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Original research article

Strong Homomorphisms of Topological Groupoids

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ABSTRACT

In this work, we generalized to topological groupoids that we known the strong morphism for groups. Also we obtained some characterizations of these homomorphisms. Namely, when there is a homomorphism between algebraic structures, there is an important result that arises from this homomorphism between the substructures of these algebraic structures: the correspondence theorem between substructures. From the perspective of groupoid theory, this theorem establishes a bijective correspondence between the subgroupoids of two related groupoids under a surjective homomorphism.

Keywords: Groupoid; Topological groupoid; Strong homomorphism

1. Introduction

Groupoids were first defined algebraically by Brandt [1] in 1926, and thanks to Ehresmann's categorical approach in the 1950s [2], they began to be studied rapidly in many branches of mathematics. It has become a valuable tool in various branches of mathematics, especially topology, algebra, and geometry[2–7]. Topological groupoids, which were introduced by incorporating the concept of continuity into the structure of groupoids, have become a useful tool especially in areas such as algebraic topology, where understanding the topological structure of spaces is of great importance [3, 4, 8–10].

Homomorphisms play a very important role in the study of mathematical structures, especially in group theory. They represent special morphisms between groups, such as epimorphisms and monomorphisms, which play an important role in understanding the structure and properties of groups. Groupoids, as generalizations of groups, also have their own special morphisms. Ivan's works on special morphisms of groupoids in the algebraic sense shed light on this issue [4, 5]. Later, Gürsoy and his colleagues examined the homomorphisms of topologi-

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cal groupoids, bringing together the rich structure of groupoids with the additional complexity of topology [8]. Understanding the relationships between these special homomorphisms is very important in solving the complex dynamics of topological groupoids.

Strong homomorphisms of algebraic structures offer a more rigorous way to study and understand mathematical structures such as groups and groupoids, with mutual preservation of both operation and algebraic properties.

On the other hand, when there is a homomorphism between algebraic structures, there is an important result that arises from this homomorphism between the substructures of these algebraic structures: the correspondence theorem between substructures. From the perspective of groupoid theory, this theorem establishes a bijective correspondence between the subgroupoids of two related groupoids under a surjective homomorphism. The most important result of the correspondence theorem for topological subgroupoids presented in the paper is that it allows us to examine in detail the topological subgroupoids of the topological groupoid G' by looking at the topological subgroupoids of the topological groupoid G by means of a strong homomorphism $\mu : G \to G'$.

2. Topological Groupoids and Homomorphisms

We here give some basic definitions and useful properties related to the groupoid theory.

Definition 2.1 ([11]). A **groupoid** consists of two sets G and G_0, called respectively the groupoid and the base, together with two maps α and β from G to G_0, called respectively the source and target maps, a map $\epsilon : G_0 \longrightarrow G, x \longmapsto \widetilde{x}$ called the object (inclusion) map, a map $i : G \longrightarrow G, a \longmapsto a^{-1}$ called the invers map and a partial composition map m defined by $m(a,b) = ab$ on the set $G * G = \{(a,b) \in G \times G \mid \alpha(a) = \beta(b)\}$, all subject to the following conditions:

i) $\alpha(ab) = \alpha(b)$ and $\beta(ab) = \beta(a)$ for all $(a,b) \in G * G$,

ii) $a(bc) = (ab)c$ for all $a,b,c \in G$ such that $\alpha(a) = \beta(b)$ and $\alpha(b) = \beta(c)$,

iii) $\alpha(\widetilde{x}) = \beta(\widetilde{x}) = x$ for all $x \in G_0$,

iv) $a\widetilde{\alpha(a)} = a$ and $\widetilde{\beta(a)}a = a$ for all $a \in G$,

v) each $a \in G$ has a (two-sided) inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $aa^{-1} = \widetilde{\alpha(a)}$, $aa^{-1} = \widetilde{\beta(a)}$.

In this paper, we will denote a groupoid G over G_0 by (G, G_0). For any objects $x, y \in G_0$ in a groupoid (G, G_0), the set $G_x = \{a \in G \mid \alpha(a) = x\}$ is called $\alpha-$fibre of G over x and the set $G^y = \{a \in G \mid \beta(a) = y\}$ is called $\beta-$fibre of G over y. Also we have $G(x,y) = G_x^y = \{a \in G \mid \alpha(a) = x, \beta(a) = y\}$. Clearly, $G(x,x) = G_x^x = G(x)$ is a group and is called the isotropy (or vertex) group at x.

Example 2.2 ([12]). For any non-empty set U, the product $U \times U$ is a groupoid over U. It is called the banal groupoid. The source and target maps are defined by the natural projections, the object map is defined by $u \mapsto (u,u)$ for any element u. For any morphism (u,v), the inverse is (v,u). Also, the partial composition is defined by $(u,v).(v,w) = (u,w)$ for morphisms $(u,v), (v,w) \in U \times U$.

Definition 2.3 ([12]). Let (G, G_0) and (H, H_0) be groupoids. A pair of $(\mu, \mu_0) : (G, G_0) \to (H, H_0)$ is called a groupoid homomorphism if $\alpha_H \circ \mu = \mu_0 \circ \alpha_G$,

$\beta_H \circ \mu = \mu_0 \circ \beta_G$ and $\mu(ab) = \mu(a)\mu(b)$ for $(a, b) \in G * G$.

