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Regularity of the Transformation Semigroups Restricted by an Equivalence Relation with a Restricted Range

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ABSTRACT

Let X be a nonempty set. Denote by $P(X)$ and $T(X)$ the partial transformation semigroup and the full transformation semigroup on X, respectively. For a nonempty subset Y of X and an equivalence relation ρ on X, let

$$P(X,Y,\rho) = \{\,\alpha \in P(X) \mid \operatorname{ran}\alpha \subseteq Y \ \text{ and } \ \forall x,y \in \operatorname{dom}\alpha, (x,y) \in \rho \Rightarrow x\alpha = y\alpha\,\},$$
$$T(X,Y,\rho) = \{\,\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y \ \text{ and } \ \forall x,y \in X, (x,y) \in \rho \Rightarrow x\alpha = y\alpha\,\}.$$

Then $P(X,Y,\rho)$ and $T(X,Y,\rho)$ are subsemigroups of $P(X)$ and $T(X)$, respectively. In this paper, we characterize the regular elements, left regular elements and right regular elements of both $P(X,Y,\rho)$ and $T(X,Y,\rho)$. In addition, the regularity, left regularity and right regularity of $P(X,Y,\rho)$ and $T(X,Y,\rho)$ are determined.

Keywords: Transformation semigroups; Regular elements; Left [Right] regular elements; Equivalence relations; Restricted ranges

1. Introduction and Preliminaries

For a set X, let $|X|$ denote the cardinality of X. An element a of a semigroup S is called *idempotent* if $a = a^2$. A semigroup S is said to be an idempotent semigroup if every element of S is idempotent. An element a in S is called *regular* if there exists some $x \in S$ such that $a = axa$, *left regular* if $a = xa^2$ for some $x \in S$ and *right regular* if $a = a^2x$ for some $x \in S$. A semigroup S is called a regular semigroup if each of its elements is regular. Left regular and right regular semigroups are defined in a similar manner. For a mapping

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α, we will denote its domain and range by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For an element x in the domain of a mapping α, the image of x under α is written as $x\alpha$. For $A \subseteq \operatorname{dom}\alpha$, we denote $\alpha|_A$ the restriction of α to A. For a nonempty set X, let $P(X)$ and $T(X)$ denote the set of all functions from a subset of X to X and the set of all functions from X to itself, respectively. It is well-known that with function composition, $P(X)$ forms a semigroup, while $T(X)$ is a subsemigroup of $P(X)$. The semigroup $P(X)$ is called the *partial transformation semigroup* on X and $T(X)$ is called the *full transformation semigroup* on X. It is a well-known fact that $P(X)$ and $T(X)$ are regular semigroups. Research on the regularity, left regularity, and right regularity of subsemigroups in $P(X)$ and $T(X)$ has been widely conducted.

For a nonempty subset Y of X, we let

$$P(X,Y) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\}.$$

Then $P(X,Y)$ and $T(X,Y)$ are subsemigroups of $P(X)$ and $T(X)$, respectively. The semigroup $T(X,Y)$ was first introduced and examined by Magill [2] in 1966. In [3], Nenthein, Youngkhong and Kemprasit provided some characterizations of regular elements in $T(X,Y)$. Sanwong and Sommanee [4] showed that $T(X,Y)$ is regular if and only if $|Y| = 1$ or $Y = X$. The characterization of left regular and right regular elements of $T(X,Y)$ are given in [1].

Let ρ be an equivalence relation on a nonempty set X. For $x \in X$, we let $[x]_\rho$ be the equivalence class of x determined by ρ. Let X/ρ denote the quotient set of X induced by ρ. In [7] and [8], Pei introduced the following semigroups.

$$P_\rho(X) = \{\, \alpha \in P(X) \mid \forall x, y \in \operatorname{dom}\alpha,\ (x,y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho \,\},$$

$$T_\rho(X) = \{\, \alpha \in T(X) \mid \forall x, y \in X,\ (x,y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho \,\}.$$

Clearly, $P_\rho(X)$ and $T_\rho(X)$ are subsemigroups of $P(X)$ and $T(X)$, respectively. The author has investigated the regular elements and Green's relations for $P_\rho(X)$ and $T_\rho(X)$ as well. In 2013, Namnak and Laysilikul [5] provided a characterization of left, right and completely regular elements in $T_\rho(X)$. In 2020, Sangkhanan and Sanwong [10] introduced a subsemigroup of $T(X,Y)$ defined by

$$T_\rho(X,Y) = \{\, \alpha \in T(X,Y) \mid \forall x, y \in X,\ (x,y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho \,\}.$$

They also gave a necessary and sufficient condition for $T_\rho(X,Y)$ to be regular and characterized Green's relations on $T_\rho(X,Y)$. In 2010, Mendes-Gonçalves and Sullivan [9] defined a subsemigroup of $T(X)$, which is given by

$$E(X,\rho) = \{\, \alpha \in T(X) \mid \forall x, y \in X,\ (x,y) \in \rho \Rightarrow x\alpha = y\alpha \,\}.$$

It is obvious that $E(X,\rho)$ is a subsemigroup of $T(X)$. We call $E(X,\rho)$ the *semigroup of transformations restricted by an equivalence relation* ρ. In 2018, Sawatraksa, Namnak and Laysirikul [6] described the left regular, right regular, and completely regular elements of $E(X,\rho)$. Moreover, they gave a necessary and sufficient condition for the semigroup $E(X,\rho)$ to be left regular, right regular or completely regular.

