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Interval Tolerance Solution for Adjusted Problem of Optimistic Interval Linear Program

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ABSTRACT

An optimistic solution to an interval linear program is a real-valued solution derived from the best-case deterministic linear program. However, relying solely on this best-case solution can be overly simplistic, as the actual realization of parameters lies within specified intervals. Instead, it is more appropriate to provide an interval vector solution that remains near the optimistic solution, particularly when the decision-maker prefers proximity to the best-case scenario. In this paper, we establish the equivalence between the weak feasible solution set of an interval equality system and the union of basic feasible solutions across all scenarios of an interval linear program with an interval inequality system, where the interval inequalities require the left-hand side to be lower than the right-hand side, but not excessively so. Furthermore, we demonstrate that, under positive variables, this set coincides with the union of basic optimal solution sets. This result enables the use of a tolerance-based approach to identify an interval vector solution near the optimistic solution. Specifically, we modify the interval linear program so that the optimistic solution becomes a tolerance solution for the adjusted problem. We then propose a method to derive the interval tolerance vector solution for the modified problem, with the goal of maximizing the total sum of the dimensions of the interval tolerance vector hyper-box. Our proposed method differs from most existing methods for finding interval solutions, as those methods typically yield interval solutions that merely include weak solutions without specifying the solution type. Even though there are existing methods for obtaining interval tolerance solutions, none of them consider interval tolerance solutions that are close to the optimistic solution.

Keywords: Interval linear problem; Optimistic problem; Optimization; Tolerance solution; Weak solution

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1. Introduction

An Interval Linear Programming (ILP) problem: max $c^{\mathrm{T}}x$ subject to $[\underline{A}, \overline{A}]x \leq [\underline{b}, \overline{b}],\ x \geq \vec{0}$, is a type of linear optimization problem where uncertainty is represented in the coefficients of the left-hand side (constraints) and right-hand side (resource limits). This uncertainty is expressed as intervals rather than precise values, allowing for flexible modeling of real-world scenarios where exact data may not be available. Given that ILP problems differ from standard linear programming due to the presence of interval parameters, the concepts of a feasible point and an optimal solution are redefined to accommodate this distinction. A point is considered feasible for the interval linear programming model if it belongs to the largest feasible region of the model. It is considered optimal if it serves as an optimal solution for the corresponding deterministic model. The literature reveals that numerous researchers have explored methods to determine interval solutions for ILP problems by employing the approach of transforming the original ILP problem into two sub-models. Moreover, the concepts of feasibility and optimality in ILP problems have been extensively studied to develop more efficient methods for identifying interval solutions.

Starting with the foundational Best and Worst Cases (BWC) method introduced by Tong in 1994 [1], the ILP model is reformulated into two sub-models: the best sub-model, representing the largest feasible region, and the worst sub-model, representing the smallest feasible region. The BWC method is employed to derive the best and worst objective function values, providing precise bounds for the optimal objective values. However, a portion of the solution space generated by the BWC method may be infeasible. In 1995, Huang et al. [2] introduced a new method called the two-step method (TSM). They integrated the concepts of grey systems and grey decision-making into a Mixed-Integer Linear Programming (MLP) framework, leading to a grey integer programming (GIP) formulation to generate the two sub-models. These sub-models differ from the BWC method in that the interval solution might not provide the best and worst objective function values directly but can achieve a larger exact optimal solution set compared to the BWC method. However, the solution space of the TSM may still include infeasible solutions. In recent years, numerous methods have been developed building upon the TSM approach. To address the issue of infeasible solutions within the solution space of the TSM, Zhou et al. (2009) [3] introduced a modified interval linear programming (MILP) method that incorporates an enhanced-interval linear programming (EILP) model and its associated solution algorithm, adding an additional constraint to the second sub-model of the TSM. The solution space derived from the EILP model is absolutely feasible compared to that of ILP. This approach facilitates a better understanding of the expected-value-oriented trade-off between system benefits and the risks of constraint violations. Although the solution space of the MILP method is absolutely feasible, some of the solutions obtained may be non-optimal. In addition, Huang and Cao (2011) [4] developed a new solution method known as the three-step method (ThSM) for solving ILP models. The main advantage of ThSM is that it guarantees no infeasible solutions will be included in the obtained results. To determine whether all solutions obtained through TSM are within the feasible decision space, they introduced a method for feasibility testing. Moreover, constricted models known as ThSMs (ThSM-I and ThSM-II) are developed to eliminate the infeasible solutions of the TSM by narrowing the solution space of the TSM into a center point. Furthermore, ThSM can generate interval solutions while maintaining low computational requirements.

Next, a robust two-step method (RTSM) proposed by Fan and Huang (2012) [5] improved upon the traditional TSM by incorporating an optimization technique in the second step to refine the solution, ensuring feasibility and minimizing the impact of uncertainty. This approach is particularly relevant for environmental management applications, such as resource allocation, pollution control, and sustainability planning, where decision-makers must account for uncertain environmental data. Compared to MILP and ThSM methods, RTSM generates a relatively larger solution space, thereby reducing the risk of significant loss of decision-related information. Moreover, an alternative solution method (SOM-2) was introduced by Lu et al. (2014) [6]. This study evaluates numerical solutions for ILP problems by comparing different methods based on coverage and validity rates. The coverage rate measures how well the obtained solution set captures the true solution space, while the validity rate assesses the feasibility of solutions within the given interval constraints. By analyzing various ILP solution approaches, the study highlights their strengths and limitations in handling uncertainty. The comparison provides insights into which methods offer a balance between solution robustness and computational efficiency. However, in some cases, the feasibility of the solution space is not guaranteed, and the SOM-2 method does not always yield an absolutely optimal solution. To ensure both the feasibility and optimality of solution spaces, the improved ILP (IILP) and improved MILP (IMILP) methods are examined. In 2016, Allahdadi et al. [7] proposed two new approaches, IILP and IMILP methods. This study focuses on enhancing the MILP method by incorporating new techniques to improve solution accuracy, feasibility, and computational efficiency. The proposed improvements aim to address limitations in existing MILP approaches, particularly in handling uncertainty and maintaining a well-balanced solution space. By introducing refined optimization strategies, the enhanced MILP method provides more reliable and interpretable results for decision-making under uncertainty. Additionally, Mishmast Nehi et al. [8] propose an improved method called the improved SOM-2 method (ISOM-2), designed to enhance the performance of existing ILP solutions. This new method addresses some of the limitations identified in traditional approaches, aiming to provide more accurate and reliable results. The solution space of the ISOM-2 method is both absolutely feasible and optimal.