Also, μ is a said to be homomorphism over μ_0. If $G_0 = H_0$ and $\mu = Id$, then μ is said to be a homomorphism over G_0. (μ, μ_0) is an **isomorphism** if both μ and μ_0 are bijections. A groupoid (G, G_0) is said to be a **subgroupoid** of (H, H_0) if the maps μ and μ_0 are the inclusions. In addition, if $G_0 = H_0$ then (G, G_0) is said to be a **wide subgroupoid** of (H, G_0). A wide subgroupoid (N, G_0) that satisfies $a\lambda a^{-1} \in N$ with $\alpha(a) = \alpha(\lambda) = \beta(\lambda)$ for any $\lambda \in N$ and any $a \in G$ is called a normal subgroupoid (G, G_0).

Definition 2.4 ([9]). A **topological groupoid** is a groupoid (G, G_0) together with topologies on G and G_0 such that the five maps which define the groupoid structure are continuous.

Example 2.5 ([12]). A topological group can be regarded as a topological groupoid with only one object.

Example 2.6 ([12]). Let G be a topological group and let U be a G-space with a continuous action $\cdot : U \times G \longrightarrow U$. Then $U \times G$ is a topological groupoid over U. Indeed, $U \times G$ has the product topology of U and G. The continuous action naturally gives rise to a map

$$(U \times G) \times G \xrightarrow{\star} U \times G$$

$$((u, a_1), a_2) \mapsto (u, a_1 a_2)$$

and this is continuous since $pr_1 \circ \star((u, a_1), a_2) = u$, which is the projection on the first factor, and $pr_2 \circ \star((u, a_1), a_2) = a_1 a_2$, which is the composite

$$(U \times G) \times G \xrightarrow{id} U \times (G \times G) \xrightarrow{pr_2} G \times G \xrightarrow{multip.} G$$

and so $pr_2 \circ \star$ is continuous, thus $\star$ is continuous. Now

$$(U \times G) * (U \times G) =$$

$$\{((u_2, a_2), (u_1, a_1)) \mid u_1 \cdot a_1 = u_2\},$$

and $m : (U \times G) * (U \times G) \to U \times G$ is defined by $(u_2, a_2)(u_1, a_1) = (u_1, a_1 a_2)$, which is the composite

$$(U \times G) * (U \times G) \to (U \times G) \times G \xrightarrow{\star} U \times G$$
$$((u_2, a_2), (u_1, a_1)) \mapsto ((u_1, a_1), a_2) \mapsto (u_1, a_1 a_2)$$

and so the partial multiplication is continuous. The continuity of other groupoid structure maps can be shown in a similar way. Consequently, $(U \times G, U)$ is a topological groupoid.

Definition 2.7 ([13]). Let (G, G_0) be a topological groupoid. A **topological subgroupoid** of (G, G_0) is a subgroupoid (H, H_0) of (G, G_0) equipped with the subspace topologies inherited from (G, G_0).

Example 2.8 ([13]). Let (G, G_0) be a topological groupoid. Then,

i) for each $x \in G_0$, the isotropy group $G(x)$ is a topological group with subspace topology under the restriction of the partial composition. A topological group bundle is the union of its isotropy groups $G(x), x \in G_0$ (here two elements may be composed iff they lie in the same fibre). We denote it by $Is(G)$, which is a topological subgroupoid.

ii) the subspace $\epsilon(G_0)$ is called the **unity space** of (G, G_0), which is a topological wide subgroupoid with the subspace topology. Obviously, we have $\epsilon(G_0) \subseteq Is(G)$.

Definition 2.9 ([13]). A topological groupoid G over G_0 is said to be **transitive** if the continuous map $\alpha \times \beta : G \longrightarrow G_0 \times G_0$ given by

$(\alpha \times \beta)(a) = (\alpha(a), \beta(a))$, $\forall a \in G$, is surjective, or equivalently if $G(x, y) \neq \emptyset$ for each pair of $x, y \in G_0$.

Definition 2.10 ([13]). By a **topological groupoid homomorphism** $(\mu, \mu_0) : (G, G_0) \to (H, H_0)$, we mean a homomorphism of groupoids which is continuous on both objects and morphisms.

Proposition 2.11 ([13]). *Let* $(\mu, \mu_0) : (G, G_0) \longrightarrow (H, H_0)$ *be topological groupoid homomorphism. Then we have* $\mu \circ \epsilon = \acute{\epsilon} \circ \mu_0$ *and* $\mu \circ i = \acute{i} \circ \mu$.

For a topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$, a set $Ker\ \mu = \{a \in G \mid \mu(a) \in \epsilon(G_0)\}$ endowed with subspace topology is said kernel of μ.

Proposition 2.12 ([9]). *For a topological groupoid homomorphism* $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$*, we have:*

1. *If* (K', K'_0) *is a topological subgroupoid of* (G', G'_0)*, then* $(\mu^{-1}(K'), \mu_0^{-1}(K'_0))$ *is also topological subgroupoid of* (G, G_0).
2. *If* (K', G'_0) *is a topological normal subgroupoid of* (G', G'_0)*, then* $\mu^{-1}(K')$ *is a topological normal subgroupoid of* (G, G_0) *such that* $Ker\mu \subset \mu^{-1}(G')$.