Let Y be a nonempty subset of X and ρ an equivalence relation on X. Define

$$P(X,Y,\rho) = \{\, \alpha \in P(X,Y) \mid \forall x, y \in \operatorname{dom}\alpha,\ (x,y) \in \rho \Rightarrow x\alpha = y\alpha \,\},$$
$$T(X,Y,\rho) = \{\, \alpha \in T(X,Y) \mid \forall x, y \in X,\ (x,y) \in \rho \Rightarrow x\alpha = y\alpha \,\}.$$

It is easy to verify that $P(X,Y,\rho)$ and $T(X,Y,\rho)$ are subsemigroups of $P(X,Y)$ and $T(X,Y)$, respectively.

In this paper, we identify the necessary and sufficient conditions for elements of $P(X,Y,\rho)$ and $T(X,Y,\rho)$ to satisfy regularity, left regularity, or right regularity and apply these conditions to determine the regularity, left regularity and right regularity of $P(X,Y,\rho)$ and $T(X,Y,\rho)$.

2. Regularity of $P(X,Y,\rho)$ and $T(X,Y,\rho)$

Before we characterize the regular elements of $P(X,Y,\rho)$ and $T(X,Y,\rho)$, it is convenient to have the following preliminary result.

Lemma 2.1. *Let $S(X,Y,\rho)$ be $P(X,Y,\rho)$ or $T(X,Y,\rho)$. If $\alpha,\beta\in S(X,Y,\rho)$ are such that $\alpha=\alpha\beta\alpha$, then the following conditions hold.*

(i) $x\alpha^{-1}\cap Y\neq\emptyset$ *for all* $x\in \operatorname{ran}\alpha$.

(ii) *For each* $x_1,x_2\in\operatorname{ran}\alpha$, *if* $(x_1,x_2)\in\rho$, *then* $x_1=x_2$.

Proof. Assume that $\alpha,\beta\in S(X,Y,\rho)$ are such that $\alpha=\alpha\beta\alpha$. Let $x\in\operatorname{ran}\alpha$. Then there is $a\in\operatorname{dom}\alpha$ such that $x=a\alpha$. Thus $x=a\alpha=a\alpha\beta\alpha$ which implies that $a\alpha\beta\in x\alpha^{-1}\cap Y$. This verifies (i). To prove (ii), let $x_1,x_2\in\operatorname{ran}\alpha$. Then $x_1=a_1\alpha$ and $x_2=a_2\alpha$ for some $a_1,a_2\in\operatorname{dom}\alpha$. Suppose that $(x_1,x_2)\in\rho$. Then we have $x_1=a_1\alpha=a_1\alpha\beta\alpha=x_1\beta\alpha=x_2\beta\alpha=a_2\alpha\beta\alpha=a_2\alpha=x_2$. Hence, the proof is complete. □

We will now present the necessary and sufficient conditions for the regularity of elements in $P(X,Y,\rho)$ and $T(X,Y,\rho)$.

Theorem 2.2. *Let $S(X,Y,\rho)$ be $T(X,Y,\rho)$ or $P(X,Y,\rho)$. For $\alpha\in S(X,Y,\rho)$, α is regular if and only if α satisfies the following conditions.*

(i) $x\alpha^{-1}\cap Y\neq\emptyset$ *for all* $x\in\operatorname{ran}\alpha$.

(ii) *For each* $x_1,x_2\in\operatorname{ran}\alpha$, *if* $(x_1,x_2)\in\rho$, *then* $x_1=x_2$.

Proof. It is immediate from Lemma 2.1 that if α is regular, then (i) and (ii) hold.

Suppose, conversely, that (i) and (ii) hold. Let $\alpha\in S(X,Y,\rho)$. For each $x\in\operatorname{ran}\alpha$, choose an element $y_x\in x\alpha^{-1}\cap Y$. Then $y_x\in\operatorname{dom}\alpha$ and $y_x\alpha=x$ for all $x\in\operatorname{ran}\alpha$.

