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Closed Form of Nearest Point of Linear Pentagonal Fuzzy Number and its Application to Course Assignment Problem with Fuzziness in Preference Level

Phantipa Thipwiwatpotjana[1], Artur Gorka[2], Worrawate Leela-apiradee[3,*]

[1]*Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand*
[2]*Department of Mathematics, Erskine College, Due West, SC 29639, USA*
[3]*Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 12120, Thailand*

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ABSTRACT

This study focuses on optimizing course assignments for instructors based on their preferences, which are inherently uncertain due to personal conflicts affecting decision making. To address this, we model each preference level using two triangular fuzzy numbers, generated based on distinct instructor personality types. When instructors experience uncertainty in selecting a precise preference level, they can choose from these predefined triangular fuzzy numbers. The MIN aggregation of these triangular fuzzy numbers results in a linear pentagonal fuzzy number, whose nearest point in closed form is derived and utilized to represent fuzzy preference levels in our course assignment model. The assignment results obtained using this approach are comparable to those derived from optimal fuzzy weight assignments for specific preference levels. Our method is validated with data from the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, demonstrating its effectiveness in handling uncertainty in course assignment decisions.

Keywords: Fuzzy preference; Nearest point; Pentagonal fuzzy number; Teaching assignment problem

***Corresponding author:** worrawate@mathstat.sci.tu.ac.th **doi:** 10.14456/scitechasia.2025.4

1. Introduction

In general, one of the main concerns in an assignment problem is to be able to put the right man on the right job. It is the same idea when we want to assign courses to instructors. The right job for instructors should be the courses that they prefer to teach. Many models [1–5] had been set up to optimize the number of courses instructors can teach by providing a binary preference ranking to verify whether a given instructor can or cannot teach those courses. However, the 0–1 ranking may not serve the real preference of many instructors. For example, some may be able to teach but do not want to teach. Hmer et al. [6] and Ismayilova et al. [7] considered preferences of instructors in their model. They used numbering scale of preference levels as $1, 2, 3, \ldots$, or proportional scale like 3 out of 5, 2 out of 9, etc. The numbering scale of preference also applied in solving the course assignment problem with interval workload data [8]. However, the results in [9] showed that these preference levels may not provide the most efficient result to all course assignment problems, even though their result optimized their model.

Teaching preference has fuzzy nature. A decision maker has to provide an appropriate membership value to each preference level. Gorka et al. [] ranked the preference into six levels: (i) prefer most to teach, (ii) prefer to teach, (iii) ok to teach, (iv) able to teach, (v) able but does not want to teach, and (vi) cannot teach. There was an explanation why fuzzy membership value assigning the weight of each level provides a better result than other weight scale. Moreover, they analyzed an optimal fuzzy weight for each preference level and achieved the smallest number of courses assigned to instructors who cannot teach them, according to their data.

In this paper, we move one step further into each of the teaching preference levels. Instead of the fuzzy weight assigned to each level of preference, we have another concerned issue. Normally, there are two types of instructors who feel a little bit more or a little bit less than the explanation of each preference level. They may feel like none of the six preference descriptions fit their feelings toward teaching a subject, due to various personal reasons. For example, even if instructor ‘A’ feels confident teaching calculus and chooses preference level (i), a heavy load of grading makes ‘A’ feels like ‘A’ does not definitely prefer to teach it the most. The fuzzy feeling of ‘A’ could be represented as the triangular fuzzy number $\tilde{a}$ in Fig. 1. On the other hand, instructor ‘B’ might not mind a heavy grading load and may think that calculus is a piece of cake. This leads to B’s feeling of more than welcome to teach calculus. However, ‘B’ may also want a more challenging subject to prove his/her ability, which will drop the feeling towards calculus. Therefore, the fuzzy feeling of most preferring to teach calculus of ‘B’ could be the triangular fuzzy number $\tilde{b}$ in Fig. 2. Note that x axis in Figs. 1–3 refers to the rank of preference level (i).

Thus, our concerned issue relates to the fuzziness in each level of preference. A feeling of an instructor may not exactly fit a preference level (i) in the sense that he/she may feel a little bit different when choosing ‘prefer most to teach’. Based on our collected data from the Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, we ask each instructor to choose the fuzzy number that fits his/her feeling the most between fuzzy numbers in Figs. 1 and 2, when expressing his/her feeling towards a subject with level (i). This applies to the similar situations of choos-

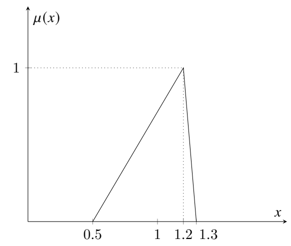

Fig. 1. $\tilde{a} = (0.5, 1.2, 1.3)$.

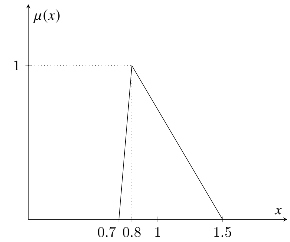

Fig. 2. $\tilde{b} = (0.7, 0.8, 1.5)$.

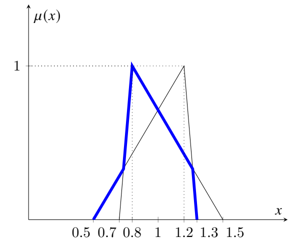

Fig. 3. $\text{MIN}\{\tilde{a}, \tilde{b}\}$.

ing the preference levels (ii)–(vi). We handle this information using MIN aggregation based on the point of view that the favor should tend to the side of behavior with more willing to teach regardless how heavy the load of the side duties like grading or answering questions. The MIN operation on these special triangular fuzzy numbers provides the linear pentagonal fuzzy number $\text{MIN}\{\tilde{a}, \tilde{b}\}$ in Fig. 3. We use the nearest point approach to represent the rank of the preference levels.

In the existing literature, no specific research articles directly apply pentagonal fuzzy numbers to course assignment problems. However, several studies have investigated their use in assignment problems, including solving them with the Hungarian method [10], optimizing assignments in industrial and managerial settings [11], and minimizing fuzzy assignment costs by matching workers with varying skill levels to jobs [12].