Proof. 1. Let (K', K'_0) be a topological subgroupoid of (G', G'_0). Firstly, let us show that $\beta(\mu^{-1}(K')) \subset \mu_0^{-1}(G'_0)$. If $y \in \beta(\mu^{-1}(K'))$, then we have $y = \beta(a)$ for any $a \in \mu^{-1}(K')$. Since μ is a topological groupoid homomorphism and (K', K'_0) is a topological subgroupoid, we have $\mu(a) \in K'$ and $\beta'(K') \subset K'_0$. Hence, it follows $\mu_0(y) = \mu_0(\beta(a)) = \beta'(\mu(a)) \in K'_0$. Therefore, it is obtained $y \in \mu_0^{-1}(K'_0)$. Similarly, it is easily shown that $\alpha(\mu^{-1}(K')) \subset \mu_0^{-1}(G'_0)$.

Now let us take $a, b \in \mu^{-1}(K')$ with $\beta(a) = \alpha(b)$. Then, we have $\mu(a), \mu(b) \in K'$ and $\beta'(\mu(a)) = \mu_0(\beta(a)) = \mu_0(\alpha(b)) = \alpha'(\mu(b))$. Hence, it follows $\mu(a).\mu(b) \in G'$. Even more, we have $\mu(a).\mu(b) \in K'$, because K' is a topological subgroupoid. Since μ is homomorphism, we have $\mu(a.b) \in K'$. So, it is obtained $a.b \in \mu^{-1}(K')$. That is, $\mu^{-1}(K')$ is closed under partial composition.

Let us see that $\varepsilon(x) \in \mu^{-1}(K')$ for all $x \in \mu_0^{-1}(K'_0)$. Since K' is a topological subgroupoid, it follows that $\mu_0(x) \in K'_0$ and $\varepsilon'(\mu_0(x)) \in K'$. Since μ is homomorphism, it follows $\mu(\varepsilon(x)) \in K'$. That is, we have $\varepsilon(x) \in \mu^{-1}(K')$.

Finally, we have to show that the inversion is closed in K'. For it, let $a \in \mu^{-1}(K')$. Then, we have $\mu(a) \in K'$. Hence it follows $(\mu(a))^{-1} \in K'$, because K' is topological subgroupoid. Also, since μ is homomorphism, we obtain $\mu(a^{-1}) \in K'$. That is, it is found $a^{-1} \in \mu^{-1}(K')$.

It is clear that the topological structure of $(\mu^{-1}(K'), \mu_0^{-1}(K'_0))$ is follows from that the continuity of (μ, μ_0) and subspace topology of (G, G_0).

2. From part 1, it is easily seen that $(\mu^{-1}(K'), G_0$ is topological subgroupoid of (G, G_0). For proof, it is enough to show that normality condition is hold and $Ker\mu \subset \mu^{-1}(G')$.

Let $\lambda \in \mu^{-1}(K')$ and $a \in G$ with $\beta(a) = \alpha(\lambda) = \beta(\lambda)$. We want to prove that $a.\lambda.a^{-1} \in \mu^{-1}(K')$. Clearly, we have $\mu(\lambda) \in K'$, $\beta'(\mu(a)) = \mu_0(\beta(a)) = \mu_0(\alpha(\lambda)) = \alpha'(\mu(\lambda))$ and $\beta'(\mu(a)) = \mu_0(\beta(a)) = \mu_0(\beta(\lambda)) = \beta'(\mu(\lambda))$. From these equalities and the normality of K', it follows $\mu(a).\mu(\lambda).(\mu(a))^{-1} \in K'$. Since μ is homomorphism, we have $\mu(a.\lambda.a^{-1}) \in K'$. Hence $a.\lambda.a^{-1} \in \mu^{-1}(K')$. Therefore, $\mu^{-1}(K')$ is normal in G.

Now let us prove that $Ker\mu \subset \mu^{-1}(G')$. For it, let us take any $a \in Ker\mu$. Then, we have $\mu(a) = \varepsilon'(x')$ for an object $x' \in G'_0$. Since $\varepsilon'(x') \in K'$, we have $\mu(a) \in K'$. Hence it follows $a \in \mu^{-1}(K')$. Therefore, we obtain $Ker\mu \subset \mu^{-1}(G')$.

□

Proposition 2.13 ([10]). *Let $(\mu, \mu_0) : (G, G_0) \longrightarrow (H, H_0)$ be a homomorphism of groupoids of G onto H, where H is a topological groupoid. Then μ induces a topology on G compatible with the groupoid structure of G and μ is then a homomorphism of topological groupoids.*

Proof. Define a set $U \subset G$ to be open if and only if $U = \mu^{-1}(V)$ for some open set $V \subset H$, and define $\overline{U} \subset G_0$ to be open if and only if $\overline{U} = \mu_0^{-1}(\overline{V})$ for some open set $\overline{V} \subset H_0$. This defines a topology on G and on G_0, and μ and μ_0 being surjective are both continuous and open maps.

