat to another. These diagrams help visualize the seating derangement process where each person is displaced to an adjacent seat following the vertical and diagonal movement restrictions.

selves in an $m \times n$ grid of seats (m rows and n columns) such that each person moves only to a vertical or diagonal neighboring seat, without any horizontal movements. Furthermore, we extend the analysis to the case of a grid with $2m$ rows and n columns under the same constraints.

2. Main Results

For any positive integers m and n, we explore seating derangements on a grid where horizontal shifts are strictly prohibited. Each individual is initially seated in a unique position within an $m \times n$ grid, where m represents the number of rows and n represents the number of columns. The only movements allowed for each person are either vertical or diagonal to adjacent seats, meaning no one can shift left or right within the same row.

This restriction on movement creates a more constrained problem compared to traditional derangements. The goal is to determine how many valid seating arrangements, or derangements, can be formed under these conditions, such that no person remains in their original seat and no horizontal movement occurs.

To solve the seating derangement problem with no horizontal movements, we need to establish the recurrence relations for the solution and provide the solution to these relations. Let $D(m, n)$ denote the number of ways to rearrange an $m \times n$ grid under the above restrictions. We first consider the problem of a $2 \times n$ seating derangement where no horizontal movement occurs. In how many ways can the seats be deranged?

To answer this, we begin by examining some special cases. When $n = 1$, there is only one way to swap the two seats in the 2×1 seating derangement, so $D(2, 1) = 1$, as shown in Fig. 1(a). When $n = 2$, there are 4 ways to rearrange the four seats in the 2×2 seating derangement, thus $D(2, 2) = 4$, as illustrated in Fig. 1(b). When $n = 3$, there are 9 ways to rearrange the six seats in the 2×3 seating derangement, giving $D(2, 3) = 9$, as shown in Fig. 1(c). Now, let us establish the recurrence relation for $D(2, n)$:

Theorem 2.1. *Let n be a positive integer. The number of seating derangements in a*

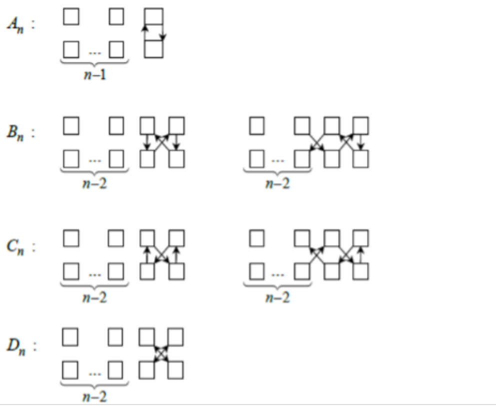

Fig. 2. The six possible endings.

$2 \times n$ grid, where no horizontal movement is allowed, satisfies the following recurrence relation:

$$D(2,n) = 2D(2,n-1) + 2D(2,n-2) - D(2,n-3),$$

for $n \geq 4$, where $D(2,1) = 1, D(2,2) = 4$ and $D(2,3) = 9$.

Proof. Let $S(2,n)$ denote the set of all possible $2 \times n$ seating arrangements such that no horizontal movement is allowed. Therefore, the cardinality of $S(2,n)$ is equal to the number of derangements of $2 \times n$, which is denoted by $D(2,n)$. The set $S(2,n)$ can be partitioned into four distinct subsets A_n, B_n, C_n and D_n according to the specific ending configurations, as illustrated by the six possible endings shown in Fig. 2. These endings capture the possible configurations at the rightmost column of the $2 \times n$ grid, which will influence the number of valid seating derangements for the rest of the grid.

Let a_n, b_n, c_n and d_n be the numbers of A_n, B_n, C_n and D_n, respectively. Hence,

$$D(2,n) = |S(2,n)| = a_n + b_n + c_n + d_n.$$

It is easy to see that there is a one-to-one correspondence between $S(2,n)$ and A_{n+1}.

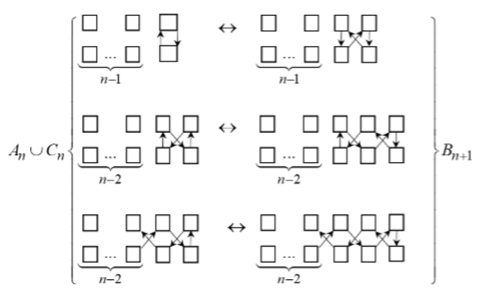

Fig. 3. A one-to-one correspondence between $A_n \cup C_n$ and B_{n+1}.

Therefore, $D(2,n) = a_{n+1}$. Next, there is a one-to-one correspondence between $A_n \cup C_n$ and B_{n+1}, shown in Fig. 3. Then, $b_{n+1} = a_n + c_n$. By similar correspondence, we have $c_{n+1} = a_n + b_n$. It is easy to see that $d_{n+1} = a_n$. Therefore, the recurrence relation for a_n can be defined as follows:

$$a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n,$$

where $a_1 = a_2 = 1$ and $a_3 = 4$. Now, $a_4 = 9$. Since $D(2,n) = a_{n+1}$, we have $D(2,n) = 2D(2,n-1) + 2D(2,n-2) - D(2,n-3)$, where $D(2,1) = 1, D(2,2) = 4$ and $D(2,3) = 9$. □

To compute the number of ways to carry out the directive for the seating derangement with no horizontal movement, we can apply the recurrence relation established in Theorem 2.1.