The novelties of this paper lie in the integration of personality-driven fuzzy modeling, pentagonal fuzzy aggregation, and a closed form nearest point derivation, all aimed at improving decision making in uncertain course assignments.

As a key step in our approach, a series of theorems are presented in Section 2 to determine the β-cuts of pentagonal fuzzy numbers. The β-cut of a linear pentagonal fuzzy number is then used to derive an explicit formula for its nearest point, providing a crisp numerical representation of the fuzzy number. In Section 3, we assign the fuzzy weight to each nearest point representation of preference level and incorporate it into our course assignment model, accounting for fuzziness in preference levels. The results in Section 4 demonstrate that our model effectively generates course assignments comparable to the results in [9], while also capturing deeper nuances of individual fuzzy preferences. This leads to higher overall instructor satisfaction with their assignments. Finally, the conclusion is presented in Section 5.

2. Linear pentagonal fuzzy number

Environmental decision making under fuzzy uncertainty has been applied in various domains, including inventory management optimization [13,14], digital transformation environments [15], air quality assessment [16], COVID-19 treatment evaluation [17], and safety modeling in the oil and gas industry [18]. In particular, pentagonal fuzzy numbers were integrated with possibility theory for decision making in [19].

A linear pentagonal fuzzy numbers is mathematically presented as follows.

Definition 2.1 (See [20]). Let $a_1 < a_2 < a_3 < a_4 < a_5$ and $r, s \in (0, 1)$ be given. A

linear pentagonal fuzzy number (LPFN), denoted by $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ has its membership function of the form

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & \text{if } x < a_1; \\ r\left(\frac{x-a_1}{a_2-a_1}\right), & \text{if } a_1 \leq x \leq a_2; \\ r + (1-r)\left(\frac{x-a_2}{a_3-a_2}\right), & \text{if } a_2 \leq x \leq a_3; \\ s + (1-s)\left(\frac{a_4-x}{a_4-a_3}\right), & \text{if } a_3 \leq x \leq a_4; \\ s\left(\frac{a_5-x}{a_5-a_4}\right), & \text{if } a_4 \leq x \leq a_5; \\ 0, & \text{if } x > a_5, \end{cases} \tag{2.1}$$

where r and s represent the left and the right picked points of $\tilde{a}$, respectively. Note for the case $0 < r = s < 1$ that it suffices to write $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r)$ instead of $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$.

The following remark ensures that the MIN aggregation of two triangular fuzzy numbers produces other types of fuzzy numbers, such as a triangular fuzzy number, which can always be represented as an LPFN.

Remark 2.2. *If* $r = \frac{a_2-a_1}{a_3-a_1}$ *and* $s = \frac{a_5-a_4}{a_5-a_3}$, *then*

$$r\left(\frac{x-a_1}{a_2-a_1}\right) = \frac{x-a_1}{a_3-a_1} = r + (1-r)\left(\frac{x-a_2}{a_3-a_2}\right)$$

and

$$s\left(\frac{a_5-x}{a_5-a_4}\right) = \frac{a_5-x}{a_5-a_3} = s + (1-s)\left(\frac{a_4-x}{a_4-a_3}\right).$$

By substituting them into Eq. (2.1), it becomes the expression below.

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & \text{if } x < a_1; \\ \frac{x-a_1}{a_3-a_1}, & \text{if } a_1 \leq x \leq a_3; \\ \frac{a_5-x}{a_5-a_3}, & \text{if } a_3 \leq x \leq a_5; \\ 0, & \text{if } x > a_5. \end{cases}$$

That is, the LPFN $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *with five parameters is transformed to be a triangular fuzzy number* $\tilde{a} = (a_1, a_3, a_5)$ *with three parameters.*

The pentagonal fuzzy number (PFN) plays the important role in many research papers. Mondal and Mandal [20] introduced symmetric/asymmetric linear and nonlinear membership of PFN then solved its associated fuzzy linear equation. A new concept of fuzzy game problem whose payoffs are represented as PFNs was proposed by Monisha and Sangeetha in [21]. Panda and Pal [22] constructed a pentagonal fuzzy matrix whose entries are considered as PFNs and studied its algebraic properties. An application of PFN in a neural network problem was published by Raj and Karthik in [23].

Moreover, the PFN with its ranking is an important tool in dealing with many real life applications, especially decision making, data analysis, artificial intelligence and various fields of operation research. Authors [24, 25] defined the ranking procedure of PFNs based on their incenter of centroids. Ponnivalavan and Pathinathan [26] defined the ranking function of PFN as the total integral value of its inverse membership function. Researchers in [26–28] used their own ranking technique to achieve an algorithm to find the shortest path problem, to find critical path in a directed acyclic graph and to solve fuzzy linear programming, respectively. In addition, a method for solving transportation problems with pentagonal fuzzy costs was found by Christi and Kasthuri in [29]. Recently, the transportation problem with pentagonal fuzzy demands and supplies has been investigated by Maheswari and Ganesan in [30].

Some basic arithmetic operations of PFN such as addition, subtraction, multiplication and division were independently established by Helen and Uma [**?**] and Selvam et al., [25] based on its β-cut. That is, the β-cut is crucial in the study of PFN. Therefore, we need to know the definition of β-

cut first.

Definition 2.3 (See [31, 32])**.** Let $\tilde{a}$ be any fuzzy numbers. The β**-cut** of $\tilde{a}$, denoted by $\tilde{a}^{\beta}$, is defined as

$$\tilde{a}^{\beta} = \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \beta\},$$

where $\mu_{\tilde{a}}$ is the membership function of $\tilde{a}$.

The study on the β-cut in the series of Theorems 2.4–2.6 together with the nearest point of linear pentagonal fuzzy number needs to be separately investigated into three types as

$$0 < r < s < 1, \quad 0 < s < r < 1$$

and

$$0 < r = s < 1,$$

in order to eventually present them in the closed forms with $r, s \in (0, 1)$.