Since the diagram

$$\begin{array}{ccc} G * G & \xrightarrow{m_G \times m_H} & H * H \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle m_H} \\ G & \xrightarrow{\mu} & H \end{array}$$

is commutative, where m_G and m_H denote the composition functions of G and H respectively, it follows that m_G is continuous, for if $U \subset G$ is open, $U = \mu^{-1}(V)$ for some open $V \subset H$, and then $m_G^{-1}(U) = m_G^{-1}(\mu^{-1}(V)) = (\mu \times \mu)^{-1}(m_H^{-1}(V))$, which is open in $G * G$. The continuity of the invers map proved similarly. Also, since the diagrams

$$\begin{array}{ccc} G & \xrightarrow{\mu} & H \\ {\scriptstyle \alpha_G}\downarrow & & \downarrow{\scriptstyle \alpha_H} \\ G_0 & \xrightarrow[\mu_0]{} & H_0 \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mu} & H \\ {\scriptstyle \beta_G}\downarrow & & \downarrow{\scriptstyle \beta_H} \\ G_0 & \xrightarrow[\mu_0]{} & H_0 \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{\epsilon_G} & G \\ {\scriptstyle \mu_0}\downarrow & & \downarrow{\scriptstyle \mu} \\ H_0 & \xrightarrow[\epsilon_H]{} & H \end{array}$$

are commutative, α_G, β_G and ϵ_G are continuous. Therefore the proof is completed. □

Definition 2.14 ([9]). Given a topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G, G'_0)$, if μ is a homeomorphism, we say it a topological groupoid isomorphism.

Example 2.15 ([9]). i) If (G, G_0) is a topological groupoid, then clearly (Id_G, Id_{G_0}) is a topological groupoid isomorphism.

ii) Let $(\mu, \mu_0) : (G, G_0) \to (H, H_0)$ and $(\nu, \nu_0) : (H, H_0) \to (K, K_0)$ be two topological groupoid homomorphisms. Then the composition $(\nu, \nu_0) \circ (\mu, \mu_0) : (G, G_0) \to (K, K_0)$ defined by $(\nu, \nu_0) \circ (\mu, \mu_0) = (\nu \circ \mu, \nu_0 \circ \mu_0)$ is a topological groupoid homomorphism.

If $(\mu, \mu_0) : (G, G_0) \to (H, H_0)$ is a topological groupoid homomorphism then for every $x, y \in G_0$ we have

$$\mu(G_x) \subseteq H_{\mu_0(x)} \ , \ \mu(G^y) \subseteq (H)^{\mu_0(y)} \ ,$$

$$\mu(G_x^y) \subseteq (H)_{\mu_0(x)}^{\mu_0(y)}.$$

Then the restrictions of μ to G_x , G^y , G_x^y respectively, defines the continuous maps

$$\mu_x : G_x \to H_{\mu_0(x)} \ , \ \mu^y : G^y \to (H)^{\mu_0(y)},$$

$$\mu_x^y : G_x^y \longrightarrow (H)_{\mu_0(x)}^{\mu_0(y)}.$$

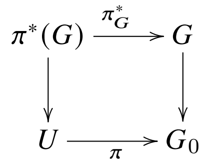
But these maps are not be topological groupoid homomorphisms.

Now let us give an important concept that will be the subject of future theorems.

Definition 2.16 ([9]). Let (G, G_0) be a topological groupoid, U a topological space and the map $\pi : U \to G_0$ continuous. Then a set
$\pi^*(G)$
$= \{(u_1, u_2, a) \mid \pi(u_1) = \alpha(a), \pi(u_2) = \beta(a)\}$
is a topological groupoid over U. The groupoid structure of $(\pi^*(G), U)$ as follows:
the projections $\alpha^*(u_1, u_2, a) = u_1$ and $\beta^*(u_1, u_2, a) = u_2$,
the object map $\epsilon^*(u) = (u, u, \epsilon(\pi(u)))$,
the partial composition $m^*((u_1, u_2, a), (u_2, u_3, b)) = (u_1, u_3, ab)$,
the inversion $i^*(u_1, u_2, a) = (u_2, u_1, i(a))$.
We denote it by $(\pi^*(G), U)$.

We say it ”the induced topological groupoid of (G, G_0) under π”.

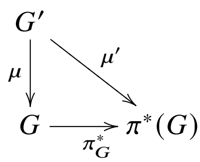
A canonical homomorphism arises between a topological groupoid and its induced groupoid. Namely; let (G, G_0) be topological groupoid and $\pi^*(G)$ its induced topological groupoid under $\pi : U \to G_0$. Then a topological groupoid homomorphism $(\pi^*_G, \pi) : (\pi^*(G), U) \to (G, G_0)$ is defined by $\pi^*(u_1, u_2, a) = a$, is said canonical homomorphism of induced topological groupoid.



As stated in the following theorem, a canonical homomorphism satisfies the universality property. Let us state the theorem without proof.

Theorem 2.17 ([13]). *Let (G, G_0) be a topological groupoid and let $(\pi^*(G), U)$ be its induced topological groupoid with a continuous map $\pi : U \to G_0$. Then the homomorphism $(\pi^*_G, \pi) : (\pi^*(G), U) \to (G, G_0)$ satisfies the universality property:*
for every topological groupoid homomorphism $(\mu, \pi) : (G', U) \to (G, G_0)$

we have one and only one $U-homomorphism\ \mu' : G' \to \pi^*(G)$ of topological groupoids such that the diagram



is commutative.

Proposition 2.18 ([8]). *For a continuous map $\pi : V \to U$, there is a functor π^* between the category $\mathcal{G}(U)$ of topological groupoids over U and the category $\mathcal{G}(V)$ of induced topological groupoids of them over V.*

Proof. For the proof, we need to show that $\pi^* : \mathcal{G}(U) \to \mathcal{G}(V)$ satisfies the conditions for being a functor.

