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Original research article

Perimetric Contraction on Quadrilaterals

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ABSTRACT

In this article, we introduce a four point analogue of the Banach contraction principle and establish sufficient conditions for such mappings to possess fixed point(s) in complete metric spaces. Notably, the classical Banach contraction principle emerges as a special case of our results. We present several non-trivial examples that not only validate our theorems but also reveal that quadrilateral perimetric contractions need not imply other well-known contraction types. Furthermore, we extend our analysis to obtain fixed point theorems in non-complete metric spaces. Lastly, we address a recent result linking mappings contracting the perimeters of triangles in metric spaces to Banach-type contractions in G-metric spaces.

Keywords: Fixed point theorems; Mapping contracting perimeter of triangles; Perimetric contraction on quadrilaterals

1. Introduction and Preliminaries

Fixed point theory plays a crucial role in mathematics, where many problems can be framed as fixed point problems. These problems involve investigating the existence and uniqueness of solutions. Applications of fixed point theory span diverse areas, including matrix equations, differential equations, integral equations, optimization, and machine learning.

The foundational work in this field dates back to Stefan Banach's introduction of the Banach contraction principle [1] in 1922. This principle guarantees the existence and uniqueness of fixed points of contraction mappings in a complete metric space. Subsequently, other prominent researchers contributed significantly to the evolution of fixed point theory. As a result, the concept of Banach contraction has been extended in various ways by relaxing the contraction condition and considering different topologies.

There are various classical results in the literature of fixed point theory. These results generalize Banach's theorem in var-

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ious ways. Nadler [2] extended Banach's theorem from single-valued mappings to multi-valued mappings. Kirk [3] has proposed the nonexpansive mapping type extension of the Banach contraction principle. Berinde [4] has introduced the enriched contractions generalizing the contraction mappings in normed linear spaces. Generalizing the underlying space, Browder [5] has initiated the fixed point result in topological vector spaces. Wardowski [6] has disclosed F-contractions using implicit functions extending the contraction mappings. Khojasteh et al. induced the idea of Z-contraction mappings by utilizing simulation functions, see [7, 8]. Kannan [9] obtained a fixed point result for a class of mappings which characterizes the completeness of the metric space. Chatterjea [10] proposed a class of mappings independent of Banach's class and identified the prerequisites for reaching fixed point. There are certain fixed point outcomes in generalized metric spaces (See [11–13]).

Recently in 2023, Petrov [14] introduced the notion of a class of mappings that can be distinguished as the mapping that contracts the perimeters of triangles and proved a fixed point result. Let us recall that.

Definition 1.1 ([14]). Let (Y, ρ) be a metric space with at least three points. Then the mapping $T : Y \to Y$ is defined as contracting perimeters of triangles if there is an $\alpha \in [0, 1)$ such that

$$\rho(Tp, Tq) + \rho(Tq, Tr) + \rho(Tr, Tp) \leq \alpha(\rho(p, q) + \rho(q, r) + \rho(r, p)), \tag{1.1}$$

for three pairwise distinct points $p, q, r \in Y$.

These mappings attain fixed points in a complete metric space if and only if it has no periodic points of order 2. There are at most two fixed points.

In 2024, Petrov along with Bisht introduced the three-point analogue of Kannan type mappings utilizing the notion of mapping contracting perimeters of triangles and developed fixed point results.

Definition 1.2 ([15]). "Let (Y, ρ) be a metric space with at least three points. Then $T : Y \to Y$ is said to be a generalized Kannan type mapping if there is a $0 \leq \delta < \frac{2}{3}$ such that

$$\rho(Tp, Tq) + \rho(Tq, Tr) + \rho(Tr, Tp) \leq \delta(\rho(p, Tp) + \rho(q, Tq) + \rho(r, Tr)), \tag{1.2}$$

for any three pairwise distinct points $p, q, r \in Y$".

Thereafter, C.M. Păcurar and O. Popescu introduced the three point analogue of the Chatterjea type mappings and proved a fixed point result.

Definition 1.3 ([16]). "Let (Y, ρ) be a metric space with at least three points. Then $T : Y \to Y$ is a generalized Chatterjea type mapping if there is $\lambda \in [0, \frac{1}{2})$ such that

$$\rho(Tp, Tq) + \rho(Tq, Tr) + \rho(Tr, Tp) \leq \lambda(\rho(p, Tq) + \rho(p, Tr) + \rho(q, Tp) + \rho(q, Tr) + \rho(r, Tp) + \rho(r, Tq)), \tag{1.3}$$

for any three pairwise distinct points $p, q, r \in Y$".

In a complete metric space, both a generalized Kannan type mapping and a generalized Chatterjea type mapping attain fixed points if they do not achieve periodic points of order 2. There are at most two fixed points in both the cases.

E. Karapınar [17] recently showed that the concept of mappings contracting the perimeters of triangles in metric spaces is equivalent to a variant of the Banach contraction in the context of G-metric spaces. The author also raised concerns about the novelty of fixed-point results derived from such contractions. However, this result contains a gap, as illustrated in the following remark:

Remark 1.4. *It has been concluded in ([17, p. 5, Theorem 3.1]) that the mapping T admits a unique fixed point. However, since T satisfies $G(Tx, Ty, Tz) \leq kG(x, y, z)$, $k \in [0, 1)$ for all pairwise distinct points of X, this does not imply the uniqueness of the fixed point of T. In addition, the absence of periodic points of order 2 is not considered here, which is necessary. For reference, see the following examples:*

Example 1.5. Let $X = \{p, q, r\}$ be a metric space endowed with the discrete metric d. Now, we define $G : X \times X \times X \to \mathbb{R}$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$. Define $T : X \to X$ be such that $Tp = p, Tq = q$, and $Tr = p$. Then $G(Tx, Ty, Tz) \leq k\ G(x, y, z)$, for all pairwise distinct points of X, where $k \in [0, 1)$. In this case, T consists of two fixed points, namely, p and q.