Mishmast Nehi et.al. [8] provided Fig. 1 which corresponding to the results of IMLP, IILP, ISOM-2 methods of the ILP model (1.1).

$$\begin{aligned} \max \quad & [3,3.5]x_1 - [1,1.2]x_2 \qquad (1.1)\\ \text{s.t.} \quad & [1,1.1]x_1 + [1.6,1.8]x_2 \le [11.6,12],\\ & [3,4]x_1 - [2,3]x_2 \le [5,7],\\ & x_1, x_2 \ge 0. \end{aligned}$$

The rectangular boxes within the yellow area in Fig. 1 are the interval vector solutions representing optimal solutions to some portion of deterministic problems of (1.1). These rectangular boxes were compared by their optimal value ranges to identify which box should be used to repre-

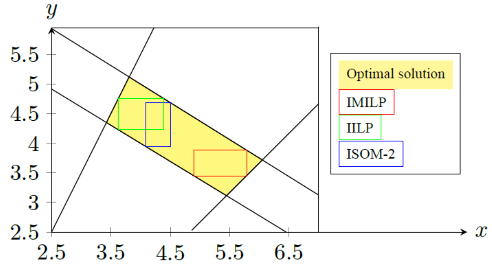

Fig. 1. The interval solution of IMILP, IILP and ISOM-2 methods.

sent the interval vector solution of (1.1). We observe that the solutions obtained from methods such as IMILP, IILP, and ISOM-2 are represented as rectangular boxes contained within the yellow area, as these methods guarantee both optimality and feasibility. In contrast, other methods reviewed such as BWC, TSM, MILP, ThSM, RTSM, and SOM-2, provide solutions where some guarantee feasibility, while others guarantee neither feasibility nor optimality. Consequently, the rectangular boxes representing the interval solutions obtained from these methods may extend beyond the yellow area. Even though methods such as IMILP, IILP, and ISOM-2 can guarantee feasible and optimal solutions, they may not be suitable for finding an interval vector solution tailored to a specific purpose. This is because these methods are designed to represent the ILP problem as a whole, and the resulting solution may encompass multiple solution types (see Definition 2.5 and Theorem 2.6 for more details).

In our case, under the special assumption that the left-hand side and the right-hand side of the inequality system overlap, the objective is to determine an interval vector solution, denoted as $[\underline{x}, \overline{x}]$, that is close to an optimistic solution of the ILP while maintaining the tolerance property. This means that any vector x within the interval vector $[\underline{x}, \overline{x}]$ will ensure that the total resources used, represented as $[\underline{A}, \overline{A}]x$ remain within the specified resource range $[\underline{b}, \overline{b}]$. None of the reviewed methods can provide an appropriate solution under this objective.

To achieve this goal, we first provide foundational knowledge about the feasible weak solution set of the equation system $[\underline{A}, \overline{A}]x = [\underline{b}, \overline{b}]$ and the associated semantics, including tolerance, control, left-localized and right-localized solutions, in Section 2. Subsequently, in Section 3, we analyze the equivalence between the feasible weak solution set of the system $[\underline{A}, \overline{A}]x = [\underline{b}, \overline{b}]$ and the union of basic feasible solution sets of deterministic problems of an ILP with constraints $[\underline{A}, \overline{A}]x \leq [\underline{b}, \overline{b}]$, $x \geq \vec{0}$ and the extra constraint $\overline{A}x \geq \underline{b}$. This extra constraint is needed for preserving the above special assumption. Furthermore, the analysis provides additional insights when considering the positivity of both sets. Specifically, we show that the positive weak solution set is equivalent to the union of basic optimal solutions of deterministic problems for an ILP with the constraint $[\underline{A}, \overline{A}]x \leq [\underline{b}, \overline{b}]$, $\overline{A}x \geq \underline{b}$, $x \geq x_0$, for a given $x_0 > \vec{0}$. Building on this result, in Section 4, we refine the interval tolerance vector solution method from [9] to achieve our objective of finding an interval vector solution close to an optimistic solution of the ILP with $\overline{A}x \geq \underline{b}$ while preserving the tolerance semantics. If the optimistic solution does not satisfy the tolerance property, adjustments to the interval $[\underline{A}, \overline{A}]$ will be made to maintain this property. In Section 5, we provide numerical examples, including a small diet problem, to illustrate the proposed approach. The final section is reserved for conclusions and remarks.

2. Preliminaries

This section focuses on the fundamental knowledge relevant to this article. It provides an overview of the interval system and presents several useful theorems.

2.1 Interval system of linear equations

Let m and n be positive integers. The set of all $m \times n$ (interval) matrices over $\mathbb{R}$ and the set of all column (interval) vectors of size n over $\mathbb{R}$ are denoted by $\mathbb{R}^{m\times n}$ ($\mathbb{IR}^{m\times n}$) and $\mathbb{R}^n$ ($\mathbb{IR}^n$), respectively. The system with vector of variables $x \in \mathbb{R}^n$ is written as the form

$$\mathbf{A}x = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$ are defined by $\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\}$ and $\mathbf{b} = [\underline{b}, \overline{b}] = \{b \in \mathbb{R}^m : \underline{b} \leq b \leq \overline{b}\}$. Moreover, $\mathbf{A}$ and $\mathbf{b}$ can be written in terms of the center matrix and the radius matrix as $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ and $\mathbf{b} = [b_c - \delta, b_c + \delta]$, where the center $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ and $b_c = \frac{1}{2}(\overline{b} + \underline{b})$ and the radius $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ and $\delta = \frac{1}{2}(\overline{b} - \underline{b})$.

Definition 2.1. A vector $x \in \mathbb{R}^n$ is called

(i) a *weak solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}, b \in \mathbf{b}$

(ii) a *tolerance solution* of $\mathbf{A}x = \mathbf{b}$ if for each $A \in \mathbf{A}$ there exists $b \in \mathbf{b}$ such that $Ax = b$

(iii) a *control solution* of $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there exists $A \in \mathbf{A}$ such that $Ax = b$.

Lemma 2.2. *[10] The tolerance solution set, denoted by $\sum_{\forall\exists}(\mathbf{A}, \mathbf{b})$, is defined as follows:*

$$\sum_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : \mathbf{A}x \subseteq \mathbf{b}\}.$$

Definition 2.3. Let $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ be any two interval vectors.

(i) If $\underline{x} \leq \underline{y} \leq \overline{x} \leq \overline{y}$, then $\mathbf{x}$ is *strictly less than or equal* to $\mathbf{y}$, denoted by $\mathbf{x} \leq_{st} \mathbf{y}$.

(ii) If $\overline{x} \leq \underline{y}$, then $\mathbf{x}$ is *strongly less than or equal* to $\mathbf{y}$, denoted by $\mathbf{x} \leq_s \mathbf{y}$.

Definition 2.4. A vector $x \in \mathbb{R}^n$ is called

(i) an *left–localized solution* of $\mathbf{A}x = \mathbf{b}$ if there is at least one $A \in \mathbf{A}$ such that $Ax \in \mathbf{b}$. For the other $A \in \mathbf{A}$, $Ax \leq_s \mathbf{b}$

(ii) a *right–localized solution* of $\mathbf{A}x = \mathbf{b}$ if there is at least one $A \in \mathbf{A}$ such that $Ax \in \mathbf{b}$. For the other $A \in \mathbf{A}$, $-Ax \leq_s -\mathbf{b}$.