For any topological groupoids (G,U) and (G',U) over U, we have the induced topological groupoids $(\pi^*(G),V)$ and $(\pi^*(G'),V)$. Let us consider topological groupoid homomorphism $\mu : (G,U) \rightarrow (G',U)$. It is easily seen that $\pi^*(\mu) : (\pi^*(G),V) \rightarrow (\pi^*(G'),V)$ defined by

$$\pi^*(\mu)(v_1,v_2,a) = (v_1,v_2,\mu(a)) \in \pi^*(G')$$

for all $(v_1,v_2,a) \in \pi^*(G)$, is a topological groupoid homomorphism.

π^* defined as above is a functor. Indeed, $\pi^*(id_G) = id_{\pi^*(G)}$, and also if $\eta : (G',U) \rightarrow (G'',U)$ is another topological groupoid homomorphism over U, then $\pi^*(\eta\circ\mu) : \pi^*(G) \rightarrow \pi^*(G'')$ defined by for every $(v_1,v_2,a) \in \pi^*(G)$

$$\pi^*(\eta\circ\mu)(v_1,v_2,a) = (v_1,v_2,(\eta\circ\mu)(a))$$

is a topological groupoid homomorphism over V such that $\pi^*(\eta\circ\mu) = \pi^*(\eta)\circ\pi^*(\mu)$. □

Proposition 2.19 ([8]). *Let $\pi : W \rightarrow V$ and $k : V \rightarrow U$ be continuous maps and let (G,U) be a topological groupoid. Then there exists an isomorphism between the induced topological groupoids $\pi^*(\sigma^*(G))$ and $(\sigma\circ\pi)^*(G)$.*

Proof. First of all, let us clearly state the induced topological groupoids mentioned in the proposition.

$\sigma^*(G)$
$= \{(v_1,v_2,a) : \sigma(v_1) = \alpha(a), \sigma(v_2) = \beta(a)\}$,
$\pi^*(\sigma^*(G))$
$= \{(w_1,w_2,(v_1,v_2,a)) : \pi(w_1) = v_1,\ \pi(w_2) = v_2\}$,
$(\sigma\circ\pi)^*(G)$
$= \{(w_1,w_2,a) : (\sigma\circ\pi)(w_1) = \alpha(a)$
$, (\sigma\circ\pi)(w_2) = \beta(a)\}$.

Now let us define the mappings

$$\Phi : (\sigma\circ\pi)^*(G) \longrightarrow \pi^*(\sigma^*(G))$$

$$(w_1,w_2,a) \mapsto (w_1,w_2,(\pi(w_1),\pi(w_2),a))$$

and

$$\Psi : \pi^*(\sigma^*(G)) \longrightarrow (\sigma\circ\pi)^*(G)$$

$$(w_1,w_2,(v_1,v_2,a)) \mapsto (w_1,w_2,a).$$

It is clear that Φ and Ψ are continuous functors such that $\Psi\circ\Phi = id_{(\sigma\circ\pi)^*(G)}$ and $\Phi\circ\Psi = id_{\pi^*(\sigma^*(G))}$. So Φ is an isomorphism between $(\sigma\circ\pi)^*(G)$ and $\pi^*(\sigma^*(G))$. □

Proposition 2.20 ([8]). *A topological groupoid (G,U) and its induced topological groupoid $id_U^*(G)$ are $U-isomorphic$.*

Proof. It is straightforward to show that the map $\Phi : G \rightarrow id_U^*(G)$ defined by $\Phi(a) = (\alpha(a),\beta(a),a)$ between the topological groupoid G and its induced topological groupoid $id_U^*(G) = \{(u_1,u_2,a) \mid \alpha(a) = u_1, \beta(a) = u_2\}$ is an isomorphism.

□

Proposition 2.21 ([8]). *If a topological groupoid (G,U) is a transitive, then induced topological groupoid $(\pi^*(G),V)$ of (G,U) under the continuous map $\pi : V \rightarrow U$ is also transitive.*

Proof. Let (G,U) be transitive. Then, we have the surjective continuous map $\alpha\times\beta : G \rightarrow U\times U$, $(\alpha\times\beta)(a) = (\alpha(a),\beta(a))$. Hence there is an element $a \in G$ such that $(\alpha\times\beta)(a) = (\pi(v_1),\pi(v_2))$ whenever $(v_1,v_2) \in V\times V$, i.e. $\alpha(a) = \pi(v_1)$ and $\beta(a) = \pi(v_2)$. Thus $(v_1,v_2,a) \in \pi^*(G)$ and $(\alpha^*\times\beta^*)(v_1,v_2,a) =$

$(\alpha^*(v_1, v_2, a), \beta^*(v_1, v_2, a)) = (v_1, v_2)$. That is, $\alpha^* \times \beta^* : \pi^*(G) \to V \times V$ is surjective. On the other hand, since α^* and β^* are projections onto the first and second factors, resp., the map $\alpha^* \times \beta^*$ is continuous. Therefore, $(\pi^*(G), V)$ is transitive. □

3. Strong Homomorphisms

In this section we give the topological version of the notion of strong homomorphism of the groupoids defined by Ivan. Also, It is well known that there is an important result that arises from a homomorphism between the substructures of algebraic structures: the correspondence theorem between substructures. From the perspective of groupoid theory, this theorem establishes a bijective correspondence between the subgroupoids of two related groupoids under a surjective homomorphism. In this section, we examine the correspondence theorem in terms of topological groupoids, taking into account the strong homomorphism. The most important result of the correspondence theorem for topological subgroupoids presented in the paper is that it allows us to examine in detail the topological subgroupoids of the topological groupoid H by looking at the topological subgroupoids of the topological groupoid G by means of a strong homomorphism $\mu : G \to H$.