Example 1.6. Let $X = \{a, b, c\}$ be a metric space endowed with the discrete metric d. Now, we define $G : X \times X \times X \to \mathbb{R}$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$. Define $T : X \to X$ be such that $Ta = b, Tb = a$ and $Tc = a$. Then $G(Tx, Ty, Tz) \leq kG(x, y, z)$, for all pairwise distinct points of X, where $k \in [0, 1)$. Here, the points a and b are periodic points of order 2. However, T does not contain any fixed point.

In a G-metric space, where the domain of G is defined by triplets, it becomes possible to derive a modified form of the three-point extension of Banach's result. However, it is not feasible to construct a four-point extension of any result within the same framework of G-metric spaces.

In this article, we would like to study the four-point analogue of Banach's result. Our goal is to establish adequate conditions ensuring the existence and uniqueness of fixed points. Additionally, we aim to compare this class of mappings with the previous classes in the literature and uncover any relationship between them.

We explore a new kind of mapping in the following section, which is defined as mapping that reduces the perimeter of quadrilaterals. We establish a fixed point result for this type of mapping in a complete metric space. Remarkably, contraction mappings are a subset of these perimeter-based mappings. We obtain Banach's fixed point theorem as a simple consequence. To validate our results, we provide a few illustrative examples.

Throughout the paper, we denote by (M, d) a metric space, $|M|$ the cardinality of the set M and $Fix(T)$ the collection of all fixed points of the mapping $T : M \to M$.

Let T be a mapping on the metric space M. A point $m \in M$ is said to be a *periodic point of order p* (or prime period p) if p is the least positive integer for which $T^p m = m$.

2. Perimetric contraction on quadrilaterals

We begin the section with the following definition:

Definition 2.1. Let (M, d) be a metric space with at least four points. Then the mapping $T : M \to M$ is said to be a perimetric contraction on quadrilaterals on M if there is an $\alpha \in [0, 1)$ such that

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp)$$

$$\leq \alpha(d(p,q) + d(q,r) + d(r,s) + d(s,p)), \tag{2.1}$$

for all distinct points $p, q, r, s \in M$.

Remark 2.2. *Perimetric contraction on quadrilaterals does not attain periodic points of order 4. Otherwise, if p is a periodic point of order 4, then $Tp = q, Tq = r, Tr = s, Ts = p$, where p, q, r, s are all distinct. Then we have,*

$$d(Tp,Tq) + d(Tq,Tr) + d(Tr,Ts) + d(Ts,Tp) = d(p,q) + d(q,r) + d(r,s) + d(s,p),$$

which contradicts Eq. (2.1).

Now, we investigate the continuity of these mappings.

Theorem 2.3. *A perimetric contraction on quadrilaterals is continuous.*

Proof. Let (M,d) be a metric space with at least four points, and let $T : M \to M$ be a perimetric contraction on quadrilaterals on M. Let $m' \in M$ be arbitrary. If m' is an isolated point of M, then there is nothing to prove. Now, suppose that m' is a limit point of M. Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ be such that $0 < \delta < \frac{\varepsilon}{6\alpha}$.

Since m' is a limit point of M, there exist $a, b \in M$ with $a \neq b \neq m'$ such that $d(m',a) < \delta$ and $d(m',b) < \delta$. Now, for all $m \in M$ with $m \neq m'$ satisfying $d(m,m') < \delta$, we have

$$\begin{aligned} d(Tm,Tm') &\leq d(Tm,Tm') + d(Tm',Ta) + d(Ta,Tb) + d(Tb,Tm) \\ &\leq \alpha(d(m,m') + d(m',a) + d(a,b) + d(b,m)) \\ &\leq 2\alpha(d(m,m') + d(m',a) + d(m',b)) \\ &< 6\alpha\delta \\ &< \varepsilon, \end{aligned}$$

and hence the result follows. □

Now, we are ready to establish a condition that is sufficient for the existence of fixed point(s) for perimetric contraction on quadrilaterals.

Theorem 2.4. *Let us suppose a complete metric space (M,d) with at least four points. Consider a mapping $T : M \to M$ to be a perimetric contraction on quadrilaterals on M. Then, T attains a fixed point in M if it does not attain periodic points of order 2 and 3. Furthermore, T can attain at most three fixed points.*

Proof. Let $T : M \to M$ be a perimetric contraction on quadrilaterals on M and let T does not attain periodic points of order 2 and 3.

Let $a_0 \in M$ be chosen arbitrarily. Define $Ta_0 = a_1, Ta_1 = a_2, \cdots, Ta_n = a_{n+1}, \cdots$. If a_n is a fixed point of T for any n, then we are done.

Now, assume that a_n is not a fixed point of T for all n. Since a_n is not a fixed point of T, it follows that $a_0 \neq a_1, a_1 \neq a_2$ and so on. Since T does not attain periodic points of order 2, then $a_0 \neq a_2, a_1 \neq a_3$ and so on. Again, since T does not attain periodic points of order 3, we have $a_0 \neq a_3$ and so on. Therefore, any four consecutive elements of $\{a_n\}$ are distinct.