Case 1 : $\alpha\in P(X,Y,\rho)$. Define $\gamma:\operatorname{ran}\alpha\to X$ by $x\gamma=y_x$ for all $x\in\operatorname{ran}\alpha$. Clearly, $\operatorname{ran}\gamma\subseteq Y$. Let $x_1,x_2\in\operatorname{dom}\gamma$ such that $(x_1,x_2)\in\rho$. Since $\operatorname{dom}\gamma=\operatorname{ran}\alpha$, we get $x_1=x_2$ by (ii). This shows that $\gamma\in P(X,Y,\rho)$. Since $(\operatorname{dom}\alpha)\alpha=\operatorname{ran}\alpha=\operatorname{dom}\gamma$ and $\operatorname{ran}\gamma\subseteq\operatorname{dom}\alpha$, we obtain that $\operatorname{dom}\alpha=\operatorname{dom}\alpha\gamma\alpha$. If $x\in\operatorname{dom}\alpha$, then $x\alpha\in\operatorname{ran}\alpha$, so $x\alpha\gamma\alpha=y_{x\alpha}\alpha=x\alpha$. Hence, $\alpha=\alpha\gamma\alpha$. Consequently, α is regular.

Case 2 : $\alpha\in T(X,Y,\rho)$. Define $\beta:X\to X$ by

$$x\beta=\begin{cases} y_z & \text{if } (x,z)\in\rho \text{ for some } z\in\operatorname{ran}\alpha,\\ x\alpha & \text{otherwise.}\end{cases}$$

If $x\in X$ is such that $(x,z_1)\in\rho$ and $(x,z_2)\in\rho$ for some $z_1,z_2\in\operatorname{ran}\alpha$, then $(z_1,z_2)\in\rho$, so $z_1=z_2$ by (ii) and hence $y_{z_1}=y_{z_2}$. Then β is well-defined. Let $x_1,x_2\in X$ such that $(x_1,x_2)\in\rho$. If $(x_1,z)\in\rho$ for some $z\in\operatorname{ran}\alpha$, then $(x_2,z)\in\rho$, so $x_1\beta=y_z=x_2\beta$. If $(x_1,z)\notin\rho$ for all $z\in\operatorname{ran}\alpha$, then $(x_2,z)\notin\rho$ for all $z\in\operatorname{ran}\alpha$ which implies that $x_1\beta=x_1\alpha=x_2\alpha=x_2\beta$. It is easy to see that $\operatorname{ran}\alpha\subseteq Y$. Hence, $\beta\in T(X,Y,\rho)$. If $x\in X$, then $x\alpha\in\operatorname{ran}\alpha$ and $(x\alpha,x\alpha)\in\rho$, so $x\alpha\beta\alpha=y_{x\alpha}\alpha=x\alpha$. We have that $\alpha=\alpha\beta\alpha$. Therefore, α is regular. □

Next, we apply Theorem 2.2 to investigate when the semigroup $P(X,Y,\rho)$ is a regular semigroup.

Theorem 2.3. *$P(X,Y,\rho)$ is regular if and only if $X = Y$ and at least one of the following conditions is true.*

(i) $|X/\rho| = 1$.

(ii) $|X/\rho| > 1$ *and ρ is the identity relation on X.*

Proof. Suppose that $P(X,Y,\rho)$ is regular. Let $x \in X$ and choose $y \in Y$. Define $\alpha : \{x\} \to X$ by $x\alpha = y$. Clearly, $\alpha \in P(X,Y,\rho)$. Then α is regular. It follows from Theorem 2.2 that $y\alpha^{-1} \cap Y \neq \emptyset$. Then $\{x\} \cap Y = y\alpha^{-1} \cap Y \neq \emptyset$ which implies that $x \in Y$. Then $X = Y$. Assume that (i) is false. Then $|X/\rho| > 1$. We aim to prove that ρ is the identity relation on X. Let $x_1, x_2 \in X$ with $(x_1, x_2) \in \rho$. Since $|X/\rho| > 1$, there exists $x_3 \in X$ such that $[x_3]_\rho \neq [x_1]_\rho$. Let $\beta : [x_1]_\rho \cup [x_3]_\rho \to X$ be defined by

$$x\beta = \begin{cases} x_1 & \text{if } x \in [x_1]_\rho, \\ x_2 & \text{if } x \in [x_3]_\rho. \end{cases}$$

Since $X = Y$, we get $\operatorname{ran}\alpha \subseteq Y$. Then $\beta \in P(X,Y,\rho)$, so β is regular. Since $x_1, x_2 \in \operatorname{ran}\beta$ and $(x_1, x_2) \in \rho$, we get $x_1 = x_2$ by the condition (ii) of Theorem 2.2. Hence, ρ is an identity relation on X.

For the converse, assume that $X = Y$ and the condition (i) or (ii) holds.

Case 1 : $X = Y$ and $|X/\rho| = 1$. Let $\alpha \in P(X,Y,\rho)$. Since $|X/\rho| = 1$, we have $\operatorname{ran}\alpha = \{y\}$ for some $y \in Y$. This implies that α satisfies the condition (ii) of Theorem 2.2. We have

$$y\alpha^{-1} \cap Y = y\alpha^{-1} \cap X = y\alpha^{-1} = \operatorname{dom}\alpha \neq \emptyset.$$

This shows that α satisfies the condition (i) of Theorem 2.2. Then α is regular and hence, $P(X,Y,\rho)$ is regular.