Definition 3.1. A strong homomorphism of the topological groupoids is a $(\mu, \mu_0) : (G, G_0) \to (G', G'_0)$ topological groupoid homomorphism that provides the following condition:

$$(\mu(a), \mu(b)) \in G' * G'$$

$$\Rightarrow$$

$$(a, b) \in G * G, a, b \in G.$$

Example 3.2.

1. Let (G, G_0) be a topological groupoid. Then $\mu = \alpha \times \beta : G \longrightarrow G_0 \times G_0$ is a strong homomorphism of the topological groupoids. For it, we must show that $(a, b) \in G * G$ while $(\mu(a), \mu(b)) = ((\alpha \times \beta)(a), (\alpha \times \beta)(b)) \in (G_0 \times G_0) * (G_0 \times G_0)$. Since $(\mu(a), \mu(b)) \in (G_0 \times G_0) * (G_0 \times G_0)$ defined, we can write

   $$\beta'(\mu(a)) = \alpha'(\mu(b)).$$

   Hence, we have the equalities

   $$\beta'(\mu(a)) = \beta'(\alpha(a), \beta(a)) = \beta(a),$$

   $$\alpha'(\mu(b)) = \alpha'(\alpha(b), \beta(b)) = \alpha(b).$$

   Thus, the equality $\beta(a) = \alpha(b)$ is obtained, which means $(a, b) \in G * G$.

2. Let us consider the induced topological groupoid $\pi^*(G)$ of a topological groupoid (G, G_0) under a continuous map $\pi : U \to G_0$. Then it is clear that we have the canonical homomorphism $(\pi^*_G, \pi) : (\pi^*(G), U) \to (G, G_0)$. Let us take a composible pair $(\pi^*_G(u, v, a), \pi^*_G(u', v', b))$ from $G * G$. We want to see that if it is $((u, v, a), (u', v', b)) \in \pi^*(G) * \pi^*(G)$ or not. If $(\pi^*_G(u, v, a), \pi^*_G(u', v', b)) \in G * G$, then we have $(a, b) \in G * G$ from the definition of canonical homomorphism. Hence, $\beta(a) = \alpha(b)$. So, the equality $\pi(y) = \pi(x')$ comes. However, since the map π is not defined injectively, we cannot obtain the equality $v = u'$. Therefore, the canonical homomorphism (π^*_Γ, π) is not a strong homomorphism of the topological groupoids.

Theorem 3.3. *1. If $(\mu, \mu_0) : (G, G_0) \to (G', G'_0)$ is a topological groupoid homomorphism such that the map μ_0 is injective, then (μ, μ_0) is a strong homomorphism of the topological groupoids.*

2. Every G_0- homomorphism of topological groupoids $\mu : G \to G'$ is a strong homomorphism.

Proof. 1. We suppose that $a, b \in G$ with $(\mu(a), \mu(b)) \in G' * G'$. Then
$\beta'(\mu(a)) = \alpha'(\mu(b))$
$\Rightarrow (\beta' \circ \mu)(a) = (\alpha' \circ \mu)(b)$
$\Rightarrow (\mu_0 \circ \beta)(a) = (\mu_0 \circ \alpha)(b)$
$\Rightarrow \mu_0(\beta(a)) = \mu_0(\alpha(b))$
$\Rightarrow \beta(a) = \alpha(b)$ (since μ_0 is injective)
$\Rightarrow (a, b) \in G * G$.

Hence (μ, μ_0) is a strong homomorphism of the topological groupoids.

2. This is a consequence of (1), since $\mu_0 = Id_{G_0}$.

□

Proposition 3.4. *Let $(\mu, \mu_0) : (G, G_0) \to (G', G'_0)$ be a strong homomorphism of topological groupoids. Then, we have:*

1. If (K, K_0) is a topological subgroupoid of (G, G_0), then $(\mu(K), \mu_0(K_0))$ is also a topological subgroupoid of (G', G'_0). Specifically, the $(Im\mu, Im\mu_0)$ forms a topological subgroupoid of G'.

2. If μ is surjective and K is a normal topological subgroupoid of G, then $\mu(K)$ is also a normal topological subgroupoid of G'.

Proof. 1. Let (K, K_0) be a topological subgroupoid of (G, G_0). For the proof, we need to satisfy the conditions for $(\mu(K), \mu_0(K_0))$ to be a topological subgroupoid.

• Let us show that$(\alpha'(\mu(K))) \subseteq \mu_0(K_0)$. In this case, if $x' \in \alpha'(\mu(K))$, then there exists $b' \in \mu(K)$ such that $x' = \alpha'(b')$. For $b' \in \mu(K)$, there exists $b \in K$ such that $\mu(b) = b'$. Consequently, $x' = \alpha'(b') = \alpha'(\mu(b)) = \mu_0(\alpha(b))$. Since $\alpha(b) \in K_0$, it follows that $x' \in \mu_0(K_0)$. Consequently, $(\alpha'(\mu(K))) \subseteq \mu_0(K_0)$. Similarly, it is shown that $(\beta'(\mu(K))) \subseteq \mu_0(K_0)$.