Theorem 2.4. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *be a LPFN such that* $0 < r < s < 1$. *The* β*-cut of* $\tilde{a}$ *can be written as* $\tilde{a}^{\beta} = [a_L^{\beta}, a_U^{\beta}]$

$$= \begin{cases} \left[a_1 + \frac{\beta}{r}(a_2 - a_1), a_5 - \frac{\beta}{s}(a_5 - a_4)\right], & \text{if } \beta \in [0, r]; \\ \left[a_2 + \frac{\beta - r}{1 - r}(a_3 - a_2), a_4 + \frac{s - \beta}{s}(a_5 - a_4)\right], & \text{if } \beta \in [r, s]; \\ \left[a_3 - \frac{1 - \beta}{1 - r}(a_3 - a_2), a_3 + \frac{1 - \beta}{1 - s}(a_4 - a_3)\right], & \text{if } \beta \in [s, 1]. \end{cases}$$

Proof. Let $\beta \in [0, 1]$. Then, the membership function $\mu_{\tilde{a}}$ shown in (2.1) implies

$$\begin{aligned} \tilde{a}^{\beta} &= \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \beta\} \\ &= X_1 \cup X_2 \cup X_3 \cup X_4, \end{aligned}$$

where

$X_1 = \{x \in [a_1, a_2] : r\left(\frac{x - a_1}{a_2 - a_1}\right) \geq \beta\}$,

$X_2 = \{x \in [a_2, a_3] : r + (1 - r)\left(\frac{x - a_2}{a_3 - a_2}\right) \geq \beta\}$,

$X_3 = \{x \in [a_3, a_4] : s + (1 - s)\left(\frac{a_4 - x}{a_4 - a_3}\right) \geq \beta\}$ and

$X_4 = \{x \in [a_4, a_5] : s\left(\frac{a_5 - x}{a_5 - a_4}\right) = s - s\left(\frac{x - a_4}{a_5 - a_4}\right) \geq \beta\}$.

Moreover, the set X_1, X_2, X_3 and X_4 can be simplified as follows:

$$\begin{aligned} X_1 &= \{x \in [a_1, a_2] : x \geq a_1 + \tfrac{\beta}{r}(a_2 - a_1)\} \\ &= \{x \in \mathbb{R} : a_2 \geq x \geq a_1 + \tfrac{\beta}{r}(a_2 - a_1) \geq a_1\} \\ &= \{x \in \mathbb{R} : x \geq a_1 + \tfrac{\beta}{r}(a_2 - a_1);\ \beta \in [0, r]\}, \\ X_2 &= \{x \in [a_2, a_3] : x \geq a_2 + \tfrac{\beta - r}{1 - r}(a_3 - a_2)\} \\ &= \{x \in \mathbb{R} : a_3 \geq x \geq a_2 + \tfrac{\beta - r}{1 - r}(a_3 - a_2) \geq a_2\} \\ &= \{x \in \mathbb{R} : x \geq a_2 + \tfrac{\beta - r}{1 - r}(a_3 - a_2); \\ &\qquad \beta \in [r, 1] = [r, s] \cup [s, 1]\}, \\ X_3 &= \{x \in [a_3, a_4] : x \leq a_4 + \tfrac{s - \beta}{1 - s}(a_4 - a_3)\} \\ &= \{x \in \mathbb{R} : a_3 \leq x \leq a_4 + \tfrac{s - \beta}{1 - s}(a_4 - a_3) \leq a_4\} \\ &= \{x \in \mathbb{R} : x \leq a_4 + \tfrac{s - \beta}{1 - s}(a_4 - a_3);\ \beta \in [s, 1]\} \end{aligned}$$

and

$$\begin{aligned} X_4 &= \{x \in [a_4, a_5] : x \leq a_5 - \tfrac{\beta}{s}(a_5 - a_4)\} \\ &= \{x \in \mathbb{R} : a_4 \leq x \leq a_5 - \tfrac{\beta}{s}(a_5 - a_4) \leq a_5\} \\ &= \{x \in \mathbb{R} : x \leq a_5 - \tfrac{\beta}{s}(a_5 - a_4); \\ &\qquad \beta \in [0, s] = [0, r] \cup [r, s]\}. \end{aligned}$$

Therefore, the values of β can be distinguished in three cases below.

Case I: If $\beta \in [0, r]$, then

$$x \in \left[a_1 + \frac{\beta}{r}(a_2 - a_1), a_5 - \frac{\beta}{s}(a_5 - a_4)\right]. \tag{2.2}$$

Case II: If $\beta \in [r, s]$, then

$$x \in \left[a_2 + \frac{\beta - r}{1 - r}(a_3 - a_2), a_4 + \frac{s - \beta}{s}(a_5 - a_4)\right]. \tag{2.3}$$

Case III: If $\beta \in [s, 1]$, then

$$x \in \left[a_3 - \frac{1 - \beta}{1 - r}(a_3 - a_2), a_3 + \frac{1 - \beta}{1 - s}(a_4 - a_3)\right]. \tag{2.4}$$

The expressions (2.2)–(2.4) complete the proof of the theorem. □

Theorem 2.5. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *be a LPFN such* *that* $0 < s < r < 1$. *The* β*-cut of* $\tilde{a}$ *can be written as* $\tilde{a}^\beta = [a_L^\beta, a_U^\beta]$

$$= \begin{cases} \left[a_1 + \frac{\beta}{r}(a_2 - a_1), a_5 - \frac{\beta}{s}(a_5 - a_4)\right], & \text{if } \beta \in [0, s]; \\ \left[a_2 - \frac{r-\beta}{r}(a_2 - a_1), a_4 - \frac{\beta - s}{1-s}(a_4 - a_3)\right], & \text{if } \beta \in [s, r]; \\ \left[a_3 - \frac{1-\beta}{1-r}(a_3 - a_2), a_3 + \frac{1-\beta}{1-s}(a_4 - a_3)\right], & \text{if } \beta \in [r, 1]. \end{cases}$$

Proof. It is similar to the proof of Theorem 2.4. □

Theorem 2.6. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r)$ *be a LPFN such that* $0 < r < 1$. *The* β*-cut of* $\tilde{a}$ *can be written as* $\tilde{a}^\beta = [a_L^\beta, a_U^\beta]$

$$= \begin{cases} \left[a_1 + \frac{\beta}{r}(a_2 - a_1), a_5 - \frac{\beta}{r}(a_5 - a_4)\right], & \text{if } \beta \in [0, r]; \\ \left[a_3 - \frac{1-\beta}{1-r}(a_3 - a_2), a_3 + \frac{1-\beta}{1-r}(a_4 - a_3)\right], & \text{if } \beta \in [r, 1]. \end{cases}$$

Proof. This is immediately obtained from replacing $s = r$ in Theorem 2.4. □

The nearest point of a fuzzy number is a representation of the fuzzy number known as a defuzzification. Asady and Zendehnam [33] defined the nearest point based on the concept of the nearest interval proposed by Dubois and Prade in [34] and Heilpern in [35] as presented in the definition below.