Case 2 : $X = Y$ and ρ is the identity relation on X. Then we have $P(X,Y,\rho) = P(X)$ is a regular semigroup. Hence, the proof is complete. □

Next, we provide a characterization of when $T(X,Y,\rho)$ forms a regular semigroup. For our required result, the following lemma is needed.

Lemma 2.4. *If $|Y| = 1$ or $|X/\rho| = 1$, then $T(X,Y,\rho)$ is an idempotent semigroup.*

Proof. Assume that $|Y| = 1$ or $|X/\rho| = 1$.

Case 1 : $|Y| = 1$. Let $\alpha \in T(X,Y,\rho)$. Since $\operatorname{ran}\alpha \subseteq Y$ and $|Y| = 1$, we get $|\operatorname{ran}\alpha| = 1$, that is, α is the constant map. This implies that α is an idempotent element of $T(X,Y,\rho)$.

Case 2 : $|X/\rho| = 1$. Let $\alpha \in T(X,Y,\rho)$ and let $x_1, x_2 \in \operatorname{ran}\alpha$. Then there exist $a_1, a_2 \in X$ such that $x_1 = a_1\alpha$ and $x_2 = a_2\alpha$. Since $|X/\rho| = 1$, we get $(a_1, a_2) \in \rho$. Thus $x_1 = a_1\alpha = a_2\alpha = x_2$. This shows that $|\operatorname{ran}\alpha| = 1$. Hence, α is idempotent. □

Theorem 2.5. *$T(X,Y,\rho)$ is regular if and only if one of the following statements holds.*

(i) $|Y| = 1$.

(ii) $|X/\rho| = 1$.

(iii) $\big|[x]_\rho \cap Y\big| = 1$ *for all $x \in X$.*

Proof. Assume that $T(X,Y,\rho)$ is regular. Suppose that (i) and (ii) are false. Then $|Y| > 1$ and $|X/\rho| > 1$. We will show that (iii) holds. Let $x \in X$ and $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Define $\alpha : X \to X$ by

$$a\alpha = \begin{cases} y_1 & \text{if } a \in [x]_\rho, \\ y_2 & \text{otherwise.} \end{cases}$$

It is clear that $\alpha \in T(X,Y,\rho)$. We see that $y_1 \in \operatorname{ran}\alpha$. Since α is regular, by the condition (i) of Theorem 2.2, $y_1\alpha^{-1} \cap Y \neq \emptyset$.

Then $[x]_\rho \cap Y = y_1\alpha^{-1} \cap Y \neq \emptyset$. Next, let $y, z \in [x]_\rho \cap Y$. Define $\beta : X \to X$ by

$$a\beta = \begin{cases} y & \text{if } \ a \in [x]_\rho, \\ z & \text{otherwise.} \end{cases}$$

Clearly, $\beta \in T(X, Y, \rho)$. Since $|X/\rho| > 1$, we get $X \neq [x]_\rho$. Then $y, z \in \operatorname{ran}\alpha$ and $(y, z) \in \rho$. Since β is regular, by the condition (ii) of Theorem 2.2, we get $y = z$. Thus $\left|[x]_\rho \cap Y\right| = 1$ as desired.

Conversely, if $|Y| = 1$ or $|X/\rho| = 1$, then $T(X, Y, \rho)$ is an idempotent semigroup by Lemma 2.4 and hence it is a regular semigroup. Suppose that $\left|[x]_\rho \cap Y\right| = 1$ for all $x \in X$. Let $\alpha \in T(X, Y, \rho)$ and let $x \in \operatorname{ran}\alpha$. Then $x = a\alpha$ for some $a \in X$. By assumption, $\left|[a]_\rho \cap Y\right| = 1$. This implies that there exists $y \in Y$ such that $(a, y) \in \rho$. Then $x = a\alpha = y\alpha$, so $y \in x\alpha^{-1} \cap Y$. Thus $x\alpha^{-1} \cap Y \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$. Next, let $x_1, x_2 \in \operatorname{ran}\alpha$ with $(x_1, x_2) \in \rho$. Then $x_1, x_2 \in [x_1]_\rho \cap Y$. Since $\left|[x_1]_\rho \cap Y\right| = 1$, we get $x_1 = x_2$. By Theorem 2.2, α is regular. Hence, $T(X, Y, \rho)$ is regular. □

3. Left regularity of $P(X, Y, \rho)$ and $T(X, Y, \rho)$

First, we provide the necessary and sufficient conditions for elements of $P(X, Y, \rho)$ to be left regular.

Theorem 3.1. *Let $\alpha \in P(X, Y, \rho)$. Then α is left regular if and only if $x\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$.*

Proof. Assume that $\alpha = \beta\alpha^2$ for some $\beta \in P(X, Y, \rho)$. Let $x \in \operatorname{ran}\alpha$. Then there is $a \in \operatorname{dom}\alpha, x = a\alpha$. Then $x = a\alpha = a\beta\alpha^2$, so $a\beta\alpha \in x\alpha^{-1}$. We have $a \in \operatorname{dom}\alpha = \operatorname{dom}\beta\alpha^2 \subseteq \operatorname{dom}\beta\alpha$. Then $a\beta \in Y \cap \operatorname{dom}\alpha$. Thus $a\beta\alpha \in x\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha$.