• Let $a'.b'$ be defined for $a', b' \in \mu(K)$. Let us prove that $a'.b' \in \mu(K)$. Indeed, for $a, b \in K$ such that $a' = \mu(a)$ and $b' = \mu(b)$, the definition of $a'.b'$ implies $(\mu(a), \mu(b)) \in G' * G'$. Since μ is a strong homomorphism of the topological groupoids, this implies $(a, b) \in G * G$. Consequently, $a.b$ is defined. Since K is a topological subgroupoid of G, we have $a.b \in K$. Thus, $a'.b' = \mu(a).\mu(b) = \mu(a.b) \in \mu(K)$.

• Let's show that for every $x' \in \mu_0(K_0)$, $\epsilon'(x') \in \mu(K)$. In that case, for each $x' \in \mu_0(K_0)$, there exists $x \in K_0$ such that $x' = \mu_0(x)$. Since $\epsilon(x) \in K$, then $\epsilon'(x') = \epsilon'(\mu_0(x)) = \mu(\epsilon(x)) \in \mu(K)$.

• Let's show that for every $a' \in \mu(K)$, $(a')^{-1} \in \mu(K)$. Indeed, $a \in K$ since $a' = \mu(a) \Rightarrow a^{-1} \in K$, then $(a')^{-1} = (\mu(a))^{-1} = \mu(a^{-1}) \in \mu(K)$.

Consequently, $(\mu(K), \mu_0(K_0))$ is a topological subgroupoid of (G', G'_0). Thus, the proof is completed.

2. From part (1), since μ_0 is surjective, $(\mu(K), G'_0)$ is a topological subgroupoid of (G', G'_0). Let $\lambda' \in \mu(K)$ and $a' \in G'$ be such that $\beta'(a') = \alpha'(\lambda) = \beta'(\lambda)$. We aim to prove that $a' \cdot \lambda' \cdot (a')^{-1} \in \mu(K)$. Indeed, since μ is surjective, for $\lambda \in K$, we have $\lambda' = \mu(\lambda)$, and for $a \in G$, we have $a' = \mu(a)$. Since μ is a strong homomorphism of the topological groupoids, from $(\mu(a), \mu(\lambda)), (\mu(\lambda), (\mu(a))^{-1}) \in G' * G'$, it follows that $(a, \lambda), (\lambda, a^{-1}) \in G * G$. As K is normal in G, $a \cdot \lambda \cdot a^{-1}$ is defined, and hence $a \cdot \lambda \cdot a^{-1} \in K$. Thus, $\mu(a \cdot \lambda \cdot a^{-1}) \in \mu(K)$, and $\mu(a).\mu(\lambda).\mu(a^{-1}) = (\mu(a))^{-1}.\mu(\lambda).\mu(a^{-1}) = (a')^{-1}.\lambda'.a' \in \mu(K)$. Therefore, $\mu(K)$ is a topological normal subgroupoid of G'. Hence, the proof is completed.

□

Remark 3.5. *In general, given any topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$, the image $(Im\mu, Im\mu_0)$ may not be a topological groupoid. However, from Proposition 3.4 (i) it is clear that if (μ, μ_0) is strong then the image $(Im\mu, Im\mu_0)$ is also a topological groupoid, And we have also $(\tilde{\mu}, \tilde{\mu_0}) : (G, G_0) \longrightarrow (Im\mu, Im\mu_0)$ is a strong homomorphism of topological groupoids, where $\tilde{\mu}$ and $\tilde{\mu_0}$ are defined by $\tilde{\mu}(a) = \mu(a)$ for each $a \in G$ and $\tilde{\mu_0}(x) = \mu_0(x)$ for each $x \in G_0$, respectively.*

If (G, G_0) is a topological groupoid, we will denote the set of its topological subgroupoids by $S(G, G_0)$, and the set of its topological normal subgroupoids by $\mathcal{N}(G)$.

If $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$ is a topological groupoid homomorphism, we will denote the set of topological subgroupoids of (G, G_0) containing the kernel of μ by $\tilde{S}(G, G_0)$, and the set of topological normal subgroupoids of (G, G_0) containing the kernel of μ by $\tilde{\mathcal{N}}(G)$.

Theorem 3.6. *{The correspondence theorem for topological subgroupoids} For any surjective strong topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$, there exists a bijection correspondence between the set $S(G', G'_0)$ of the topological subgroupoids of (G', G'_0) and the set $\tilde{S}(G, G_0)$ of the topological subgroupoids of (G, G_0) containing the kernel of μ.*

Proof. Let us consider continuous mappings

$$\varphi : \tilde{S}(G, G_0) \to S(G', G'_0)$$
$$K \mapsto \varphi(K) = \mu(K)$$

and

$$\psi : S(G', G'_0) \to \tilde{S}(G, G_0)$$
$$K' \mapsto \psi(K') = \mu^{-1}(K').$$

From Proposition 3.4(i), it is observed that for every $K \in \tilde{S}(G, G_0)$, $\mu(K)$ is a topological subgroupoid of G'. Hence, φ is well-defined. Additionally, due to Proposition 2.12 (1), for every $K' \in S(G')$, $\mu^{-1}(K')$ is a topological subgroupoid of G. Therefore, ψ is well-defined.

The continuous mappings φ and ψ defined above hold the following equalities:

$$\psi \circ \varphi = Id_{\tilde{S}(G)} \text{ and } \varphi \circ \psi = Id_{S(G')} \quad (3.1)$$

The equalities are equivalent to the statements:

$$\mu^{-1}(\mu(K)) = K \ \ and \ \ \mu(\mu^{-1}(K')) = K'$$

$\forall K \in \tilde{S}(G), \forall K' \in S(G')$, respectively.