Definition 2.7 (See [33–36])**.** Let $\tilde{a}$ be a fuzzy number with its β-cut $\tilde{a}^\beta = [a_L^\beta, a_U^\beta]$. The **nearest interval** to $\tilde{a}$, denoted by $EI(\tilde{a})$, is represented as

$$EI(\tilde{a}) = \left[\int_0^1 a_L^\beta \, d\beta, \int_0^1 a_U^\beta \, d\beta\right] := [\underline{u}, \overline{u}],$$

i.e., $\underline{u}$ and $\overline{u}$ are lower and upper bounds of the interval $EI(\tilde{a})$, respectively.

The **nearest point** to $\tilde{a}$, denoted by $N(\tilde{a})$, is defined as the middle point of the interval $EI(\tilde{a})$, i.e.,

$$N(\tilde{a}) = \frac{\underline{u} + \overline{u}}{2} = \frac{1}{2}\int_0^1 (a_L^\beta + a_U^\beta) \, d\beta.$$

To accomplish a general formula of the nearest point of a linear pentagonal fuzzy number as provided in Corollary 2.11, let us first work on the three cases of r and s in Theorems 2.8–2.10 as follows.

Theorem 2.8. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *be a LPFN such that* $0 < r < s < 1$. *The nearest point to* $\tilde{a}$ *is*

$$\begin{aligned} N(\tilde{a}) = \tfrac{1}{2}\big[& r(a_1 + a_5) + (s - r)(a_2 + a_4) + 2(1 - s)a_3 \\ &+ \tfrac{r}{2}(a_2 - a_1) + \tfrac{2s - r - 1}{2}(a_3 - a_2) \\ &+ \tfrac{1-s}{2}(a_4 - a_3) + \tfrac{s - 2r}{2}(a_5 - a_4)\big]. \end{aligned} \tag{2.5}$$

Proof. According to Theorem 2.4, the β-cut of $\tilde{a}$ implies $N(\tilde{a})$ as $\frac{1}{2}\int_0^1 (a_L^\beta + a_U^\beta) \, d\beta$

$$\begin{aligned} &= \frac{1}{2}\int_0^r \left[a_1 + \tfrac{\beta}{r}(a_2 - a_1) + a_5 - \tfrac{\beta}{s}(a_5 - a_4)\right] d\beta \\ &+ \frac{1}{2}\int_r^s \left[a_2 + \tfrac{\beta - r}{1-r}(a_3 - a_2) + a_4 + \tfrac{s-\beta}{s}(a_5 - a_4)\right] d\beta \\ &+ \frac{1}{2}\int_s^1 \left[a_3 - \tfrac{1-\beta}{1-r}(a_3 - a_2) + a_3 + \tfrac{1-\beta}{1-s}(a_4 - a_3)\right] d\beta \\ &= \frac{1}{2}\Big[r(a_1 + a_5) + \tfrac{r}{2}(a_2 - a_1) - \tfrac{r^2}{2s}(a_5 - a_4) \\ &+ (s - r)(a_2 + a_4) + \left(\tfrac{\beta^2}{2} - r\beta\right)\Big|_{\beta=r}^{\beta=s} \cdot \tfrac{1}{1-r}(a_3 - a_2) \\ &+ \left(s\beta - \tfrac{\beta^2}{2}\right)\Big|_{\beta=r}^{\beta=s} \cdot \tfrac{1}{s}(a_5 - a_4) + 2(1 - s)a_3 \\ &- \left(\beta - \tfrac{\beta^2}{2}\right)\Big|_{\beta=s}^{\beta=1} \cdot \tfrac{1}{1-r}(a_3 - a_2) \\ &+ \left(\beta - \tfrac{\beta^2}{2}\right)\Big|_{\beta=s}^{\beta=1} \cdot \tfrac{1}{1-s}(a_4 - a_3)\Big] \\ &= \frac{1}{2}\Big[r(a_1 + a_5) + (s - r)(a_2 + a_4) + 2(1 - s)a_3 \\ &+ \tfrac{r}{2}(a_2 - a_1) + \tfrac{2s - r - 1}{2}(a_3 - a_2) + \tfrac{1-s}{2}(a_4 - a_3) \\ &+ \tfrac{s - 2r}{2}(a_5 - a_4)\Big], \end{aligned}$$

where the above simplification derives from the reason that

$$\left(\tfrac{\beta^2}{2} - r\beta\right)\Big|_{\beta=r}^{\beta=s} = \tfrac{(s-r)^2}{2} = \left(s\beta - \tfrac{\beta^2}{2}\right)\Big|_{\beta=r}^{\beta=s}$$

and
$$\left(\beta - \frac{\beta^2}{2}\right)\Big|_{\beta=s}^{\beta=1} = \frac{(1-s)^2}{2}.$$
This completes the proof of the theorem. □

Theorem 2.9. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *be a LPFN such that* $0 < s < r < 1$. *The nearest point to* $\tilde{a}$ *is*

$$\begin{aligned} N(\tilde{a}) = \tfrac{1}{2}\big[& s(a_1 + a_5) + (r - s)(a_2 + a_4) \\ &+ 2(1 - r)a_3 + \tfrac{2s-r}{2}(a_2 - a_1) - \tfrac{1-r}{2}(a_3 - a_2) \\ &+ \tfrac{s-2r+1}{2}(a_4 - a_3) - \tfrac{s}{2}(a_5 - a_4)\big]. \end{aligned} \tag{2.6}$$