Conversely, suppose that $x\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$. For each $x \in \operatorname{ran}\alpha$, choose $y_x \in Y \cap \operatorname{dom}\alpha$ such that $y_x\alpha \in x\alpha^{-1}$. Let $\beta : \operatorname{dom}\alpha \to X$ be defined by

$$x\beta = y_{x\alpha} \ \text{ for all } \ x \in \operatorname{dom}\alpha.$$

Clearly, $\operatorname{ran}\beta \subseteq Y$. Let $x_1, x_2 \in \operatorname{dom}\beta$ such that $(x_1, x_2) \in \rho$. Then $x_1\alpha = x_2\alpha$, so $x_1\beta = y_{x_1\alpha} = y_{x_2\alpha} = x_2\beta$. Thus $\beta \in P(X, Y, \rho)$. Let $x \in \operatorname{dom}\alpha$. Then $x \in \operatorname{dom}\beta$ and $x\beta = y_{x\alpha} \in \operatorname{dom}\alpha$, so $x \in \operatorname{dom}\beta\alpha$. We have $x\beta\alpha = y_{x\alpha}\alpha \in (x\alpha)\alpha^{-1} \subseteq \operatorname{dom}\alpha$. Then $x \in \operatorname{dom}\beta\alpha^2$. This shows that $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta\alpha^2$. For the reverse inclusion, we have $\operatorname{dom}\beta\alpha^2 \subseteq \operatorname{dom}\beta = \operatorname{dom}\alpha$. Hence, $\operatorname{dom}\alpha = \operatorname{dom}\beta\alpha^2$. For any $x \in \operatorname{dom}\alpha$, $x\beta\alpha^2 = y_{x\alpha}\alpha^2 = x\alpha$. Then α is left regular. □

Next, we apply Theorem 3.1 to investigate when the semigroup $P(X, Y, \rho)$ is a left regular semigroup.

Theorem 3.2. *$P(X, Y, \rho)$ is left regular if and only if $|X| = 1$.*

Proof. Assume that $|X| = 1$. Then $P(X, Y, \rho)$ contains exactly two elements namely 0 and 1, where 0 is an empty map and 1 is an identity map. We have $0 = 0^3$ and $1 = 1^3$. Thus $P(X, Y, \rho)$ is left regular.

Conversely, suppose that $P(X, Y, \rho)$ is left regular. Let $x \in X$. Choose $y \in Y$ and define $\alpha : \{x\} \to X$ by $x\alpha = y$. Clearly, $\alpha \in P(X, Y, \rho)$. Since α is left regular, by Theorem 3.1, $y\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha \neq \emptyset$. Then we have $\{x\} \cap (Y \cap \{x\})\alpha \neq \emptyset$. This forces $(Y \cap \{x\})\alpha \neq \emptyset$, that is, $x \in Y$. Thus $\{x\} \cap \{y\} = \{x\} \cap \{x\}\alpha = \{x\} \cap (Y \cap \{x\})\alpha \neq \emptyset$. It follows that $x = y$. Since x is arbitrary, we get $|X| = 1$. □

Now, we provide the necessary and sufficient conditions for the left regularity of the elements in $T(X, Y, \rho)$.

Theorem 3.3. *Let $\alpha \in T(X,Y,\rho)$. Then α is left regular if and only if $x\alpha^{-1} \cap Y\alpha \neq \emptyset$ for all $x \in ran\,\alpha$.*

Proof. If $\alpha \in T(X,Y,\rho)$ is left regular, then α is left regular in $P(X,Y,\rho)$. We have $x\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$, by Theorem 3.1. Since $\operatorname{dom}\alpha = X$, we have $x\alpha^{-1} \cap Y\alpha \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$.

For the converse assume that the latter statement holds. Since $\operatorname{dom}\alpha = X$, we have $x\alpha^{-1} \cap (Y \cap \operatorname{dom}\alpha)\alpha \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$. Now construct a map β as in Theorem 3.1. Since $\operatorname{dom}\beta = \operatorname{dom}\alpha = X$, we get $\beta \in T(X,Y,\rho)$ and $\alpha = \beta\alpha^2$ as desired. □

Before characterizing the left regularity of $T(X,Y,\rho)$, it is useful to establish the following lemma.

Lemma 3.4. *Let $|Y| > 1$ and $|X/\rho| > 1$. If $|X/\rho| > 2$ or $\left|[a]_\rho \cap Y\right| \neq 1$ for some $a \in X$, then there exists a proper subset C of X such that $|C \cap Y| > 1$ and for each $x \in X$, if $(x,c) \in \rho$ for some $c \in C$, then $x \in C$.*

Proof. Suppose that $|X/\rho| > 2$ or $\left|[a]_\rho \cap Y\right| \neq 1$ for some $a \in X$.