Let us show that the first equality is hold.

(i) If $a \in K$, then we have $\mu(a) \in \mu(K)$ and $a \in \mu^{-1}(\mu(K))$. Therefore, $K \subseteq \mu^{-1}(\mu(K))$.

(ii) If $a \in \mu^{-1}(\mu(K))$, then we have $\mu(a) \in \mu(K)$ and there exists at least one $b \in K$ such that $\mu(a) = \mu(b)$. Hence, $\mu(a) \cdot (\mu(b))^{-1} \in \epsilon'(G_0')$. Since μ is a homomorphism, $\mu(a \cdot b^{-1}) \in \epsilon'(G_0')$. Then, it follows $a \cdot b^{-1} \in \text{Ker}\,\mu$. If we denote $a \cdot b^{-1}$ by c, then $a = c \cdot b$. Since $c \in \text{Ker}\,\mu \subseteq K$ and $b, c \in K$, we obtain $a \in K$. Thus, $\mu^{-1}(\mu(K)) \subseteq K$.

Consequently, the first equality comes from (i) and (ii).

(iii) Let us now prove that $\mu(\mu^{-1}(K')) \subseteq K'$. Let us take any $a' \in \mu(\mu^{-1}(K'))$. Then there exists at least one $a \in \mu^{-1}(K')$ such that $\mu(a) = a'$. Since $a \in \mu^{-1}(K')$, then $\mu(a) \in K'$. Hence, it follows $a' \in K'$. Therefore, $\mu(\mu^{-1}(K')) \subseteq K'$ is obtained.

(iv) Let $a' \in K'$. Since μ is surjective, there is at least one $a \in G$ such that $\mu(a) = a'$. Here, since $\mu(a) \in K'$, it follows that $a \in \mu^{-1}(K')$. Consequently, $\mu(a) \in \mu(\mu^{-1}(K'))$ and thus $a' \in \mu(\mu^{-1}(K'))$ is obtained. Therefore, $K' \subseteq \mu(\mu^{-1}(K'))$.

From (iii) and (iv), our second equality is obtained.

From Equality 3.1 1, it can be seen that the ψ is invertible. Thus, ψ is a bijection. □

Corollary 3.7. *{The correspondence theorem for topological subgroupoids via a G_0 homomorphism} For a continuous surjective G_0- homomorphism $\mu : G \longrightarrow G'$, there exists a bijective correspondence from the set $S(G', G_0)$ of topological subgroupoids of (G', G_0) to the set $S(G, G_0)$ of topological subgroupoids of (G, G_0).*

Proof. The proof is immediately from Theorem 3.3 and Theorem 3.6. □

If we consider the Proposition 2.12 (ii) and 3.4 (ii) together, the proof of the following theorem will appear automatically.

Theorem 3.8. *{The correspondence theorem for topological normal subgroupoids} For any surjective strong topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G_0')$, there exists a continuous bijective correspondence from the set of topological normal subgroupoids of (G', G_0'), denoted as $\mathcal{N}(G')$, to the set $\tilde{\mathcal{N}}(G)$ of topological normal subgroupoids of (G, G_0) containing $Ker\,\mu$.*

Corollary 3.9. *{The correspondence theorem for topological normal subgroupoids via a G_0- homomorphism} For a surjective G_0- homomorphism $\mu : G \longrightarrow G'$, there exists a continuous bijective correspondence from the set of topological normal subgroupoids of (G', G), denoted as $\mathcal{N}(G')$, to the set $\tilde{\mathcal{N}}(G)$ of topological normal subgroupoids of (G, G_0) containing $Ker\,\mu$.*

Proof. The proof comes immediately from Theorem 3.3 (ii) and Theorem 3.8. □

Remark 3.10.

1) *Theorems 3.6 and 3.8 are generalizations of correspondence theorems for topological subgroups and topological normal subgroups via surjective strong homomorphisms of topological groups.*

2) Theorems 3.6 and 3.8 are not valid for arbitrary continuous surjective homomorphisms of topological groupoids. The strongness of the topological groupoid homomorphism plays a key role in the correspendence.

Corollary 3.11. *For any strong topological groupoid homomorphism $(\mu, \mu_0) : (G, G_0) \longrightarrow (G', G'_0)$, there exists a continuous bijective correspondence from the set $S(Im\mu, Im\mu_0)$ of topological subgroupoids of $(Im\mu, Im\mu_0)$ to the set $S(G, G_0)$ of topological subgroupoids of (G, G_0).*

Proof. From Remark 3.5, if we apply Theorem 3.6 to the strong topological groupoid homomorphism $(\tilde{\mu}, \tilde{\mu_0}) : (G, G_0) \longrightarrow (Im\mu, Im\mu_0)$ corresponding to (μ, μ_0) then the proof is immediately follows. □

4. Conclusion

Given a homomorphism between algebraic structures, we encounter some important results arising from this homomorphism between the substructures of these algebraic structures: the substructure correspondence theorem. In this study, the well-known concept of strong homomorphism between groups is extended to topological groupoids, and the substructure correspondence theorem is examined from the perspective of groupoid theory. More precisely, we prove that there is a bijective correspondence between the subgroupoids (and normal subgroupoids) of two related groupoids under a surjective homomorphism. It becomes clear that this correspondence holds not only between topological groupoids but also between different types of groupoids.

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