Let $\lambda_n = d(a_n,a_{n+1}) + d(a_{n+1},a_{n+2}) + d(a_{n+2},a_{n+3}) + d(a_{n+3},a_n)$ for all $n \in \mathbb{N} \cup \{0\}$ so that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, for any $\mathbb{N} \cup \{0\}$ by $\mathbb{N}$, we have $\lambda_n \leq \alpha\lambda_{n-1}$. Then, it is clear that

$$\begin{aligned} d(a_0,a_1) &\leq \lambda_0, \\ d(a_1,a_2) &\leq \lambda_1 \leq \alpha\lambda_0, \\ &\vdots \end{aligned}$$

$$d(a_n, a_{n+1}) \leq \lambda_n \leq \alpha^n \lambda_0. \qquad (2.2)$$

Now, for all $n \in \mathbb{N} \cup \{0\}$ and for any $p = 1, 2, 3, \cdots$, we have

$$\begin{aligned} d(a_n, a_{n+p}) &\leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \\ &\quad \cdots + d(a_{n+p-1}, a_{n+p}) \\ &\leq \alpha^n \lambda_0 + \alpha^{n+1} \lambda_0 + \cdots + \\ &\quad \alpha^{n+p-1} \lambda_0 \\ &\leq \alpha^n (1 + \alpha + \cdots + \alpha^{p-1}) \lambda_0 \\ &\leq \alpha^n \frac{1 - \alpha^p}{1 - \alpha} \lambda_0. \end{aligned}$$

This implies that $d(a_n, a_{n+p}) \to 0$ as $n \to \infty$ and for any $p = 1, 2, 3, \cdots$. Hence, $\{a_n\}$ is a Cauchy sequence in M and therefore convergent, as M is complete. Let $a_n \to a^* \in M$. Then, by the continuity of T, $a^* \in Fix(T)$.

Let us suppose that T has four distinct fixed points, say, p, q, r, s. Then

$$\begin{aligned} &d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ &\leq \alpha(d(p, q) + d(q, r) + d(r, s) + d(s, p)), \end{aligned}$$

which again implies that $\alpha \geq 1$ - a contradiction to Eq. (2.1). Hence, the result follows. □

In relation to the converse of the preceding theorem, we derive the following remark.

Remark 2.5. *A perimetric contraction on quadrilaterals cannot simultaneously admit periodic points of prime periods* 2 *and* 3*, irrespective of the existence of fixed points.*

Below, we present the following examples in support of Theorem 2.4. The first is an example of a mapping contracting perimeter of quadrilaterals with exactly three fixed points.

Example 2.6. Let (M, d) be a metric space where $M = \{w, x, y, z\}$ and let the metric d be defined on M as $d(x, y) = d(x, z) = d(y, z) = 1$ and $d(w, x) = d(w, y) = d(w, z) = 2$. Let $T : M \to M$ be defined as $Tw = x, Tx = x, Ty = y, Tz = z$. Then, T is a perimetric contraction on quadrilaterals on M. Note that T does not contain periodic points of order 2 and 3. Thus, Theorem 2.4 guarantees that T has a fixed point. Clearly $Fix(T) = \{x, y, z\}$.

Next, we provide examples to show that neither of the conditions T has periodic points of order 2 nor periodic points of order 3 can be dropped for the existence of fixed points.

Example 2.7. Let $M = \{a, b, c, d\}$ be a metric space endowed with the discrete metric d. Let $T : M \to M$ be defined by $Ta = c, Tb = c, Tc = b, Td = b$. Then T is a perimetric contraction on quadrilaterals on M. But, since b and c are periodic points of order 2, therefore by Theorem 2.4, $Fix(T) = \phi$.

Example 2.8. Let (M, d) be a metric space with $M = \{p, q, r, s\}$ and the metric δ is defined by $\delta(q, r) = \delta(q, s) = \delta(r, s) = 1$ and $\delta(p, q) = \delta(p, r) = \delta(p, s) = 2$. Let $T : M \to M$ be defined by $Tp = r, Tq = r, Tr = s, Ts = q$. Then T is a perimetric contraction on quadrilaterals on M. But, since q, r, s are periodic points of order 3, it follows from Theorem 2.4, $Fix(T) = \phi$.

From Example 2.6, we observe that a mapping contracting the perimeter of quadrilaterals may have multiple fixed points. On the other hand, we can create a situation where it is guaranteed that there will be a unique fixed point for this kind of mappings.

Proposition 2.9. *Let (M, d) be a complete metric space and let $T : M \to M$ be a perimetric contraction on quadrilaterals on M*

and let T have no periodic points of order 2 *and* 3. *If M contains infinitely many points such that the iterative sequence $m_0, m_1 = Tm_0, m_2 = Tm_1, \ldots,$ converges to a point $\xi \in M$ with $\xi \neq m_i$; for all $i \in \mathbb{N} \cup \{0\}$, then $Fix(T) = \{\xi\}$.*

Proof. That ξ is a fixed point of T follows from Theorem 2.4. Let η be another fixed point of T. Then $\eta \neq m_i$, for all $i \in \mathbb{N} \cup \{0\}$, otherwise we have $\xi = \eta$. Therefore ξ, η, and m_i are all distinct, for all $i \in \mathbb{N} \cup \{0\}$. Now, for all $i \in \mathbb{N} \cup \{0\}$ by $\mathbb{N}$

$$K_i = \frac{d(T\xi, T\eta) + d(T\eta, Tm_{i-1}) + d(Tm_{i-1}, Tm_i) + d(Tm_i, T\xi)}{d(\xi, \eta) + d(\eta, m_{i-1}) + d(m_{i-1}, m_i) + d(m_i, \xi)} = \frac{d(\xi, \eta) + d(\eta, m_i) + d(m_i, m_{i+1}) + d(m_{i+1}, \xi)}{d(\xi, \eta) + d(\eta, m_{i-1}) + d(m_{i-1}, m_i) + d(m_i, \xi)}.$$

Then $K_i \leq \alpha$ for all $i \in \mathbb{N} \cup \{0\}$ by $\mathbb{N}$. Now, letting $i \to \infty$, we get $K_i \to 1$ - which is a contradiction to Eq. (2.1). Hence, $Fix(T) = \{\xi\}$. □

Now, we provide an alternative proof of the Banach Contraction Principle using Theorem 2.4.