Theorem 2.2. *Let n be a positive integer. The number of seating derangements in a $2 \times n$ grid, where no horizontal movement is allowed, is given by*

$$D(2,n) = \frac{1}{5}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + 2(-1)^n\right).$$

Proof. By Theorem 2.1, the number of the $2 \times n$ seating derangement with no horizontal movement is

$$D(2,n) = 2D(2,n-1) + 2D(2,n-2) - D(2,n-3), \quad (2.1)$$

for $n \geq 4$, where $D(2,1) = 1, D(2,2) = 4$, and $D(2,3) = 9$. The characteristic equation is

$$x^3 - 2x^2 - 2x + 1 = 0.$$

Hence the roots are

$$\alpha = \frac{3+\sqrt{5}}{2}, \beta = \frac{3-\sqrt{5}}{2} \text{ and } \gamma = -1.$$

The general solution of Eq. (2.1) is

$$D(2,n) = A\alpha^n + B\beta^n + C\gamma^n.$$

By using the initial conditions $D(2,1) = 1, D(2,2) = 4$ and $D(2,3) = 9$, we get the result. This completes the proof.

□

Next, we show that the number of seating derangements in a $2 \times n$ grid, with the restriction of no horizontal movement is equal to the square of the $(n+1)$-th Fibonacci number. Recall that the Fibonacci sequence $\{F_n\}_{n \in N}$ is defined recursively by

$$F_1 = 1, F_2 = 1,$$

and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Thus, the square of the Fibonacci numbers are 1, 1, 4, 9, 25,. . . .

Theorem 2.3. *The number of seating derangements in a $2 \times n$ grid, with the restriction of no horizontal movement is equal to the square of the Fibonacci number F_{n+1}^2.*

Proof. By Theorem 2.1, the number of the $2 \times n$ seating derangement with no horizontal moved is $D(2,n)$, which satisfies the recurrence relations:

$$D(2,n) = 2D(2,n-1) + 2D(2,n-2) - D(2,n-3), \text{ for } n \geq 4,$$

where $D(2,1) = 1, D(2,2) = 4$ and $D(2,3) = 9$. We will prove that $D(2,n) = F_{n+1}^2$ by induction on n. Clearly that the equation holds for $n = 1$ and $n = 2$. Next, assume the equation is true for all $n \leq k$. Then,

$$\begin{aligned}
&D(2,k+1) \\
&= 2D(2,k) + 2D(2,k-1) - D(2,k-2) \\
&= 2F_{k+1}^2 + 2F_k^2 - F_{k-1}^2 \\
&= (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2) \\
&\quad + (F_{k+1}^2 - 2F_{k+1}F_k + F_k^2) - F_{k+1}^2 \\
&= (F_{k+1} + F_k)^2 + (F_{k+1} - F_k)^2 - F_{k-1}^2 \\
&= F_{k+2}^2 + F_{k-1}^2 - F_{k-1}^2 \\
&= F_{k+2}^2.
\end{aligned}$$

Therefore, $D(2,n) = F_{n+1}^2$ for all $n \geq 1$.

□

Now, we extend our analysis to include the solution for the $2m \times n$ seating derangement with no horizontal movement. By utilizing Theorem 2.3, we can compute the number of ways to carry out the seating derangement for any given $2m \times n$ grid, where no horizontal movement is allowed.

Theorem 2.4. *Let m and n be positive integers. The number of seating derangements in a $2m \times n$ grid, where no horizontal movement occurs, is equal to F_{n+1}^{2m}, where F_n is the n-th Fibonacci numbers.*

Proof. Let n be a fixed positive integer. For any positive integer m, let $D(2m,n)$ denote the number of seating derangements in a

$2m \times n$ grid with no horizontal movement. We will show that $D(2m, n) = F_{n+1}^{2m}$ by induction on m.

When $m = 1$, it is true by Theorem 2.3 that $D(2, n) = F_{n+1}^{2}$. Assume that $D(2k, n) = F_{n+1}^{2k}$ holds for all positive integer k≤m. We now prove the formula for $m + 1$.

Let A represent the block of the first $2m$ rows and let represent the block of the remaining 2 rows.

By the condition of the problem, the movement of individuals in A and B cannot interfere with each other. By the inductive hypothesis, the number of ways to seat people in block A is F_{n+1}^{2m}, and the number of ways to seat people in block B is F_{n+1}^{2}. Therefore, the total number of ways to arrange seating in the $2(m + 1) \times n$ grid is

$$\begin{aligned} D(2(m+1), n) &= (F_{n+1}^{2m})(F_{n+1}^{2}) \\ &= F_{n+1}^{2(m+1)}. \end{aligned}$$

This completes the proof. □

Example 2.5. The number of seating derangements in a 4×3 grid, where no horizontal movement occurs, is 81.

Proof. Divide the 4×3 grid into two independent blocks: the first two rows form one 2×3 block, and the last two rows form another 2×3 block. The number of ways to arrange seating in a 2×3 grid while satisfying the derangement condition is $D(2, 3) = 9$, as shown in Fig. 1(c). Since horizontal movement is restricted, the arrangements in the two blocks are independent of each other. Thus, the total number of seating derangements in the 4×3 grid is:

$$D(4, 3) = (9)(9) = 81.$$

Notably, $D(4, 3) = 81 = F_4^4$. □

3. Conclusion

In this paper, we analyzed the problem of seating derangements on a grid, focusing on cases where horizontal shifts are not allowed, and movements are restricted to vertical or diagonal shifts. We derived recurrence relations to determine the number of valid seating derangements for grids of various sizes, particularly focusing on $2 \times n$ and $2m \times n$ grids.

Our results provide a framework for calculating seating permutations under these movement constraints, extending previous work on derangements with more general movement rules.

Acknowledgements

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