Definition 2.5. [11] Let T, C, L and R be row index subsets of M such that

$$T = \{i \in M : (\mathbf{A}x)_i \subseteq \mathbf{b}_i\},$$
$$C = \{i \in M : (\mathbf{A}x)_i \supseteq \mathbf{b}_i\},$$
$$L = \{i \in M : (\mathbf{A}x)_i \leq_{st} \mathbf{b}_i\},$$
$$R = \{i \in M : (-\mathbf{A}x)_i \leq_{st} -\mathbf{b}_i\}.$$

A vector $x \in \mathbb{R}^n$ is called a *tolerance-control-localized solution* ($TCLR$) of $\mathbf{A}x = \mathbf{b}$ if $T \cup C \cup L \cup R = M$.

Theorem 2.6. *[11] Given $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$. Then*

$$\sum_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \sum_{TCLR}(\mathbf{A}, \mathbf{b}),$$

for which $\sum_{\exists\exists}(\mathbf{A}, \mathbf{b})$ and $\sum_{TCLR}(\mathbf{A}, \mathbf{b})$ are the weak solution set and the set containing all tolerance-control-localized solutions of $\mathbf{A}x = \mathbf{b}$, respectively.

Definition 2.7. A vector $x \in \mathbb{R}^n$ is called a *weak feasible solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$ and $x \geq \vec{0}$.

Definition 2.8. A vector $x \in \mathbb{R}^n$ is called a *positive weak solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$ and $x > \vec{0}$.

Theorem 2.9. *[12] A vector $x \in \mathbb{R}^n$ is a weak feasible solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies the following system*

$$\begin{aligned} \underline{A}x &\leq \overline{b}, \\ \overline{A}x &\geq \underline{b}, \\ x &\geq \vec{0}. \end{aligned}$$

More details on characteristics of each solution type and its application can be found in [10, 12–20].

2.2 Linear program and interval linear program

A linear programming problem is written as

$$\min\ c^{\mathrm{T}}x \text{ s.t. } Ax = b,\ x \geq \vec{0},$$

where c and x are $n \times 1$ matrices, A is an $m \times n$ matrix of rank m and b is an $m \times 1$ matrix.

Definition 2.10. For any nonsingular $m \times m$ sub-matrix B of A, we call $x = \begin{pmatrix} x_B & \vec{0} \end{pmatrix}^{\mathrm{T}}$ a *basic solution* with respect to B, where $\vec{0}$ in x is the zero vector of all leftover components of x associated with the $n-m$ leftover columns of A, if $Bx_B = b$ and x satisfies $Ax = b$. Moreover, B is referred to as a *basis* and the components of x associated with the column of B, x_B, are called *basic variables*.

Definition 2.11. A vector x satisfying the system $Ax = b,\ x \geq \vec{0}$ is said to be a *feasible solution* of the system. A feasible solution that is also basic is called a *basic feasible solution*, i.e., $x_B \geq \vec{0}$. A basic feasible solution x^* is said to be optimal if it yields the minimum objective value among all feasible solutions, that is,

$$c^T x^* \leq c^T x \quad \text{for all feasible } x.$$

In such a case, x^* is called a *basic optimal solution*.

Let $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$. An ILP problem is defined as

$$\begin{aligned} \min\ & c^{\mathrm{T}}x \\ \text{s.t.}\ & \mathbf{A}x \leq \mathbf{b} \\ & x \geq \vec{0}. \end{aligned} \tag{2.1}$$

In this paper, we do not consider intervals in the objective function of an ILP, since it can be transformed into an equivalent ILP without interval coefficients in the objective function. We further assume that each deterministic problem of (2.1) is bounded. For cases involving infeasible or unbounded deterministic problems, appropriate adjustments are necessary; see [21, 22] for details.

By considering specific values $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_i \in [\underline{b}_i, \overline{b}_i]$ in the ILP problem (2.1), a deterministic model is obtained as follows:

$$\begin{aligned} \min\ & z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} a_{ij}x_j \leq b_i,\ i = 1, 2, ..., m, \\ & x_j \geq 0,\ j = 1, 2, ..., n. \end{aligned} \tag{2.2}$$

Definition 2.12. The feasible solution set $\mathcal{S}$ of an ILP is defined to be the union of the feasible solution sets of each deterministic problem of that ILP; i.e.,

$$\mathcal{S} = \bigcup_{A\in\mathbf{A},\ b\in\mathbf{b}} \left\{ x \mid Ax \leq b,\ x \geq \vec{0} \right\}.$$

The optimal solution set of an ILP is defined as the union of the optimal solutions of all corresponding deterministic problems.

The basic feasible solution set of an ILP is defined as the union of the basic feasible solutions of all deterministic instances.

The basic optimal solution set of an ILP is defined as the union of the basic optimal solutions obtained from each deterministic problem.

Additionally, the lower and upper bounds of the optimal values are obtained by solving the following problems, respectively:

$$\begin{aligned} \min \ & \underline{z} = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} \underline{a}_{ij} x_j \le \overline{b}_i, \ i = 1, 2, ..., m, \\ & x_j \ge 0, \ j = 1, 2, ..., n, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \min \ & \overline{z} = \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} \overline{a}_{ij} x_j \le \underline{b}_i, \ i = 1, 2, ..., m, \\ & x_j \ge 0, \ j = 1, 2, ..., n. \end{aligned} \tag{2.4}$$

Problems (2.3) and (2.4) are described as the optimistic (best) and the pessimistic (worst) sub-models of the ILP problem.

Several research articles [11, 14, 23–29] define the main constraint of an ILP as an equality constraint, $\mathbf{A}x = \mathbf{b}$, which exhibits a weak solution property. This means that not all combinations of $A \in \mathbf{A}$ and $b \in \mathbf{b}$ need to be solvable. Instead, it suffices for at least one pair A and b to produce a solvable equation $\mathbf{A}x = \mathbf{b}$. On the other hand, in many other papers [1–8, 30–33], the inequality constraint $\mathbf{A}x \le \mathbf{b}$ is interpreted more strictly, requiring a solution for each scenario of $\mathbf{A}x \le \mathbf{b}$. In the next section, we will demonstrate that the inequality constraint $\mathbf{A}x \le \mathbf{b}$ can be transformed into an equivalent weak solution set $\mathbf{A}x = \mathbf{b}$ by incorporating additional assumption that $\mathbf{A}x$ should not be too much (strongly) less than $\mathbf{b}$ into the ILP problem (2.1).

3. Weak solution set as union of basic feasible solutions of deterministic problems of interval linear program

The primary objective of this paper is to determine a tolerance interval vector solution that is close to an optimistic solution of an ILP. Notably, the tolerance behavior is associated with an interval equality system, whereas the constraints of an ILP are defined as an interval inequality system. Consequently, it is crucial to analyze the relationship between these two systems.