Proof. The proof of the theorem can be verified in similar fashion as the proof of Theorem 2.8. □

Theorem 2.10. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r)$ *be a LPFN such that* $0 < r < 1$. *The nearest point to* $\tilde{a}$ *is*

$$\begin{aligned} N(\tilde{a}) = \tfrac{1}{2}\big[& r(a_1 + a_5) + 2(1 - r)a_3 + \tfrac{r}{2}(a_2 - a_1) \\ &- \tfrac{1-r}{2}(a_3 - a_2) + \tfrac{1-r}{2}(a_4 - a_3) - \tfrac{r}{2}(a_5 - a_4)\big]. \end{aligned} \tag{2.7}$$

Proof. By the reason that $\tilde{a}$ is a LPFN $(a_1, a_2, a_3, a_4, a_5; r, s)$ where $r = s$, we immediately obtain the theorem from replacing $s = r$ in Theorem 2.8. □

Corollary 2.11. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4, a_5; r, s)$ *be a LPFN such that* $0 < r, s < 1$. *If* $m = \min\{r, s\}$ *and* $M = \max\{r, s\}$, *then the nearest point to* $\tilde{a}$ *is*

$$\begin{aligned} N(\tilde{a}) = \frac{1}{2}\Bigg[& m(a_1 + a_5) + (M - m)(a_2 + a_4) \\ &+ 2(1 - M)a_3 + A_1^T \begin{bmatrix} a_2 - a_1 \\ a_5 - a_4 \end{bmatrix} + A_2^T \begin{bmatrix} a_3 - a_2 \\ a_4 - a_3 \end{bmatrix}\Bigg]. \end{aligned} \tag{2.8}$$

The terms A_1 *and* A_2 *are the first and the second column-vectors of the matrix*

$$A = \begin{bmatrix} \frac{\hat{s}+1}{4} & \frac{\hat{s}-1}{4} \\ \frac{\hat{s}-1}{4} & \frac{\hat{s}+1}{4} \end{bmatrix} \begin{bmatrix} m & 2M - m - 1 \\ M - 2m & 1 - M \end{bmatrix},$$

where

$$\hat{s} = \operatorname{sgn}(s - r) = \begin{cases} -1, & \text{if } s < r; \\ 0, & \text{if } s = r; \\ 1, & \text{if } s > r. \end{cases}$$

Proof. The nearest point $N(\tilde{a})$ in (2.8) can be expressed as

$$\begin{aligned} \tfrac{1}{2}\big[& m(a_1 + a_5) + (M - m)(a_2 + a_4) + 2(1 - M)a_3 \\ &+ \big[\big(\tfrac{\hat{s}+1}{4}\big)m + \big(\tfrac{\hat{s}-1}{4}\big)(M - 2m)\big](a_2 - a_1) \\ &+ \big[\big(\tfrac{\hat{s}+1}{4}\big)(2M - m - 1) + \big(\tfrac{\hat{s}-1}{4}\big)(1 - M)\big](a_3 - a_2) \\ &+ \big[\big(\tfrac{\hat{s}-1}{4}\big)(2M - m - 1) + \big(\tfrac{\hat{s}+1}{4}\big)(1 - M)\big](a_4 - a_3) \\ &+ \big[\big(\tfrac{\hat{s}-1}{4}\big)m + \big(\tfrac{\hat{s}+1}{4}\big)(M - 2m)\big](a_5 - a_4)\big]. \end{aligned}$$

Furthermore, we can derive the above $N(\tilde{a})$ in the following three cases below.

Case I: If $0 < r < s < 1$, then $m = \min\{r, s\} = r, M = \max\{r, s\} = s$ and $\hat{s} = 1$. Thus, (2.5) holds.

Case II: If $0 < s < r < 1$, then $m = \min\{r, s\} = s, M = \max\{r, s\} = r$ and $\hat{s} = -1$. Thus, (2.6) holds.

Case III: If $0 < r = s < 1$, then $m = \min\{r, s\} = r = \max\{r, s\} = M$ and $\hat{s}) = 0$. Thus, (2.7) holds. Hence, the proof of the corollary is done. □

Ranking of fuzzy numbers is crucial for decision analysis. Asady and Zendehnam [33] used the concept of the nearest point in ranking any two fuzzy number described in Definition 2.12.

Definition 2.12 (See [33])**.** Let $\tilde{a}$ and $\tilde{b}$ be any two fuzzy numbers. The ranking of $\tilde{a}$ and $\tilde{b}$ is defined by their nearest points as follows:

i) $\tilde{a}$ is **greater than** $\tilde{b}$, denoted by $\tilde{a} \succ \tilde{b}$, if $N(\tilde{a}) > N(\tilde{b})$,

ii) $\tilde{a}$ is **less than** $\tilde{b}$, denoted by $\tilde{a} \prec \tilde{b}$, if $N(\tilde{a}) < N(\tilde{b})$,

iii) $\tilde{a}$ is **equal to** $\tilde{b}$, denoted by $\tilde{a} \approx \tilde{b}$, if $N(\tilde{a}) = N(\tilde{b})$,

where the nearest points $N(\tilde{a})$ and $N(\tilde{b})$ to $\tilde{a}$ and $\tilde{b}$ are defined as Definition 2.7.

3. Course assignment problem with fuzzy in preference ranking

Let us first provide the general details of our course assignment problem using the data collected by the department of Mathematics and Computer Science, Chulalongkorn University. The aim of the department is to assign courses to instructors in such a way that instructors get courses they prefer to teach and their teaching workloads do not differ too much from their requested workloads. The teaching workload contains two parts: course teaching and thesis advising. The instructors know their amount of advising duty based on how many students are under their supervision, which means they can provide the amount of their requested workloads so that the department could assign to them the right proportion of course teaching workload.