Corollary 2.10. *Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping with contraction constant $\alpha \in [0, 1)$. Then T has a unique fixed point in M.*

Proof. If $|M| = 1$ or $|M| = 2$, then there is nothing to prove.

Now, let $|M| = 3$. If $Fix(T) = \phi$, then it is easy to verify that there exists $m \in M$ such that $T^2m = m$ or $T^3m = m$.

If there exists $m \in M$ such that $T^2m = m$, then

$$d(Tm, T^2m) = d(Tm, m) = d(m, Tm),$$

which contradicts the fact that $\alpha \in [0, 1)$.

Now, if there exists $m \in M$ such that $T^3m = m$, then

$$d(m, Tm) = d(T^3m, T^4m) \leq \alpha^3 d(m, Tm),$$

again a contradiction to the fact that $\alpha \in [0, 1)$. Thus, $Fix(T) \neq \phi$.

Furthermore, it is evident that, in general, there is no element $m \in M$ such that $T^2m = m$ and $T^3m = m$.

Now, let $|M| \geq 4$. Since, there exists no $m \in M$ such that $T^2m = m$ and $T^3m = m$ then, T has no periodic points of order 2 and 3.

Now, for all distinct points $p, q, r, s \in M$, we have

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \leq \alpha(d(p, q) + d(q, r) + d(r, s) + d(s, p)).$$

Thus, T is a perimetric contraction on quadrilaterals on M. Hence, by Theorem 2.4, it follows that T admits at most three fixed points in M. The contraction condition confirms the uniqueness of the fixed point. □

We present the following examples demonstrating the existence of perimetric contractions on quadrilaterals. These examples do not fall under mappings that contract the perimeter of triangles, nor do they align with Kannan type, Chatterjea type, generalized Kannan type, or generalized Chatterjea type mappings.

Example 2.11. Let (M, d) be a metric space with $M = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ where d is the Euclidean metric.

Now, define the mapping T as follows:

$$T(m) = \begin{cases} 0, & \text{if } m = \{0, \frac{1}{3}\}, \\ \frac{1}{3}, & \text{if } m = \frac{2}{3}, \\ \frac{2}{3}, & \text{if } m = 1. \end{cases}$$

For, $a = \frac{2}{3}, b = 1$, we have,

$$d(Ta, Tb) = \frac{1}{2}(d(a, Ta) + d(b, Tb)),$$

and hence T is not a Kannan type mapping.

For, $a = \frac{2}{3}, b = 1$, we have,

$$d(Ta, Tb) = \frac{1}{2}(d(a, Tb) + d(b, Ta)),$$

and hence T is not a Chatterjea type mapping.

Again, for $a = \frac{1}{3}, b = \frac{2}{3}, c = 1$, we have

$$d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) = \frac{4}{3} = d(a, b) + d(b, c) + d(c, a),$$

and therefore, T is not a mapping contracting perimeters of triangles.

Again, for $a = 0, b = \frac{2}{3}, c = 1$, we have

$$d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) = \frac{4}{3} = d(a, Ta) + d(b, Tb) + d(c, Tc),$$

and therefore, T is not a generalized Kannan type mapping.

Again, for $a = \frac{1}{3}, b = \frac{2}{3}, c = 1$, we have

$$d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) = \frac{4}{3} = d(a, b) + d(b, c) + d(c, a)$$

and therefore, T is not a generalized Chatterjea type mapping.

Now, for any four distinct points of M, the condition (2.1) holds with $\alpha \in \left[\frac{2}{3}, 1\right)$. As a result, T is a perimetric contraction on quadrilaterals on M. Note that T has no periodic points of order 2 and 3. Thus, Theorem 2.4 implies that $Fix(T) \neq \phi$. Note that $Fix(T) = \{0\}$.

In the next example, we consider a countably infinite metric space.

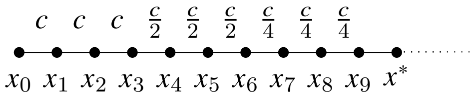

Fig. 1. The points in the space (M, d) that are separated by the consecutive lengths.

Example 2.12. Let $M = \{x^*, x_0, x_1, \dots\}$ with cardinality $\aleph_0$ and let $c \in \mathbb{R}^+$. The metric d is defined on M as follows:

$$d(x, y) = \begin{cases} \frac{c}{2^{[\frac{n}{3}]}}, & \text{if } x = x_n, y = x_{n+1}, \\ & n = 0, 1, \dots, \\ \sum_{a=n}^{m-1} \frac{c}{2^{[\frac{a}{3}]}}, & \text{if } x = x_n, y = x_m, n+1 \\ & < m, m, n = 0, 1, \dots, \\ 6c - \sum_{a=0}^{n-1} \frac{c}{2^{[\frac{a}{3}]}}, & \text{if } x = x_n, y = x^*, \\ & n = 0, 1, \dots, \\ 0, & \text{if } d(y, x), \forall x, y. \end{cases}$$

where $[\cdot]$ is the box function. Then, (M, d) is a complete metric space.