Allahdadi and Mishmast Nehi [33] obtained that the optimal solution of an ILP with constraint $[\underline{A}, \overline{A}]x \le [\underline{b}, \overline{b}], \ x \ge x_0$, for a given $x_0 > \vec{0}$ is equal to the positive weak solution set of $[\underline{A}, \overline{A}]x = [\underline{b}, \overline{b}]$. Building on this concept, we extend their idea, simplify the proof, and provide a more general statement. Specifically, we demonstrate that the weak solution set is equivalent to the basic feasible solution set of the ILP with the extra constraint $\overline{A}x \ge \underline{b}$. The details are provided in Theorem 3.1 and Corollary 3.2.

A general ILP problem in the literature is given by

$$\text{Problem } ILP\text{: max } c^{\mathrm{T}}x \text{ subject to } [\underline{A}, \overline{A}]x \le [\underline{b}, \overline{b}], \ x \ge \vec{0}.$$

In the worst deterministic problem (2.4), if its optimal solution x^* provides $\overline{A}x^* < \underline{b}$, it may be more reasonable to update the lower bound to $\underline{b}_{\text{new}} = \overline{A}x^*$. This adjustment reflects the modeling interpretation of the lower bound as a limit on the available resources and ensures the constraints more accurately capture the situation. Therefore, in this paper we assume that $\underline{b}$ in an ILP problem satisfies $\overline{A}x^* = \underline{b}$. Now, consider an ILP with extra constraint $\overline{A}x \geq \underline{b}$,

$$\text{Problem } ILP_1\text{: max } c^{\mathrm{T}}x \text{ subject to}$$
$$[\underline{A}, \overline{A}]x \leq [\underline{b}, \overline{b}],\ \overline{A}x \geq \underline{b},\ \ x \geq \vec{0}.$$

For any $A \in [\underline{A}, \overline{A}]$ and $b \in [\underline{b}, \overline{b}]$, the corresponding deterministic problem of ILP is not equivalent to the one of ILP_1, in general. However, this extra constraint may be able to capture some hidden behavior that cannot be captured by the main inequality constraint. For example, if the i-th worker has to produce x_j units of the j-th product and the production time of each unit of the j-th product that this worker could manage is in the range $[\underline{a}_{ij}, \overline{a}_{ij}]$ while the total production time is in the range $[\underline{b}_i, \overline{b}_i]$. The worker would try to produce the goods as fast as possible, which results in the following constraint:

$$[\underline{a}_{i1}, \overline{a}_{i1}]x_1 + ... + [\underline{a}_{in}, \overline{a}_{in}]x_n \leq [\underline{b}, \overline{b}].$$

To maintain the quality, the manager may not want the total production time to be too much lower than $\underline{b}$, otherwise there is no need to set up the range $[\underline{b}, \overline{b}]$ in the first place. By adding the constraint $\overline{a}_{i1}x_1 + \overline{a}_{i2}x_2 + ... + \overline{a}_{in}x_n \geq \underline{b}$, it would guarantee the overlapping of the worker's finishing time and the manager's range of total production time, while still allowing the worker to finish earlier. For this reason, we state new version of an ILP problem as Problem ILP_1.

We prove in Theorem 3.1 that the union of feasible sets of all deterministic problems, $\mathcal{T}$, equals the set of all feasible weak solutions of $\mathbf{A}x = \mathbf{b}$, Ω. Moreover $\mathcal{T}$ is also $\mathcal{T}_1$, the union of the basic feasible solution sets for each deterministic problem of Problem ILP_1 and even $\mathcal{T}_2$, the union of the basic optimal solution sets, under some assumptions. Note here that we avoid stating the main constraint of an ILP problem as $\mathbf{A}x = \mathbf{b}$, when interval parameters are involved. This is because each deterministic ILP problem requires optimality, yet it is unlikely that the system $Ax = b$ will remain feasible for all $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

Theorem 3.1. *(Weak solution set as the set $\mathcal{T}$) Let Ω denote the weak feasible solution set of the system $\mathbf{A}x = \mathbf{b}$. Let $\mathcal{T}$ represent The feasible solution set of Problem ILP_1. Finally, let $\mathcal{T}_1$ be the union of all basic feasible solution sets for each deterministic problem associated with Problem ILP_1. The following statements are true.*

1. *Let $\mathbf{A} \in \mathbb{IR}^{m \times n}$. Then $\Omega = \mathcal{T}$.*
2. *Let $\mathbf{A} \in \mathbb{IR}^{m \times n}$ where $m \geq n$. Then $\Omega = \mathcal{T}_1$.*

Proof. Let $x \in \mathcal{T}$. Then, there exists $A \in \mathbf{A}$ and $b \in \mathbf{b}$, such that $Ax \leq b$, $\overline{A}x \geq \underline{b}$ and $x \geq \vec{0}$. Thus, $\underline{A}x \leq Ax \leq b \leq \overline{b}$. Hence, $x \in \Omega$. Now let $x \in \Omega$. Then, $\underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}$ and $x \geq \vec{0}$. Hence $x \in \mathcal{T}$. So $\mathcal{T} = \Omega$.

Now, we will show that $\Omega \subseteq \mathcal{T}_1$. Let $x^{'} \in \Omega$, there exists $A^{'} \in \mathbf{A}$ and $b^{'} \in \mathbf{b}$ such that $A^{'}x^{'} = b^{'}$, $x^{'} \geq \vec{0}$, where $m \geq n$. This implies that $x^{'}$ provides a basic feasible solution of the deterministic problem

$$\text{min } c^{\top}x \text{ subject to } A^{'}x \leq b^{'}, x \geq \vec{0},$$

for any $c \in [\underline{c}, \overline{c}]$. The vector $x^{'}$ also satisfies $\overline{A}x^{'} \geq \underline{b}$, from the definition of Ω. Therefore, $x^{'}$ becomes a part of a basic feasible solution of Problem I_1:

$$\text{Problem } I_1 : \quad \min\ c^\top x \text{ subject to} \quad A^{'}x \le b^{'},\ \overline{A}x \ge \underline{b},\ x \ge \vec{0}.$$

Hence, $\Omega \subseteq \mathcal{T}_1$. Since $\Omega = \mathcal{T}$ and $\mathcal{T}_1 \subseteq \mathcal{T}$, we have $\Omega = \mathcal{T} = \mathcal{T}_1$. □

Corollary 3.2. *Let* $\mathbf{A} \in \mathbb{IR}^{m\times n}$ *where* $m \ge n$. *If* $x > \vec{0}$ *for all* $x \in \Omega$, *then* $\Omega = \mathcal{T}_2$, *where* Ω *denotes the weak feasible solution set of the system* $\mathbf{A}x = \mathbf{b}$, *and* $\mathcal{T}_2$ *denotes the set of basic optimal solutions to Problem* ILP_1.