3.1 Instructor preference

In order to assign the courses each instructor prefers to teach, the department did a survey on instructors preference on each subject. According to the teaching preference in [9], the department provided six levels of subjective feeling as presented in Table 1. The weight assigned to each preference level should be scaled down to a number between $[0, 1]$. However, it is still fuzzy to decide the weight values, in general. Gorka et al. analyzed in [9] an optimal fuzzy weight of each preference level based on the objective of minimizing the number of courses with preference levels (v) and (vi) but still tried to balance the overload and underload of each instructor. This optimal preference weight is presented in the last column of Table 1 and the corresponding fuzzy number of preference is presented on the left of Fig. 4. It may surprise some readers, who have the feeling that these weights should be a bit more different from each other like $1, 0.8, 0.6, 0.4, 0.2, 0$, to see that the optimal fuzzy weights of preference levels (i)–(v) are very high and closer to each other than they expected.

The assignment obtained in [9] provided the smallest number of courses assigned to instructors who cannot teach them. However, as mentioned in the Introduction section, this paper deals with the fuzziness in each level of preferences, not just the fuzziness of the weight representing each preference level. For example, an instructor may not be certain in his/her feeling to choose an exact preference level (ii) but also sure that he/she will not choose preference level (i) or (iii), since he/she may think that if there is any choice of rank ‘2.3’, it may suit his/her personal reason/feeling better than the precise preference rank 2. We want to investigate more insight into this personal feeling. If we have fuzziness in each level of preferences on top of the fuzziness of assigning the weights to each preference, would it have any good effect in the overall assignment comparing with the assignment in [9] and is it worth spending a bit more time collecting associated data, or not.

We ask instructors to provide more insight in their preference feeling towards each subject, by creating two fuzzy numbers for each preference level. These two fuzzy numbers are created based on two types of instructor personality. One is feeling a bit more than that rank but still having some negative feeling of choosing that rank. Another is feeling a bit less than that rank due to some related work needing to be done in that particular course which discourages

Table 1. Teaching preference description.

Preference level	Level description	Ranking of each level	Fuzzy weight of preference
(i)	Prefers most to teach the subject	1	1
(ii)	Prefers to teach the subject	2	0.95
(iii)	OK to teach the subject	3	0.9
(iv)	Able to teach the subject	4	0.85
(v)	Able but does not want to teach the subject	5	0.8
(vi)	Cannot teach the subject	6	0

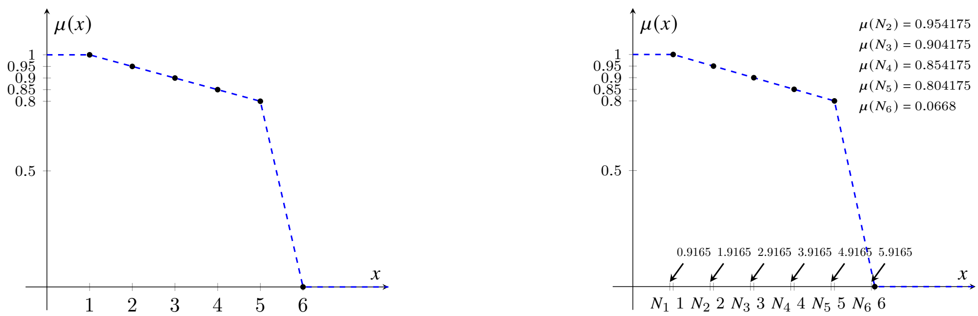

Fig. 4. Left: An optimal fuzzy weight of instructors preference. Right: Fuzzy weight of preference ranked by nearest points, where N_k is the nearest point of preference level (k).

the instructor from teaching the course. The first two columns of Fig. 5 show these two fuzzy numbers of each preference rank 1–6, respectively.

3.2 Nearest point representation of preference level

The MIN operator is preferable to use as the aggregation operator in our course assignment, since it has the idea of rewarding the personality with more positive/optimistic attitude toward teaching the courses. For each preference level, after aggregating these two fuzzy numbers using the MIN operator, we obtain six pentagonal fuzzy numbers (see the last column of Fig. 5). The ranking of these six pentagonal fuzzy numbers is very clear, no matter which representation approach we use. In this course assignment application, we choose the nearest point approach to represent each pentagonal fuzzy number based on its simple closed form derived in the previous section. Each individual could have a nearest point of its own and the fuzzy weight of the nearest point will come along. However, by using the MIN aggregation to make these specific triangular fuzzy numbers become a linear pentagonal fuzzy number before finding nearest points helps saving time on the process of finding an individual nearest point and its weight.

We now are able to provide the nearest point of each pentagonal fuzzy numbers in Table 2. These nearest points will be used in our course assignment model instead of the standard ranking 1–6. We use the fuzzy number (the left of Fig. 4) from [] which is optimized under the standard ranking as a guideline to assign the fuzzy weight of preference ranked by nearest points, as shown on the right of Fig. 4 and in the last column of Table 2. By using the MIN operator