Define a mapping $T : M \to M$ as $Tx_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $Tx^* = x^*$.

Since $d(Tx_{3n}, Tx_{3n+1}) = d(x_{3n}, x_{3n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, then T is not a contraction.

Again, $d(Tx_0, Tx_1) = \frac{1}{2}(d(x_0, Tx_0) + d(x_1, Tx_1))$ and hence T is not a Kannan type mapping.

Again, $d(Tx_0, Tx_1) = \frac{1}{2}(d(x_0, Tx_1) + d(x_1, Tx_0))$ and hence T is not a Chatterjea type mapping.

Also, for all $n \in \mathbb{N} \cup \{0\}$,

$$d(Tx_{3n}, Tx_{3n+1}) + d(Tx_{3n+1}, Tx_{3n+2}) + d(Tx_{3n+2}, Tx_{3n}) = d(x_{3n}, x_{3n+1}) + d(x_{3n+1}, x_{3n+2}) + d(x_{3n+2}, x_{3n}).$$

Therefore, T fails to be a mapping contracting perimeters of triangles.

Also,

$$d(Tx_0, Tx_1) + d(Tx_1, Tx_2) + d(Tx_2, Tx_0) = \frac{4}{3}(d(x_0, Tx_0) + d(x_1, Tx_1) + d(x_2, Tx_2)).$$

Thus, T is not a generalized Kannan type mapping.

Again,

$$d(Tx_0,Tx_1)+d(Tx_1,Tx_2)+d(Tx_2,Tx_0)$$
$$=\frac{1}{2}(d(x_0,Tx_1)+d(x_0,Tx_2)+d(x_1,Tx_0)$$
$$+d(x_1,Tx_2)+d(x_2,Tx_0)+d(x_2,Tx_1)).$$

Thus, T is not a generalized Chatterjea type mapping.

We now show that T is a perimetric contraction on quadrilaterals on M.

Consider the points $x_k,x_l,x_m,x^* \in M$ with $0\le k<l<m$. Then, we have

$$d(x_k,x_l)+d(x_l,x_m)+d(x_m,x^*)+d(x^*,x_k)$$
$$=2d(x_k,x^*)=12c-2\sum_{a=0}^{k-1}\frac{c}{2^{[\frac{a}{3}]}},$$

and

$$d(Tx_k,Tx_l)+d(Tx_l,Tx_m)+d(Tx_m,Tx^*)$$
$$+d(Tx^*,Tx_k)=2d(Tx_k,Tx^*)$$
$$=2d(x_{k+1},x^*)=12c-2\sum_{a=0}^{k}\frac{c}{2^{[\frac{a}{3}]}}.$$

Now, we have

$$d(x_0,x_n)=\begin{cases}6c(1-(\frac{1}{2})^p), & \text{if } n=3p,\\ 6c(1-(\frac{1}{2})^p)-\frac{c}{2^{p-1}}, & \text{if } n=3p-1,\\ 6c(1-(\frac{1}{2})^p)-\frac{c}{2^{p-2}}, & \text{if } n=3p-2.\end{cases} \tag{2.3}$$

Consider the ratio,

$$R_k=\frac{d(Tx_k,Tx_l)+d(Tx_l,Tx_m)+d(Tx_m,Tx^*)+d(Tx^*,Tx_k)}{d(x_k,x_l)+d(x_l,x_m)+d(x_m,x^*)+d(x^*,x_k)}$$
$$=\frac{12c-2\sum_{a=0}^{k}\frac{c}{2^{[\frac{a}{3}]}}}{12c-2\sum_{a=0}^{k-1}\frac{c}{2^{[\frac{a}{3}]}}}$$
$$=1-\frac{\frac{c}{2^{\left[\frac{k}{3}\right]}}}{6c-\sum_{a=0}^{k-1}\frac{c}{2^{[\frac{a}{3}]}}}$$
$$=\begin{cases}1-\frac{\frac{c}{2^p}}{6c-6c(1-(\frac{1}{2})^p)}, & \text{if } k=3p,\\ 1-\frac{\frac{c}{2^{p-1}}}{6c-6c(1-(\frac{1}{2})^p)+\frac{c}{2^{p-1}}}, & \text{if } k=3p-1,\\ 1-\frac{\frac{c}{2^{p-1}}}{6c-6c(1-(\frac{1}{2})^p)+\frac{c}{2^{p-2}}}, & \text{if } k=3p-2,\end{cases}$$
$$=\begin{cases}\frac{5}{6}, & \text{if } k=3p,\\ \frac{3}{4}, & \text{if } k=3p-1,\\ \frac{4}{5}, & \text{if } k=3p-2.\end{cases}$$

Now, consider the points $x_k,x_l,x_m,x_n\in M$ with $0\le k<l<m<n$. Then, we have,

$$d(x_k,x_l)+d(x_l,x_m)+d(x_m,x_n)+d(x_n,x_k)$$
$$=2d(x_k,x_n)=2\sum_{a=k}^{n-1}\frac{c}{2^{[\frac{a}{3}]}},$$

and

$$d(Tx_k,Tx_l)+d(Tx_l,Tx_m)+d(Tx_m,Tx_n)$$
$$+d(Tx_n,Tx_k)$$
$$=2d(Tx_k,Tx_n)$$
$$=2d(x_{k+1},x_{n+1})$$
$$=2d(x_k,x_n)-2[d(x_k,x_{k+1})-d(x_n,x_{n+1})]$$
$$=2\sum_{a=k}^{n-1}\frac{c}{2^{[\frac{a}{3}]}}-2\left(\frac{c}{2^{[\frac{k}{3}]}}-\frac{c}{2^{[\frac{n}{3}]}}\right).$$