Proof. It is sufficiently enough to show that $\Omega \subseteq \mathcal{T}_2$. Let $x^* \in \Omega$. Since $\Omega = \mathcal{T}_1$, x^* is a basic feasible solution to some deterministic problems of ILP_1. Let S be the set of all deterministic problems of ILP_1 containing x^* as their basic feasible solution. Since $x^* > 0$, all components $x_1, x_2, \dots, x_n$ of x^* are basic variables. A basis matrix corresponding to x^* of each deterministic problem in S contains coefficient columns of basic variables $x_1, x_2, \dots, x_n$ and some other slack variables. Since all elements in Ω are assumed to be greater than $\vec{0}$, all basis matrices of deterministic problems in S must also correspond to columns of variables $x_1, x_2, \dots, x_n$ and some other slack variables. Hence, x^* is a basic optimal solution to a problem in S. □

4. Interval solutions of interval linear program

In this section, we propose a novel approach for determining a tolerance based interval solution near optimistic solution of Problem ILP_1. We begin by discussing a method for identifying subsets of feasible interval vectors within the set of tolerance solution, as introduced by Beaumont and Philippe [9].

4.1 Interval tolerance solution

This section reviews the method proposed by Beaumont and Philippe [9] for identifying a subset of the interval tolerance solution.

In their 2001 work, Beaumont and Philippe introduced two polyhedrons that characterize subsets of the interval vectors contained within the tolerance solution set. They also proposed a refined definition of optimality for an interval vector within this context. Moreover, they demonstrated how the Simplex algorithm can be utilized to identify an optimal interval vector belonging to the tolerance solution set.

They begin by defining the set of all possible interval vectors included in the tolerance solution set, denoted by S. The practical characterization of the tolerance solution set is then presented in Theorem 4.1.

$$S = \left\{ \begin{array}{c} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n} \mid x_1 \le x_2 \\ \text{and } [x_1, x_2] \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \end{array} \right\}.$$

Theorem 4.1 ([9, 34])**.** *Let* I_{2n} *denote the* $2n \times 2n$ *identity matrix. For any vector* $y \in \mathbb{R}^n$, $y \in \sum_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *if and only if there exist* $(y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that* $y = y_1 - y_2$, *where* y_1 *and* y_2 *cannot simultaneously take positive values. Then,* (y_1, y_2) *satisfies the following system of linear inequalities:*

$$\begin{pmatrix} B \\ -I_{2n} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le \begin{pmatrix} b' \\ 0 \end{pmatrix},$$

where

$$B = \begin{pmatrix} A_c + \Delta & -(A_c - \Delta) \\ -(A_c - \Delta) & A_c + \Delta \end{pmatrix}$$

and

$$b' = \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix},$$

with $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$.

However, this theorem provides a practical description of the solution set without involving interval sets. In Theorem 4.2, the concept of Theorem 4.1 will be used to generate a subset of S.

Theorem 4.2 ([9]). *Let x_0 be prior known element of $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$, i.e., $x_0 \in \mathbf{x} \subseteq \Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$. Let $|B|$ represent the matrix obtained by taking the absolute value of each element of B individually. The notations in Theorem 4.1 remain valid. Define*

$$x_0 \in \sum_{\forall\exists}(\mathbf{A},\mathbf{b}), \quad b'' = b' - B\begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix},$$

where $x_0^+ = \frac{|x_0|+x_0}{2}$, $x_0^- = \frac{|x_0|-x_0}{2}$, and

$$S_1 = \left\{ \begin{matrix} X \mid X = \begin{pmatrix} x_0 - 2y_2 \\ x_0 + 2y_1 \end{pmatrix} \text{ such that} \\ \begin{pmatrix} B + |B| \\ -I_{2n} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} b'' \\ 0 \end{pmatrix} \end{matrix} \right\}.$$

Then, $S_1 \subseteq S$.

The interval tolerance solution of the system $\mathbf{A}x = \mathbf{b}$ can now be obtained using the above theorem. However, when the left-hand side is slightly lower than the right-hand side, we refer to the ILP problem as Problem ILP_1. As proven in Section 3, the feasible weak solution set of the system $\mathbf{A}x = \mathbf{b}$ is equivalent to the union of the basic optimal solutions of deterministic problems for an ILP_1, subject to $x \geq x_0$, for a given $x_0 > \vec{0}$. By this relationship, we can now use the concept of a tolerance solution for interval linear programming in an inequality system.

According to the objective of this paper, we aim to obtain an interval vector solution that exhibits tolerance behavior closely aligned with an optimistic solution. Initially, we must determine the optimistic solution to the problem. Should the optimistic solution fail to satisfy the tolerance conditions, the interval $[\underline{A}, \overline{A}]$ will be adjusted accordingly. In the next subsection, we present an approach to adjust an interval problem if the obtained solution is not a tolerance solution. We then generalize the method proposed in [9] to find an interval tolerance solution for the adjusted problem.

4.2 Left-hand side adjustment

The method for finding interval tolerance solutions from [9] is valid only when a tolerance solution exists for the given problem. However, we seek an interval vector solution that is close to the optimistic solution x^* and also captures the semantics of tolerance. If the optimistic solution x^* does not satisfy the tolerance criteria, for example, if x^* is a control, left-localized, or right-localized solution, we adjust the left-hand side of the problem to induce a tolerance behavior in x^*.

In this section, we introduce a new approach for modifying the left-hand side of the problem to ensure that optimistic solution x^* exhibits tolerance semantics with respect to the adjusted problem. Furthermore, this approach ensures that the left-hand side of the adjusted interval problem remains within the original problem's interval, while minimizing the total difference between their interval parameters $\mathbf{A}$.

4.2.1 The control optimistic solution

Consider the control solution $x^* \geq \vec{0}$. It is known that if x^* is a control solution for an ILP problem, then $\mathbf{b} \subseteq \mathbf{A}x^*$, shown in Fig. 2. Nevertheless, if we shrink the interval $\mathbf{A}x^*$ to $\mathbf{A}'x^* = [\underline{A}'x^*, \overline{A}'x^*]$, the system $\mathbf{A}'x^* = \mathbf{b}$ will exhibit tolerance behavior, shown in Fig. 3.

If x^* is a control solution of $\mathbf{A}x^* = \mathbf{b}$, then by solving Model (4.1), x^* is modified into a tolerance solution of $\mathbf{A}'x^* = \mathbf{b}$, minimizing the difference between the original ($\mathbf{A}$) and adjusted left-hand side inter-

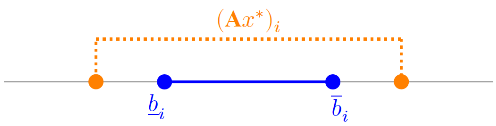

Fig. 2. $(\mathbf{A}x^*)_i \supseteq \mathbf{b}_i$, where $x^* \in \mathbb{R}^n$.