Case 1 : $|X/\rho| > 2$. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Choose $C = [y_1]_\rho \cup [y_2]_\rho$. Since $|X/\rho| > 2$, we have C is a proper subset of X.

Case 2 : $|X/\rho| = 2$ and $\left|[a]_\rho \cap Y\right| \neq 1$ for some $a \in X$.

Subcase 2.1 : $[a]_\rho \cap Y = \emptyset$. Since $|Y| > 1$ and $|X/\rho| = 2$, there exists $b \in X$ such that $\left|[b]_\rho \cap Y\right| > 1$. Choose $C = [b]_\rho$. Since $|X/\rho| = 2$, C is a proper subset of X.

Subcase 2.2 : $\left|[a]_\rho \cap Y\right| > 1$. Then choose $C = [a]_\rho$. □

Theorem 3.5. *The semigroup $T(X,Y,\rho)$ is left regular if and only if one of the following conditions holds.*

(i) $|Y| = 1$.

(ii) $|X/\rho| = 1$.

(iii) $|Y| > 1, |X/\rho| = 2$ *and* $\left|[x]_\rho \cap Y\right| = 1$ *for all* $x \in X$.

Proof. To prove necessity, assume that $T(X,Y,\rho)$ is left regular. Suppose that (i), (ii) and (iii) are false. Then $|Y| > 1$ and $|X/\rho| > 2$ or $\left|[a]_\rho \cap Y\right| \neq 1$ for some $a \in X$. By Lemma 3.4, there is a proper subset C of X such that $|C \cap Y| > 1$ and for each $x \in X$, if $(x,c) \in \rho$ for some $c \in C$, then $x \in C$. Let $y_1, y_2 \in C \cap Y$ such that $y_1 \neq y_2$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} y_1 & \text{if } x \in C, \\ y_2 & \text{otherwise.} \end{cases}$$

Clearly, $\operatorname{ran}\alpha = \{y_1, y_2\} \subseteq Y$. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in \rho$. By the property of C, $x_1 \in C$ if and only if $x_2 \in C$. Then $x_1\alpha = x_2\alpha$. This shows that $\alpha \in T(X,Y,\rho)$. Thus α is left regular. By Theorem 3.3, $y_2\alpha^{-1} \cap Y\alpha \neq \emptyset$. Since $Y\alpha \subseteq \{y_1, y_2\}$ and $y_2\alpha^{-1} \cap \{y_1, y_2\} = (X - C) \cap \{y_1, y_2\} = \emptyset$, we get $y_2\alpha^{-1} \cap Y\alpha = \emptyset$, a contradiction.

To demonstrate sufficiency, we first assume that either (i) or (ii) is true. Then by Lemma 2.4, we get $T(X,Y,\rho)$ is an idempotent semigroup which implies that $T(X,Y,\rho)$ is a left regular semigroup. Next, suppose that (iii) holds. It is easy to see that $|Y| = 2$, say $Y = \{y_1, y_2\}$. Thus $X = [y_1]_\rho \cup [y_2]_\rho$, a disjoint union. Let $\alpha \in T(X,Y,\rho)$. Since $\operatorname{ran}\alpha \subseteq Y$, we have $|\operatorname{ran}\alpha| = 1$ or $|\operatorname{ran}\alpha| = 2$. If $|\operatorname{ran}\alpha| = 1$, then $\alpha = \alpha^3$, so α is left regular. Suppose that $|\operatorname{ran}\alpha| = 2$. Since $\operatorname{ran}\alpha \subseteq Y$, we have $\operatorname{ran}\alpha = Y$. Since $y_1\alpha \neq y_2\alpha$, we have

$|Y\alpha| = 2$. Since $Y\alpha \subseteq Y$, we get $Y\alpha = Y$. Then we have

$$y_1\alpha^{-1} \cap Y\alpha = y_1\alpha^{-1} \cap Y \neq \emptyset \text{ and}$$
$$y_2\alpha^{-1} \cap Y\alpha = y_2\alpha^{-1} \cap Y \neq \emptyset.$$

Therefore, α is left regular. □

4. Right regularity of $P(X,Y,\rho)$ and $T(X,Y,\rho)$

To characterize the right regular elements of $P(X,Y,\rho)$ and $T(X,Y,\rho)$, the following Lemma is needed.