Consider the ratio,

$$R_{k,n}=\frac{d(Tx_k,Tx_l)+d(Tx_l,Tx_m)+d(Tx_m,Tx_n)+d(Tx_n,Tx_k)}{d(x_k,x_l)+d(x_l,x_m)+d(x_m,x_n)+d(x_n,x_k)}$$
$$=\frac{2\sum_{a=k}^{n-1}\frac{c}{2^{[\frac{a}{3}]}}-2\left(\frac{c}{2^{[\frac{k}{3}]}}-\frac{c}{2^{[\frac{n}{3}]}}\right)}{2\sum_{a=k}^{n-1}\frac{c}{2^{[\frac{a}{3}]}}}$$
$$=1-\frac{\frac{c}{2^{[\frac{k}{3}]}}-\frac{c}{2^{[\frac{n}{3}]}}}{\sum_{a=k}^{n-1}\frac{c}{2^{[\frac{a}{3}]}}}.$$

It is to be noted that $n\ge k+3$. Therefore

$$\left[\frac{n}{3}\right]\ge\left[\frac{k}{3}\right]+1\implies 2^{[\frac{n}{3}]}\ge 2.2^{[\frac{k}{3}]}$$
$$\implies \frac{1}{2^{[\frac{n}{3}]}}\le\frac{1}{2.2^{[\frac{k}{3}]}}\implies\frac{c}{2^{[\frac{n}{3}]}}\le\frac{c}{2.2^{[\frac{k}{3}]}}. \tag{2.4}$$

Now, from (2.3), we can write,

$$d(x_n,x^*)=\begin{cases}\frac{6c}{2^p}, & \text{if } n=3p,\\ \frac{6c}{2^p}+\frac{c}{2^{p-1}}, & \text{if } n=3p-1,\\ \frac{6c}{2^p}+\frac{c}{2^{p-2}}, & \text{if } n=3p-2.\end{cases} \tag{2.5}$$

Therefore, from (2.5), we get,

$$\begin{aligned} & d(x_n, x^*) \leq 6d(x_n, x_{n+1}) \\ \implies & d(x_k, x^*) \leq 6d(x_k, x_{k+1}) \\ \implies & d(x_k, x_n) \leq d(x_k, x^*) \leq 6d(x_k, x_{k+1}) \\ \implies & \sum_{a=k}^{n-1} \frac{c}{2^{[\frac{a}{3}]}} \leq 6\frac{c}{2^{[\frac{k}{3}]}}. \end{aligned} \tag{2.6}$$

Consequently, from Eq. (2.4) and Eq. (2.6) we have

$$R_{k,n} \leq 1 - \frac{\frac{c}{2^{[\frac{k}{3}]}} - \frac{1}{2}\frac{c}{2^{[\frac{k}{3}]}}}{6\frac{c}{2^{[\frac{k}{3}]}}} = \frac{11}{12}.$$

Thus, the inequality (2.1) holds for any four distinct points from M with $\alpha = \frac{11}{12} = \max\{\frac{5}{6}, \frac{3}{4}, \frac{4}{5}, \frac{11}{12}\}$.

Therefore, T is a perimetric contraction on quadrilaterals on M. Also, T does not contain any periodic points of order 2 and 3. Hence, by Theorem 2.4, T has a fixed point in M. Note that $x^* \in Fix(T)$.

In the next example, we consider an uncountably infinite metric space.

Example 2.13. Let $M = \{-1, -\frac{2}{3}, -\frac{1}{3}\} \cup [0,1] \subset \mathbb{R}$ be a metric space equipped with the Euclidean metric d and let $T : M \to M$ be a mapping defined as follows:

$$Tm = \begin{cases} \frac{m}{2}, & \text{if } m \in [0,1], \\ 0, & \text{if } m = -\frac{1}{3}, \\ -\frac{1}{3}, & \text{if } m = -\frac{2}{3}, \\ -\frac{2}{3}, & \text{if } m = -1. \end{cases}$$

Now, for $a = \frac{2}{3}, b = -\frac{2}{3}$, we get, $d(Ta, Tb) = \frac{2}{3} = d(a, Ta) + d(b, Tb)$, and hence T is not a Kannan type mapping.

Again, for $a = -\frac{1}{3}, b = -\frac{2}{3}$, we get, $d(Ta, Tb) = \frac{1}{3} = \frac{1}{2}(d(a, Tb) + d(b, Ta))$, and so T is not a Chatterjea type mapping.

Now, for $a = -\frac{1}{3}, b = -\frac{2}{3}, c = -1$, we have

$d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) = \frac{4}{3} = d(a, b) + d(b, c) + d(c, a)$.

Thus, T is not a mapping contracting perimeters of triangles.

Also, $d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) = \frac{4}{3} = \frac{4}{3}(d(a, Ta) + d(b, Tb) + d(c, Tc))$.

Therefore, T is not a generalized Kannan type mapping.

On the other hand,

$$\begin{aligned} & d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) \\ & = \frac{4}{7}(d(a, Tb) + d(a, Tc) + d(b, Ta) + \\ & \quad d(b, Tc) + d(c, Ta) + d(c, Tb)). \end{aligned}$$

Thus, T is not a generalized Chatterjea type mapping.