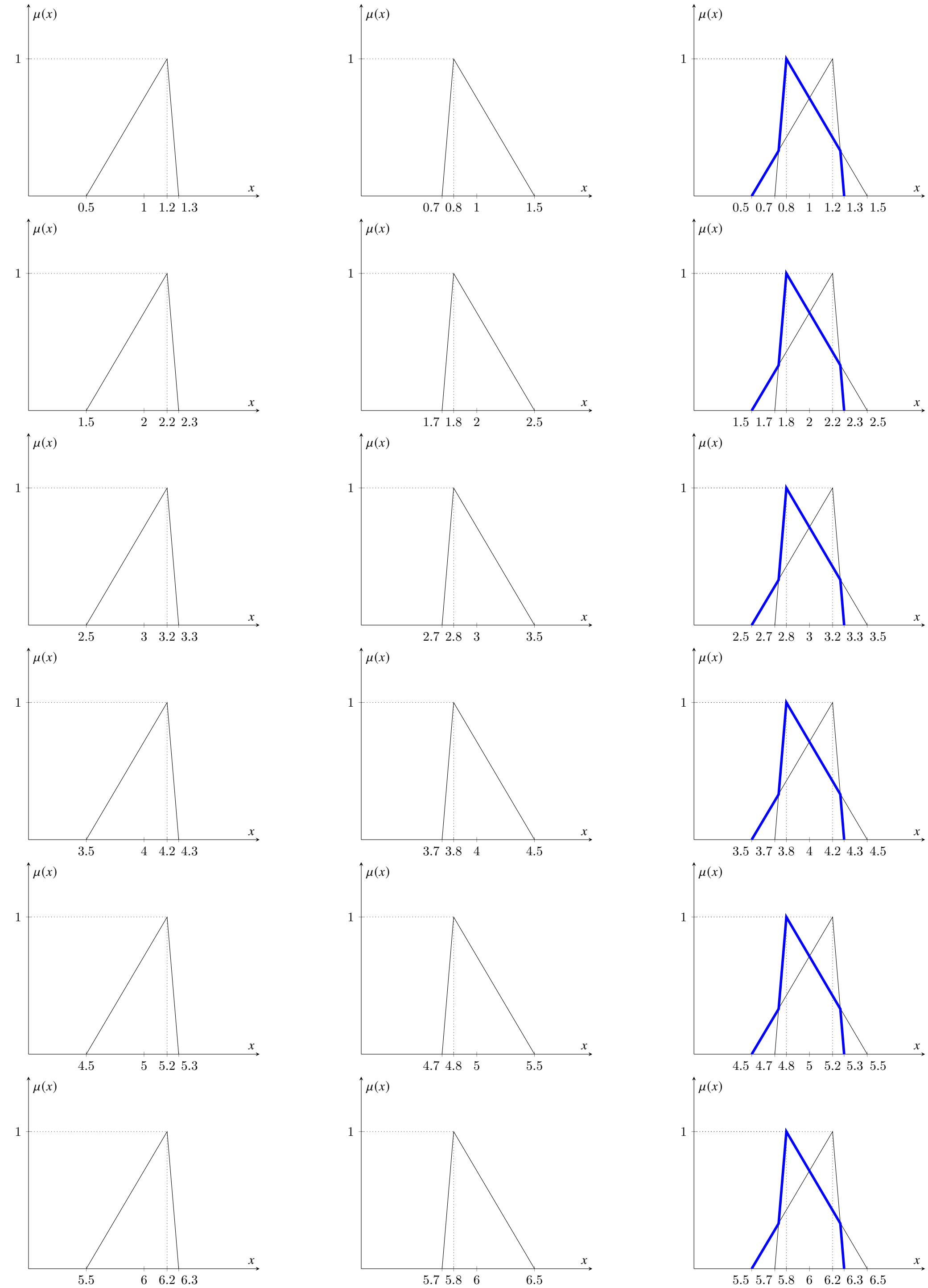

Fig. 5. Two fuzzy numbers for each preference level based on two types of individual personality and MIN of these two fuzzy numbers.

Table 2. Nearest point representation of each preference level.

Preference level	Level description	Nearest point of each level	Fuzzy weight by nearest point
(i)	Prefers most to teach the subject	0.9165	1
(ii)	Prefers to teach the subject	1.9165	0.954175
(iii)	OK to teach the subject	2.9165	0.904175
(iv)	Able to teach the subject	3.9165	0.854175
(v)	Able but does not want to teach the subject	4.9165	0.804175
(vi)	Cannot teach the subject	5.9165	0.0668

and the nearest point to obtain the representation of each preference level, we carry the message of fuzzy information of each individual to our course assignment model.

3.3 Mathematical model of course assignment problem

There are 98 subjects taught by the department of Mathematics and Computer Science, Chulalongkorn University. Some subjects have more than one section. Some subjects/sections require two instructors, so our model will consider them as two subsections with a half size workload for each subsection. The data set contains the total of $m = 120$ sections/subsections and $n = 58$ instructors. Table 3 provides the notation needed to set up the course assignment model. The following mathematical model could be seen also in []. The difference is that $\tilde{c}_{ij}$ in this model is the fuzzy weight coefficient generated by the nearest point of each preference level.

$$\max \sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{c}_{ij}x_{ij} - M_1\sum_{i=1}^{n}\delta_i - M_2\sum_{i=1}^{n}\beta_i$$

s.t.

Const. 1: $\sum_{j\in J^1} x_{ij} + \sum_{j\in J^2} x_{ij} \le 3, \quad \forall i \in I$

Const. 2: $\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in J^1$

Const. 3: $\sum_{i=1}^{n} x_{ij} = \frac{1}{2}, \quad \forall j \in J^2$

Const. 4: $\sum_{j\in J_k\cap J^1} x_{ij} \le 1, \quad \forall i \in I, k \in K$

Const. 5: $\sum_{j\in J_k\cap J^2} x_{ij} \le \frac{1}{2}, \quad \forall i \in I, k \in K$

Const. 6: $\sum_{j=1}^{m} a_j x_{ij} - \delta_i + \beta_i = b_i - d_i, \quad \forall i \in I.$

The objective function of the above mathematical model is constructed to maximize the overall preference and reduce the overall extra and remaining workload at the same time. We use $M_1 = 0.0015$ and $M_2 = 10$ as in []. By using M_2 much higher than M_1, the remaining workload should not appear too large, which means the requested workload of each instructor should be fulfilled as much as possible. In Const. 1, the model limits the number of sections/subsections each instructor should teach to be no more than 3 sections/subsections. Const. 2 and Const. 3 serve the restriction of only one instructor allowed in each section/subsection. To prevent the situation that some instructors might be assigned to teach multiple sections/subsections of the same subject, Const. 4 and Const. 5 are needed. In Const. 6, each instructor should meet his/her own requested amount of workload, otherwise an excessive workload or a remaining amount of workload appears.

In summary, our method consists of the following steps:

Table 3. Notation and meaning of variables, parameters and sets.