Lemma 4.1. *Let $S(X,Y,\rho)$ be $P(X,Y,\rho)$ or $T(X,Y,\rho)$. If $\alpha,\beta \in S(X,Y,\rho)$ are such that $\alpha = \alpha^2\beta$, then $ran\,\alpha \subseteq dom\,\alpha$ and $\alpha|_{ran\,\alpha}$ is injective.*

Proof. Assume that $\alpha,\beta \in S(X,Y,\rho)$ are such that $\alpha = \alpha^2\beta$. Let $x \in \operatorname{ran}\alpha$, say $x = a\alpha$ for some $a \in \operatorname{dom}\alpha$. We have $a \in \operatorname{dom}\alpha^2\beta$, that is, $x = a\alpha \in \operatorname{dom}\alpha\beta \subseteq \operatorname{dom}\alpha$. This shows $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$. Let $x_1, x_2 \in \operatorname{ran}\alpha$ such that $x_1\alpha = x_2\alpha$. Then $x_1 = a_1\alpha$ and $x_2 = a_2\alpha$ for some $a_1, a_2 \in \operatorname{dom}\alpha$. Hence, $x_1 = a_1\alpha = a_1\alpha^2\beta = x_1\alpha\beta = x_2\alpha\beta = a_2\alpha^2\beta = a_2\alpha = x_2$. Therefore, $\alpha|_{\operatorname{ran}\alpha}$ is injective. □

Next, we describe the right regular elements of $P(X,Y,\rho)$.

Theorem 4.2. *Let $\alpha \in P(X,Y,\rho)$. Then α is right regular if and only if $ran\,\alpha \subseteq dom\,\alpha$ and $\alpha|_{ran\,\alpha}$ is injective.*

Proof. Necessity follows immediately from Lemma 4.1.

Conversely, suppose $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha|_{\operatorname{ran}\alpha}$ is injective. Define

$$A = \{x \in X \mid \exists\, z \in \operatorname{ran}\alpha^2, (x,z) \in \rho\}.$$

For each $z \in \operatorname{ran}\alpha^2$, there is a unique $y_z \in \operatorname{ran}\alpha$ with $y_z\alpha = z$. Define $\beta : A \to X$ by

$$x\beta = y_z \text{ for all } x \in A.$$

We must show that β is well-defined. If $x \in A$ satisfies $(x,z) \in \rho$ and $(x,w) \in \rho$ for some $z,w \in \operatorname{ran}\alpha^2$, then $(z,w) \in \rho$. Since $\operatorname{ran}\alpha^2 \subseteq \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$, we get $z,w \in \operatorname{dom}\alpha$. So $z\alpha = w\alpha$. Since $\alpha|_{\operatorname{ran}\alpha}$ is injective, we have $z = w$, so $y_z = y_w$. Clearly, $\operatorname{ran}\beta \subseteq Y$. Let $x_1, x_2 \in \operatorname{dom}\beta$ such that $(x_1,x_2) \in \rho$. If $(x_1,z) \in \rho$ for some $z \in \operatorname{ran}\alpha^2$, then $(x_2,z) \in \rho$. Thus $x_1\beta = y_z = x_2\beta$. Then $\beta \in P(X,Y,\rho)$. Since $(\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\operatorname{ran}(\alpha^2) \subseteq A = \operatorname{dom}\beta$, we obtain that $\operatorname{dom}\alpha = \operatorname{dom}(\alpha^2\beta)$. Let $x \in \operatorname{dom}\alpha$. Then $x\alpha \in \operatorname{ran}\alpha$ and $(x\alpha)\alpha = x\alpha^2 = y_{x\alpha^2}\alpha$. Since $\alpha|_{\operatorname{ran}\alpha}$ is injective, we get $y_{x\alpha^2} = x\alpha$. Then $x\alpha^2\beta = y_{x\alpha^2} = x\alpha$. This shows that $\alpha = \alpha^2\beta$. Hence, α is a right regular element of $P(X,Y,\rho)$. □

Now, we use Theorem 4.2 to determine the condition under which the semigroup $P(X,Y\rho)$ is a right regular semigroup.

Theorem 4.3. *$P(X,Y,\rho)$ is right regular if and only if $|X| = 1$.*

Proof. If $|X| = 1$, then $P(X,Y,\rho)$ contains exactly two elements namely 0 and 1, where 0 is an empty map and 1 is an identity map. Then $P(X,Y,\rho)$ is right regular.

Conversely, suppose that $P(X,Y,\rho)$ is right regular. Let $x \in X$. Choose $y \in Y$ and define $\alpha : \{x\} \to X$ by $x\alpha = y$. Since α is right regular, by Theorem 4.2, $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$. Then $\{y\} = \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha = \{x\}$, that is, $x = y$. Since x is arbitrary, we have $|X| = 1$ as desired. □

Next, we describe the right regular elements of $T(X,Y,\rho)$.