For any four distinct points of M, the condition (2.1) holds for $\alpha \in [\frac{2}{3}, 1)$. Thus, T is a perimetric contraction on quadrilaterals on M. Also, T does not contain any periodic points of orders 2 and 3. Therefore, by Theorem 2.4, $Fix(T) \neq \phi$. Note that $Fix(T) = \{0\}$.

In the following flowchart, we summarize the implications as observed in the above examples:

"The question of whether the class of mappings that contract the perimeters of triangles constitutes a subclass of perimetric contractions on quadrilaterals remains an open problem for researchers."

Now, we establish fixed-point theorems without considering the completeness of the underlying space in the presence of different sets of adequate conditions.

Theorem 2.14. *Let (M, d) be a metric space with at least four points, and let $T : M \to M$ be a perimetric contraction on quadrilaterals on M satisfying the following conditions:*

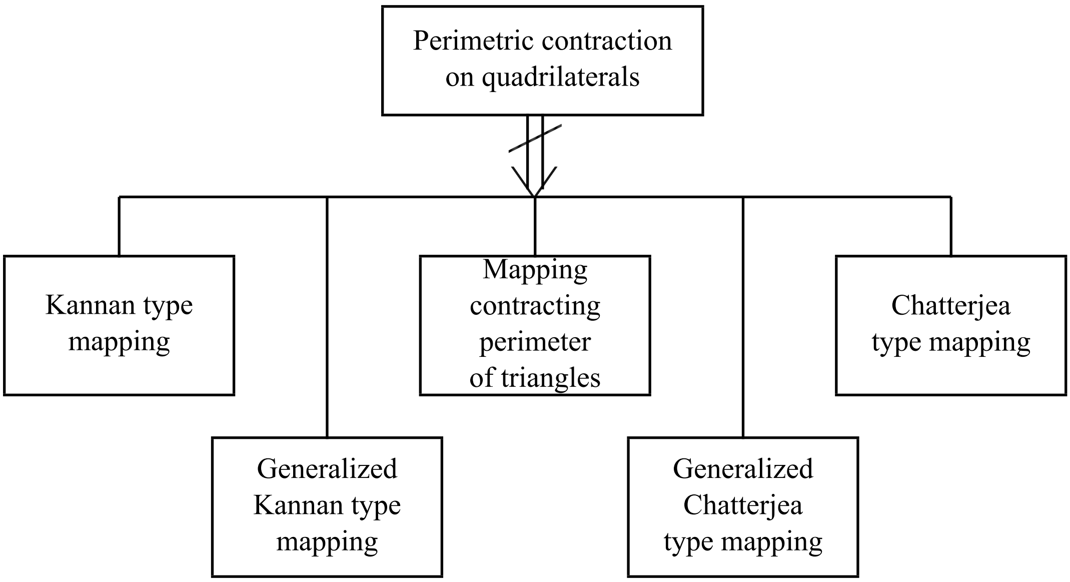

Fig. 2. Perimetric contractions on quadrilaterals do not imply those classes of mappings.

(i) T does not attain periodic points of order 2 and 3.

(ii) There exists $x_0 \in M$ such that the sequence of iterations $x_0, x_1 = Tx_0, \ldots, x_n = Tx_{n-1}, \ldots$ has a convergent subsequence $\{x_{n_k}\}$ converging to x^*.

Then x^* is a fixed point of T. The number of fixed points is at most three.

Proof. Since T is continuous and $x_{n_k} \to x^*$, we have $Tx_{n_k} = x_{n_k+1} \to Tx^*$.

If possible, suppose that $Tx^* \neq x^*$. Now, consider two open balls $B = B(x^*, r)$ and $B^* = B(Tx^*, r)$, with $0 \leq r < \frac{1}{3}d(x^*, Tx^*)$.

Thus, there exists $N \in \mathbb{N}$ such that $x_{n_i} \in B$ and $x_{n_i+1} \in B^*$ for all $i > N$.

Hence,

$$d(x_{n_i}, x_{n_i+1}) > r \text{ for all } i > N. \tag{2.7}$$

If the sequence $\{x_n\}$ does not contain a fixed point of T, then from (2.2) of Theorem 2.4, we get

$$d(x_n, x_{n+1}) \leq \alpha^n \lambda_0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

where $\lambda_0 = d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_0)$ and $\alpha \in [0, 1)$.

Thus, we have

$$d(x_{n_i}, x_{n_i+1}) \leq \alpha^{n_i} \lambda_0 \text{ for all } n_i \in \mathbb{N} \cup \{0\}.$$

$$\implies \lim_{i\to\infty} d(x_{n_i}, x_{n_i+1}) = 0,$$

which contradicts (2.7). Hence, $Tx^* = x^*$.

The existence of at most three fixed points follows from Theorem 2.4. □

In the following theorem, we assume that T is a perimetric contraction on quadrilaterals and is defined on a dense subset of M.

Theorem 2.15. *Let (M, d) be a metric space with at least four points. Suppose T satisfies the following conditions:*

(i) T does not attain periodic points of order 2 and 3.

(ii) T is a perimetric contraction on quadrilaterals on D, where D is a dense subset of M.

(iii) There exists $x_0 \in M$ such that the sequence of iterations $x_0, x_1 =$

$Tx_0, \ldots x_n = Tx_{n-1}, \ldots$ has a convergent subsequence x_{n_k} converging to x^*.

Then x^* is a fixed point of T. The number of fixed points is at most three.

Proof. Suppose that, w, x, y, z be four distinct points of M. Here, four cases may occur.