Symbol	Meaning
Set	
I	$I = \{1, 2, 3, \ldots, n\}$ be the set of all n instructors.
J	$J = \{1, 2, 3, \ldots, m\}$ be the set of all m course sections/subsections.
K	$K = \{1, 2, 3, \ldots, s\}$ be the set of all s subjects. Each k^{th} subject contains the total of $\lvert J_k \rvert$ sections/subsections, $k \in K$.
J_k	$J_k = \left\{ j \in J \mid \text{the } j^{\text{th}} \text{ section/subsection is a section/subsection of the } k^{\text{th}} \text{ subject} \right\}$.
J^1	$J^1 = \left\{ j \in J \mid \text{the } j^{\text{th}} \text{ section requires only one instructor} \right\}$.
J^2	$J^2 = \left\{ j \in J \mid \text{the } j^{\text{th}} \text{ subsection requires two instructors} \right\}$. Note that $J^1 \cap J^2 = \emptyset$ and $J^1 \cup J^2 = J$.
Parameter	
a_j	the amount of workload earned by teaching the j^{th} section/subsection.
b_i	the i^{th} instructor's requested workload.
$\tilde{c}_{ij}$	the fuzzy weight of the nearest point representation of preference of the i^{th} instructor to the j^{th} section/subsection.
d_i	the seminar, project and thesis advisor duties of the i^{th} instructor. The i^{th} instructor needs to fulfill the amount of teaching workload of $b_i - d_i$.
Variable	
x_{ij}	$x_{ij} = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ instructor teaches the } j^{\text{th}} \text{ section, } j \in J^1; \\ \frac{1}{2}, & \text{if the } i^{\text{th}} \text{ instructor teaches the } j^{\text{th}} \text{ subsection, } j \in J^2; \\ 0, & \text{otherwise.} \end{cases}$
δ_i	$\delta_i \geq 0$ be an extra workload that the i^{th} instructor has on the top of the requested workload.
β_i	$\beta_i \geq 0$ be a remaining amount of the requested workload after deducting from the actual assigned workload.

1. Construct two triangular fuzzy numbers for each preference level (i)–(vi).

2. Use the MIN operator to aggregate these fuzzy numbers for each preference level, resulting in six LPFNs: $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_6$.

3. Compute the nearest points $N(\tilde{a}_1), N(\tilde{a}_2), \ldots, N(\tilde{a}_6)$ using Corollary 2.11. We denote these points as $N_1, N_2, \ldots, N_6$.

4. Determine the fuzzy weight of each preference based on the nearest points.

5. Use the fuzzy weight coefficients as parameters in the course assignment model.

6. Solve the model using CPLEX.

4. Result

The mathematical model with weight coefficient of the nearest point representation of preference is run by CPLEX ver-

Table 4. Number of subjects/subsections of each preference level and the statistical value of extra workload of all instructors, where Model I is the model with optimal weight assigned to each preference level, and Model II is the model with weight coefficient of the nearest point representation of preference.

Preference level	Number of sections/subsections evaluated by	
	Model I	Model II
(i)	83	81
(ii)	13	17
(iii)	9	7
(iv)	3	3
(v)	5	4
(vi)	7	8
Statistical value of extra workload	Model I	Model II
Min value	0	0
Max value	11.75	11.75
Average	2.85	2.90

sion 12.6.3. We compare the results of our model that integrates the fuzziness of each preference level to the model and the one in [9] which has an optimal weight assigned to each preference level in Table 4. The number of assigned sections/subsections based on each preference level in Table 4 shows that the overall performance of our model is acceptable, or even better in some cases. The total number of assigned subjects/subsections with preference levels (v) and (vi) in both models are the same which equals 12 classes. Even though Model II has two sections less in classes of preference (i), four more instructors are assigned to subjects with preference (ii), and fewer sections are assigned with preference (iii). Moreover, there is no remaining amount of workload in either model, and the total amount of overload is not significantly different. On the other hand, by using the result from Model II, we are able to integrate more real fuzzy feeling of instructors in the model. As the fuzzy weight in preference levels (ii)–(v) by the nearest point approach provides a little higher values than the optimal fuzzy weight in [9], Model II definitely uses the fuzzy feeling of instructors to evaluate the result, which provides more sections with preference level (ii) as expected.

At this end, let us explore in more detail on the 12 sections of preferences (v) and (vi), which turn out to be 9 different subjects listed below.

2301117 (sec 1) : this is calculus. There are more than 30 sections of calculus (distinguished by students majors such as biology, chemistry, engineer, finance, etc). So, it is common not to have enough instructors to serve this basic course.

2301170 (sec 2) : this is the first programming course. Also, there are many sections, which could encounter the same problem as calculus classes.

2301286 (sec 2), 2301286 (lab 1), 2301286 (lab 5), and 2301286 (lab 6) : these are lecture and labs of the first statistics course. Since many majors need to have basic statistics, this subject also splits into many sections. As a result, the department does not have enough instructors to serve all sections.

2301370 (sec 1), 2301482 (sec 1), 2301645 (sec 1), 2301686 (sec 1), 2301736 (sec 1A), and 2301762 (sec 1) : these are higher level courses. Very few instructors provide preference level (i)–(iv) to these courses. Most of them cannot teach the subjects. Some instructors who can teach the subjects have already filled up their workload.

The department may have to work manually on changing the assignment result got from the model so that all these higher courses are taught by the instructors who can teach them and move their previously assigned courses to those instructors who were assigned to the courses but cannot teach them. The department could also use this result to plan on hiring some temporary/permanent positions in some specific fields of study.

5. Conclusion

This study introduced two triangular fuzzy numbers to represent the inherent uncertainty in preference levels, effectively modeling human perception of preference ambiguity. By aggregating these fuzzy numbers, we explored the concept of pentagonal fuzzy numbers. Through the β-cuts established in Theorems 2.4–2.6, we derived explicit nearest point formulas in Theorems 2.8–2.10, leading to a closed-form expression for the nearest point of any linear pentagonal fuzzy number, as demonstrated in Corollary 2.11. These nearest points serve as precise representations of preference levels, allowing for their integration into the course assignment process. The assigned values align with the optimal fuzzy preference numbers outlined in [9], demonstrating the effectiveness of our model in handling uncertainty in course assignments.

However, this study operates under the assumption that instructors can reliably express their preferences using predefined triangular fuzzy numbers, which may not always reflect real-world variations in decision making. This limitation should be acknowledged in the research.

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