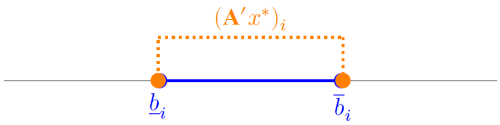

Fig. 3. $(\mathbf{A}'x^*)_i \subseteq \mathbf{b}_i$, where $x^* \in \mathbb{R}^n$.

val matrices $(\mathbf{A}')$. Let $\underline{a}'_{ij}$ and $\overline{a}'_{ij}$ be unknown variables, for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}(\underline{a}'_{ij} - \underline{a}_{ij}) + (\overline{a}_{ij} - \overline{a}'_{ij}) \\ \text{s.t.} \quad & \underline{A}'x^* = \underline{b}, \\ & \overline{A}'x^* = \overline{b}, \\ & \underline{A} \le \underline{A}', \\ & \overline{A}' \le \overline{A}. \end{aligned} \tag{4.1}$$

4.2.2 The left-localized optimistic solution

Consider the left-localized solution $x^* \geq \vec{0}$. It is known that if x^* is a left-localized solution to an ILP problem, then $\underline{A}x^* \leq \underline{b} \leq \overline{A}x^* \leq \overline{b}$, as illustrated in Fig. 4. However, by shifting $\underline{A}$ to $\underline{A}'$ while keeping $\overline{A}'$ fixed as $\overline{A}$, the system $\mathbf{A}'x^* = \mathbf{b}$ exhibits tolerance characteristics, with the total change from the original interval $\mathbf{A}$ to the new interval $\mathbf{A}'$ minimized. Fig. 5 shows the interval system $\mathbf{A}'x^*$ obtained by shifting the interval $\mathbf{A}$ according to the method described above.

If x^* is a left-localized solution of $\mathbf{A}x^* = \mathbf{b}$, then by solving Model (4.2), x^* is transformed into a tolerance solution

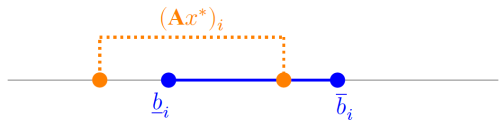

Fig. 4. $(\mathbf{A}x^*)_i \leq_{st} \mathbf{b}_i$, where $x^* \in \mathbb{R}^n$.

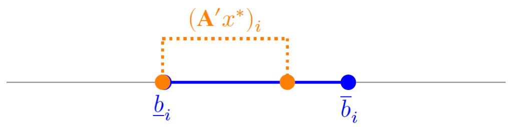

Fig. 5. Fixing $\overline{A}'$ as $\overline{A}$, $(\mathbf{A}'x^*)_i \subseteq \mathbf{b}_i$, where $x^* \in \mathbb{R}^n$.

of $\mathbf{A}'x^* = \mathbf{b}$, while minimizing the difference between the original $(\mathbf{A})$ and adjusted left-hand side $(\mathbf{A}')$. Let $\underline{a}'_{ij}$ and $\overline{a}'_{ij}$ be unknown variables, for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\underline{a}'_{ij} - \underline{a}_{ij} \\ \text{s.t.} \quad & \underline{A}'x^* = \underline{b}, \\ & \overline{A}' = \overline{A}, \\ & \underline{A} \le \underline{A}'. \end{aligned} \tag{4.2}$$

4.2.3 The right-localized optimistic solution

Consider the right-localized solution $x^* \geq \vec{0}$. It is known that if x^* is a right-localized solution for an ILP problem, then $\underline{b} \leq \underline{A}x^* \leq \overline{b} \leq \overline{A}x^*$, as illustrated in Fig. 6. Nevertheless, if we shift $\overline{A}$ to $\overline{A}'$ and fix $\underline{A}'$ as $\underline{A}$, the system $\mathbf{A}'x^* = \mathbf{b}$ will possess tolerance properties, while minimizing the total interval change from $\mathbf{A}$ to $\mathbf{A}'$. The interval system $\mathbf{A}'x^*$, shown in Fig. 7, is obtained by shifting the interval $\mathbf{A}$ in accordance with the method described above.

If x^* is a right-localized solution of $\mathbf{A}x^* = \mathbf{b}$, then by solving Model (4.3), x^* is modified into a tolerance solution of

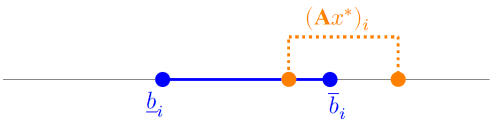

Fig. 6. $(-\mathbf{A}x^*)_i \leq_{st} (-\mathbf{b})_i$, where $x^* \in \mathbb{R}^n$.

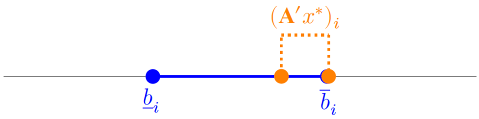

Fig. 7. Fixing $\underline{A}'$ as $\underline{A}$, $(\mathbf{A}'x^*)_i \subseteq \mathbf{b}_i$, where $x^* \in \mathbb{R}^n$.

$\mathbf{A}'x^* = \mathbf{b}$, minimizing the difference between the original ($\mathbf{A}$) and adjusted left-hand side ($\mathbf{A}'$). Let $\underline{a}'_{ij}$ and $\overline{a}'_{ij}$ be unknown variables, for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{a}_{ij} - \overline{a}'_{ij} \qquad (4.3) \\ \text{s.t.} \quad & \overline{A}' x^* = \overline{b}, \\ & \underline{A}' = \underline{A}, \\ & \overline{A}' \leq \overline{A}. \end{aligned}$$

4.3 Interval tolerance solution near optimistic solution

Steps of the approach: Find interval tolerance solution for an adjusted problem, related to the optimistic solution of the original problem. If $x > \vec{0}$ for all $x \in \Omega$, then

1. Find an optimistic solution of the original problem: in this step, we have to solve the best deterministic problem, $\min\{\underline{c}^{\mathrm{T}}x | \, \underline{A}x \leq \overline{b}\}$. The solution x^*, obtained from this problem is an optimistic solution of the problem.

2. Adjusting process: if x^* is a tolerance solution of $\mathbf{A}x^* = \mathbf{b}$, proceed to Step 3. Else x^* is not a tolerance solution, it must be adjusted to become the tolerance solution of a new system $\mathbf{A}'x^* = \mathbf{b}$. This new problem is obtained by adjusting the left-hand side of the original problem, following Subsection 4.2. Once x^* becomes a tolerance solution for the adjusted problem $\mathbf{A}'x^* = \mathbf{b}$, proceed to Step 3.

3. To obtain the interval tolerance solution that maximizes the total width of the interval, let M be a sufficiently large constant and w denote the width of the interval solution. We solve Model (4.4) to determine the unknown variables x_0, y_1, and y_2, which are then used to construct the interval tolerance solution. Let $J = \{1, 2, \dots, n\}$.