Case- I:- Let $w, x, y \in D$ and $z \in M \setminus D$. Then there exists a sequence $\{d_n\}$ in M such that $d_n \to z$. Also, suppose that each element of the sequence and w, x, y, z are distinct. Now,

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& d(Tw,Tx) + d(Tx,Ty) + d(Ty,Td_n) + d(Td_n,Tz)\\
&+ d(Tz,Td_n) + d(Td_n,Tw)\\
\leq& d(Tw,Tx) + d(Tx,Ty) + d(Ty,Td_n) + d(Td_n,Tw)\\
&+ 2d(Td_n,Tz)\\
\leq& \alpha[d(w,x) + d(x,y) + d(y,d_n) + d(d_n,w)]\\
&+ 2d(Td_n,Tz).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& \alpha[d(w,x) + d(x,y) + d(y,z) + d(z,w)].
\end{aligned}$$

Case- II:- Let $w, x \in D$ and $y, z \in M \setminus D$. Then there exists a sequence $\{c_n\}, \{d_n\}$ in M such that $c_n \to y$ and $d_n \to z$. Also, suppose that each element of the sequences and w, x, y, z are distinct. Now,

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& d(Tw,Tx) + d(Tx,Tc_n) + d(Tc_n,Ty) + d(Ty,Tc_n)\\
&+ d(Tc_n,Td_n) + d(Td_n,Tz)\\
&+ d(Tz,Td_n) + d(Td_n,Tw)\\
\leq& d(Tw,Tx) + d(Tx,Tc_n) + d(Tc_n,Td_n) + d(Td_n,Tw)\\
&+ 2d(Tc_n,Ty) + 2d(Td_n,Tz)\\
\leq& \alpha[d(w,x) + d(x,c_n) + d(c_n,d_n) + d(d_n,w)]\\
&+ 2d(Tc_n,Ty) + 2d(Td_n,Tz).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& \alpha[d(w,x) + d(x,y) + d(y,z) + d(z,w)].
\end{aligned}$$

Case- III:- Let $w \in D$ and $x, y, z \in M \setminus D$. Then there exists a sequence $\{b_n\}, \{c_n\}, \{d_n\}$ in M such that $b_n \to x, c_n \to y$ and $d_n \to z$. Also, suppose that each element of the sequences and w, x, y, z are distinct. Now,

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& d(Tw,Tb_n) + d(Tb_n,Tx) + d(Tx,Tb_n)\\
&+ d(Tb_n,Tc_n) + d(Tc_n,Ty) + d(Ty,Tc_n)\\
&+ d(Tc_n,Td_n) + d(Td_n,Tz) + d(Tz,Td_n)\\
&+ d(Td_n,Tw)\\
\leq& d(Tw,Tb_n) + d(Tb_n,Tc_n) + d(Tc_n,Td_n)\\
&+ d(Td_n,Tw) + 2d(Tb_n,Tx) + 2d(Tc_n,Ty)\\
&+ 2d(Td_n,Tz)\\
\leq& \alpha[d(w,b_n) + d(b_n,c_n) + d(c_n,d_n) + d(d_n,w)]\\
&+ 2d(Tb_n,Tx) + 2d(Tc_n,Ty) + 2d(Td_n,Tz).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& \alpha[d(w,x) + d(x,y) + d(y,z) + d(z,w)].
\end{aligned}$$

Case- IV:- Let $w, x, y, z \in M \setminus D$. Then there exists a sequence $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ in M such that $a_n \to w, b_n \to x, c_n \to y$ and $d_n \to z$. Also, suppose that each element of the sequences and w, x, y, z are distinct. Now,

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& d(Tw,Ta_n) + d(Ta_n,Tb_n) + d(Tb_n,Tx)\\
&+ d(Tx,Tb_n) + d(Tb_n,Tc_n) + d(Tc_n,Ty)\\
&+ d(Ty,Tc_n) + d(Tc_n,Td_n) + d(Td_n,Tz)\\
&+ d(Tz,Td_n) + d(Td_n,Ta_n) + d(Ta_n,Tw)\\
\leq& d(Ta_n,Tb_n) + d(Tb_n,Tc_n) + d(Tc_n,Td_n)\\
&+ d(Td_n,Ta_n) + 2d(Ta_n,Tw) + 2d(Tb_n,Tx)\\
&+ 2d(Tc_n,Ty) + 2d(Td_n,Tz)\\
\leq& \alpha[d(a_n,b_n) + d(b_n,c_n) + d(c_n,d_n) + d(d_n,a_n)]\\
&+ 2d(Ta_n,Tw) + 2d(Tb_n,Tx) + 2d(Tc_n,Ty)\\
&+ 2d(Td_n,Tz).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
&d(Tw,Tx) + d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tw)\\
\leq& \alpha[d(w,x) + d(x,y) + d(y,z) + d(z,w)].
\end{aligned}$$

Thus, in each of the above cases, T is a perimetric contraction on quadrilaterals on M. Therefore, by Theorem 2.14 the result follows. □

3. Conclusion

In this article, extending the contraction mappings in four point analogue, we have proposed the idea of a mapping that characterizes the contraction of perimeters of quadrilaterals. A fixed point result of such mapping is obtained in a complete metric space. Notably, achieving a fixed point requires avoiding periodic points of order 2 and 3 and these mappings can attain a maximum of three fixed points. Thus, we derive a necessary condition for the fixed point to be unique. In addition, a subset of these perimeter-based mappings is the class of contraction mappings. As a result, we prove Banach's fixed point theorem in an alternative way using our result. Furthermore, we provide nontrivial examples to validate our findings, distinguishing perimetric contraction on quadrilaterals from different well-known classes. Finally, we establish fixed point results of these mappings in a metric space without considering its completeness.

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