Theorem 4.4. *Let $\alpha \in T(X,Y,\rho)$. Then α is right regular if and only if $\alpha|_{ran\,\alpha}$ is injective.*

Proof. The necessity follows straight from Lemma 4.1. For sufficiency, assume that $\alpha|_{\operatorname{ran}\alpha}$ is injective. For each $x \in \operatorname{ran}(\alpha^2)$, there is exactly one $y_x \in \operatorname{ran}\alpha$ for which $y_x\alpha = x$. Let $\beta : X \to X$ be defined by

$$x\beta = \begin{cases} y_z & \text{if } (x,z) \in \rho \text{ for some} \\ & z \in \operatorname{ran}(\alpha^2), \\ x\alpha & \text{otherwise.} \end{cases}$$

Clearly, $\operatorname{ran}\beta \subseteq Y$. For each $x \in X$, if $(x, z_1) \in \rho$ and $(x, z_2) \in \rho$ for some $z_1, z_2 \in \operatorname{ran}(\alpha^2)$, then $(z_1, z_2) \in \rho$. Thus $z_1\alpha = z_2\alpha$. Since $\alpha|_{\operatorname{ran}\alpha}$ is injective, we get $z_1 = z_2$. Then β is well-defined. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in \rho$. If $(x_1, z) \in \rho$ for some $z \in \operatorname{ran}(\alpha^2)$, then $(x_2, z) \in \rho$. Thus $x_1\beta = y_z = x_2\beta$. If $(x_1, z) \notin \rho$ for all $z \in \operatorname{ran}(\alpha^2)$, then $(x_2, z) \notin \rho$ for all $z \in \operatorname{ran}(\alpha^2)$. So $x_1\beta = x_1\alpha = x_2\alpha = x_2\beta$. This shows that $\beta \in T(X,Y,\rho)$. Let $x \in X$. We have $x\alpha \in (x\alpha^2)\alpha^{-1}$. By the uniqueness of $y_{x\alpha^2}$, we get $y_{x\alpha^2} = x\alpha$. Then $x\alpha^2\beta = y_{x\alpha^2} = x\alpha$. Therefore, α is right regular. □

Finally, we apply Theorem 4.4 to investigate the conditions under which $T(X,Y,\rho)$ becomes a right regular semigroup.

Theorem 4.5. *$T(X,Y,\rho)$ is right regular if and only if one of the following conditions holds.*

(i) $|Y| = 1$.

(ii) $|X/\rho| = 1$.

(iii) $|Y| > 1, |X/\rho| = 2$ *and* $\big|[x]_\rho \cap Y\big| = 1$ *for all* $x \in X$.

Proof. Suppose that $T(X,Y,\rho)$ is right regular. Assume (i), (ii) and (iii) are false. Then $|Y| > 1$ and $|X/\rho| > 2$ or $\big|[a]_\rho \cap Y\big| \neq 1$ for some $a \in X$. By Lemma 3.4, there exists a proper subset C of X such that $|C \cap Y| > 1$ and for each $x \in X$, if $(x, c) \in \rho$ for some $c \in C$, then $x \in C$. Now let $y_1, y_2 \in C \cap Y$ such that $y_1 \neq y_2$ and define α as in Theorem 3.5. Since α is right regular, by Theorem 4.4, $\alpha|_{\operatorname{ran}\alpha}$ is injective. We see that $y_1\alpha = y_1 = y_2\alpha$, but $y_1 \neq y_2$. Then $\alpha|_{\operatorname{ran}\alpha}$ is not injective, a contradiction.

Conversely, we assume that (i) or (ii) is true. Then by Lemma 2.4, we get $T(X,Y,\rho)$ is an idempotent semigroup which implies that $T(X,Y,\rho)$ is right regular. Next, we assume that (iii) holds. Then $|Y| = 2$, say $Y = \{y_1, y_2\}$. Thus $X = [y_1]_\rho \cup [y_2]_\rho$, a disjoint union. Let $\alpha \in T(X,Y,\rho)$. Since $\operatorname{ran}\alpha \subseteq Y$, we have $|\operatorname{ran}\alpha| = 1$ or $|\operatorname{ran}\alpha| = 2$. If $|\operatorname{ran}\alpha| = 1$, then $\alpha = \alpha^3$, so α is right regular. Suppose that $|\operatorname{ran}\alpha| = 2$. We see that $y_1 \neq y_2$ and $y_1\alpha \neq y_2\alpha$. Therefore, α is right regular. □

5. Conclusion

In this work, we presented necessary and sufficient conditions when elements of the semigroups $P(X,Y,\rho)$ and $T(X,Y,\rho)$ are regular, left regular and right regular and used them to analyze the regularity of $P(X,Y,\rho)$ and $T(X,Y,\rho)$.

Note that when ρ is the identity relation, we have $T(X,Y,\rho) = T(X,Y)$. In this case, Theorem 2.5 shows that $T(X,Y,\rho)$ is regular if and only if $|Y| = 1$ or $Y = X$, which aligns with the regularity condition for $T(X,Y)$ in [4]. Thus, our results extend the known results for $T(X,Y)$.

Semigroup theory forms the foundation for more advanced algebraic structures such as monoids, groups, rings, and algebras. A fundamental result in algebraic semigroups theory is Cayley's Theorem for semigroups, which states that every semigroup can be embedded into a full transformation semigroup. This result highlights the importance of studying subsemigroups of full transformation semigroups to understand the structure of arbitrary semigroups. Consequently, the characterization of these subsemigroups has become a central topic of interest among researchers in semigroup theory.

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