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} w_j \qquad (4.4) \\ \text{s.t.} \quad & (B + |B|)y \leq \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix} - B \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix} \\ & 2(y_1^j + y_2^j) \geq w_j, \; j \in J, \\ & M z_j \geq y_1^j, \; j \in J, \\ & M(1 - z_j) \geq y_2^j, \; j \in J, \\ & y_p^j \geq 0, \; p = 1, 2, \; j \in J, \\ & x_0^+, \; x_0^- \geq \vec{0}, \\ & z_j \in \{1, 2\}, \; j \in J, \\ & w_j \geq 0, \; j \in J, \end{aligned}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2n}, \; y_p = \begin{pmatrix} y_p^1 \\ y_p^2 \\ \vdots \\ y_p^n \end{pmatrix},$$

for all $p = 1, 2$, and

$$B = \begin{pmatrix} A_c + \Delta & -(A_c - \Delta) \\ -(A_c - \Delta) & A_c + \Delta \end{pmatrix}.$$

By Theorem 4.2, an interval tolerance solution can be expressed as

$$\mathbf{x} = \begin{pmatrix} x_0 - 2y^2 \\ x_0 + 2y^1 \end{pmatrix}.$$

Unlike the approach in [9], the variable x_0 in Model (4.4) does not require prior specification. Instead, an optimal x_0 that maximizes the total distance of x_i for all $i = 1, \dots, n$ can be determined by solving this model. Thus, after this step, we obtain an interval tolerance solution for $\mathbf{A}'x = \mathbf{b}$, corresponding to the optimistic solution of the original problem.

Furthermore, a sensitivity analysis approach can be applied to directly examine how much the input intervals, represented as deterministic values in B, $\underline{b}$ and $\overline{b}$ in Model 4.4, may vary without altering the interval tolerance solution.

5. Numerical examples

In this section, two numerical examples will be presented to illustrate the usage of the proposed approach.

Example 5.1. Based on Problem (1.1) shown in Section 1, determine an interval tolerance solution that approximates the optimistic solution of the problem.

Since $x > \vec{0}$ for all $x \in \Omega$, our proposed approach can be applied to the ILP problem. Consider the best sub-model:

$$\begin{aligned} \max \quad & 3.5\, x_1 - 1\, x_2 \\ \text{s.t.} \quad & x_1 + 1.6\, x_2 \leq 12, \\ & 3x_1 - 3\, x_2 \leq 7, \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

The optimal solution is $x^* = (x_1^*, x_2^*) = \left(\frac{236}{39}, \frac{145}{39}\right)$. Then $\mathbf{A}_i x^*$ for each $i = 1, 2$ can be expressed as the following:
$\mathbf{A}_1 x^* = [12.000, 13.348]$, then x^* is a right-localized solution to the first constraint of (1.1),
$\mathbf{A}_2 x^* = [7.000, 16.769]$, then x^* is also a right-localized solution to the second constraint.
Since x^* is a right-localized solution, we need to adjust $\mathbf{A}$ to $\mathbf{A}'$. Thus, x^* becomes a tolerance solution to the system $\mathbf{A}'x^* = \mathbf{b}$. Therefore, the adjusted problem (5.1) can be rewritten as follows, ensuring that x^* is a tolerance solution to problem.

$$\begin{aligned} \max \quad & 3.5\, x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 1.6\, x_2 = [11.6, 12], \\ & 3\, x_1 - 3\, x_2 = [5, 7], \\ & x_1,\ x_2 \geq 0. \end{aligned} \tag{5.1}$$

Some deterministic coefficients in the problem can be considered as degenerate interval values. That is, a deterministic value (e.g., 5) can be represented as an interval with zero width, such as $[5, 5]$. Therefore, it is not necessary for all input data to be expressed as intervals.

Next, we determine an interval tolerance solution for Problem (5.1) using the idea in Step 3. By solving Model (4.4) with $M = 100$, the unknown variables $x_0 = (x_{0_1},\ x_{0_2}) = (5.4872,\ 3.8205)$, $w = (0.4,\ 0)$, $y_1 = (0.2,\ 0)$ and $y_2 = (0,\ 0)$. By Theorem 4.2, the interval tolerance solution to Problem (5.1) is given by $\mathbf{x}_1 = [5.4872, 5.8872]$ and $x_2 = 3.8205$. As shown in Fig. 8, the obtained solution is represented by a blue line within the green area, where the green area corresponds to the weak and tolerance solution set of the adjusted problem (5.1). By adding a constraint that the width of $\mathbf{x}_1$ is twice the width of $\mathbf{x}_2$, we obtained an interval solution given by $\mathbf{x}_1 = [5.56, 5.78]$ and $\mathbf{x}_2 = [3.78, 3.89]$, as shown by the red box in Fig. 8.

In addition, Fig. 8 shows that the weak solution set of the adjusted problem (5.1) is smaller compared to the weak solu-

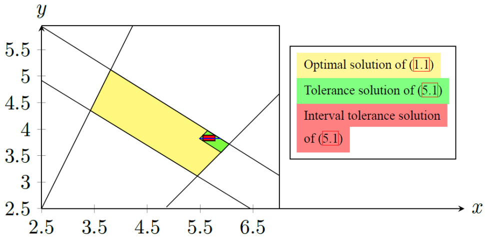

Fig. 8. The interval tolerance solution of an adjusted Problem (5.1)

tion or optimal solution set of the original problem (1.1). Narrowing the set of weak solutions can help eliminate solutions that are only feasible under very specific or unrealistic parameter realizations, potentially leading to reduced complexity and faster convergence in certain algorithms.

Example 5.2. Diet problem.
A farmer formulates two different chicken feed blends, Formula I and Formula II, each with distinct nutrient compositions. The nutrient content (measured in kilograms per 100 kilograms of chicken feed) required for laying hens is outlined in Table 1, as referenced in [14, 35]. Given that the production cost per unit of Formula I is approximately 5 to 6 times higher than that of Formula II, the farmer aims to mix these formulas while ensuring that the quantity of Formula II is at least 5 times that of Formula I, resulting in a total of 100 kg of feed. This scenario leads to the formulation of an interval linear programming problem with the objective of cost minimization, as follows.

$$\begin{aligned}
\min\ & [5,6]\ x_1 + \ x_2 \qquad (5.2)\\
\text{s.t. } & [0.4921, 0.50]\ x_1 + [0.1072, 0.11]\ x_2 \\
& \geq [16, 16.227],\\
& [0.007, 0.007]\ x_1 + [0.004, 0.004]\ x_2 \\
& \geq [0.35, 0.442924],\\
& [0.0098, 0.0105]\ x_1 + [0.0074, 0.008]\ x_2 \\
& \geq [0.75, 0.7743392],\\
& [0.0577, 0.058]\ x_1 + [0.0316, 0.032]\ x_2 \\
& \geq [3.5, 3.5334388],\\
& [0.008, 0.008]\ x_1 + [0.0033, 0.0033]\ x_2 \\
& \geq [0.35, 0.3972476],\\
& 0.04\ x_1 + 0.04\ x_2 \geq 4,\\
& x_1 + x_2 = 100,\\
& x_2 \geq 5\ x_1,\\
& x_1,\ x_2 \geq 1.
\end{aligned}$$

Since $x_1, x_2 \geq 1$, then $x > \vec{0}$ for all $x \in \Omega$, which means our proposed approach is applicable to the ILP problem. Consider the best sub-model:

$$\begin{aligned}
\min\ & 5\ x_1 + \ x_2\\
\text{s.t. } & 0.5x_1 + 0.11x_2 \geq 16,\\
& 0.007\ x_1 + 0.004\ x_2 \geq 0.35,\\
& 0.0105\ x_1 + 0.008\ x_2 \geq 0.75,\\
& 0.058\ x_1 + 0.032\ x_2 \geq 3.5,\\
& 0.008\ x_1 + 0.0033\ x_2 \geq 0.35,\\
& 0.04\ x_1 + 0.04\ x_2 \geq 4,\\
& x_1 + x_2 = 100,\\
& x_2 \geq 5\ x_1,\\
& x_1,\ x_2 \geq 1.
\end{aligned}$$

The optimal solution is $x^* = (x_1^*, x_2^*) = (12.821, 87.179)$. Accordingly, for each $i = 1, \ldots, 6$, $\mathbf{A}_i x^*$ is given by:

$$\begin{aligned}
\mathbf{A}_1 x^* &= [15.6548, 16.0002],\\
\mathbf{A}_2 x^* &= [0.4385, 0.4385],\\
\mathbf{A}_3 x^* &= [0.7707, 0.8321],\\
\mathbf{A}_4 x^* &= [3.4946, 3.5333],\\
\mathbf{A}_5 x^* &= [0.3962, 0.3902],\\
\mathbf{A}_6 x^* &= 4.
\end{aligned}$$

Thus, x^* satisfies the first to sixth constraints of Problem (5.2) as a left-localized, tolerance, right-localized, left-localized, tolerance, and tolerance solution, respectively.

By applying the approach from Subsection 4.2, we can adjust $\mathbf{A}$ to $\mathbf{A}'$ such that x^* becomes a tolerance solution for the system $\mathbf{A}'x^* = \mathbf{b}$. Therefore, the adjusted problem (5.3) can be rewritten as follows, ensuring that x^* is a tolerance solution to the problem.

$$\begin{aligned} \min\ & [5,6]\ x_1 + \ x_2 \qquad (5.3) \\ \text{s.t.}\ & 0.50\ x_1 + 0.11\ x_2 = [16, 16.227], \\ & 0.007\ x_1 + 0.004\ x_2 = [0.35, 0.443], \\ & [0.0098, 0.010052]\ x_1 + 0.0074\ x_2 \\ & \ = [0.75, 0.774], \\ & [0.0577, 0.058]\ x_1 + [0.03167, 0.032]\ x_2 \\ & \ = [3.5, 3.534], \\ & 0.008\ x_1 + 0.0033\ x_2 \\ & \ = [0.35, 0.397], \\ & 0.04\ x_1 + 0.04\ x_2 = 4, \\ & x_1 + x_2 = 100, \\ & x_2 \geq 5\ x_1, \\ & x_1,\ x_2 \geq 1. \end{aligned}$$

In Problem (5.3), the nutrient contents, protein, methionine, lysine, calcium, phosphorus, and fat of Formula I and II are adjusted to determine the optimal mix of chicken feed that minimizes cost as shown in Table 1, while also ensuring that the chickens receive all required nutrients within the specified intervals. After applying Model (4.4) with $M = 100$, to solve Problem (5.3), the decision variables $x_0 = (12.821,\ 87.179), w = (0,\ 0), y_1 = (0,\ 0)$ and $y_2 = (0,\ 0)$. By Theorem 4.2, we obtain an interval tolerance solution with zero width $(x_1,\ x_2) = (12.821,\ 87.179)$.

However, we found that adjusting the nutritional concentrations in the formulas may reduce the flexibility of the data. Nevertheless, this adjustment allows us to obtain a tolerance solution that is close to the optimistic solution, meaning that the resulting feed mix ensures an appropriate cost that is near the minimum cost of the original problem, while still satisfying the required nutrient levels as discussed in [35].

6. Conclusions

Under the assumption that the ILP problem is considered as Problem ILP_1, we initially analyzed the equivalence between the feasible weak solution set of the interval linear equation system and the union of basic feasible solutions obtained from corresponding deterministic problems of ILP_1. This analysis further establishes that the feasible weak solution set is equivalent to the union of basic optimal solutions for an ILP_1 with $x \geq x_0$, where $x_0 > \vec{0}$. Building on these findings, we determined an interval tolerance vector solution by adapting the method from [9] to obtain a solution that remains close to an optimistic ILP solution while maintaining the tolerance property. If the optimistic solution is not a tolerance solution of the ILP problem, an adjustment to the left-hand side matrix $\mathbf{A}$ is implemented. However, unlike the method from [9], our proposed approach does not require specifying x_0 in advance. Instead, an optimal x_0 that maximizes the total width of x_i for all $i = 1, \ldots, n$ can be determined by solving Model (4.4). Although adjusting the left-hand side of the problem may reduce data flexibility, it enables us to obtain an interval vector solution that is close to the optimistic solution while still preserving tolerance behavior. Apart from the diet problem, the proposed technique can be applied to various areas, such as supply chain management, energy planning, and healthcare, where solution strategies often involve optimistic or pessimistic viewpoints, and constraints naturally align with tolerance semantics.

However, tolerance semantics alone may be insufficient to fully characterize solutions in real-world applications. For ex-

Table 1. Typical nutrition concentrations for in-production laying hens (kilograms per 100 kilograms of chicken food) and available nutrients in chicken formulas I and II (kilograms per 100 kilograms of chicken food).

	Protein	Methionine	Lysine	Calcium	Phosphorous	Fat
Formula I	[49.21, 50.00]	[0.70, 0.70]	[0.98, 1.05]	[5.77, 5.80]	[0.80, 0.80]	4
Formula II	[10.72, 11.00]	[0.40, 0.40]	[0.74, 0.80]	[3.16, 3.20]	[0.33, 0.33]	4
Adjusted formula I	[50, 50]	[0.70, 0.70]	[0.9800, 1.0052]	[5.77, 5.80]	[0.80, 0.80]	4
Adjusted formula II	[11, 11]	[0.40, 0.40]	[0.74, 0.74]	[3.167, 3.200]	[0.33, 0.33]	4
Typical Nutrition Concentrations for production-laying hen	[1600.0, 1622.7]	[35.0, 44.3]	[75.0, 77.4]	[350, 354]	[35.0, 39.7]	400

ample, in the course assignment problem, the number of assigned courses can exceed the requirement but must not fall below it. In such cases, a right-localized solution may be more appropriate. Therefore, future research will focus on developing approaches for obtaining interval vectors that capture specific semantic interpretations, such as control, left-localized, and right-localized solutions.

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