E-ISSN 2586-9027
Homepage : *https://tci-thaijo.org/index.php/SciTechAsia* Science & Technology Asia
Vol.30 No.4 October - December 2025 Page: [90-98]
Original research article

On the Convergence and Stability of a Hybrid Iteration Scheme in Uniformly Convex Banach Space

Nehjamang Haokip

Department of Mathematics, Churachandpur College Churachandpur, Manipur 795128, India

Received 19 June 2025; Received in revised form 17 August 2025
Accepted 9 September 2025; Available online 17 December 2025

ABSTRACT

A new iterative algorithm for approximating fixed points is considered on the lines of the iterative algorithm considered by Pansuwan and Sintunavarat [1]. The convergence of the considered iterative algorithm is established. Finally, the convergence rate of the new iterative algorithm is compared with that of the iterative algorithm considered in [1].

Keywords: Convergence of iteration; Fixed point; Hybrid iteration scheme

1. Introduction

For a contraction mapping T on a complete metric space (X, d), Banach showed the existence of a unique fixed point and the convergence of the Picard iterates $\{x_n\}$, where for x_0 in X,

$$x_n = Tx_{n-1}, \quad n = 1,\ 2,\ 3,\ \ldots$$

to the fixed point of T. However, for non-contraction mappings, the Picard iteration need not converge to the fixed point, if it exists. For instance, in the complete metric space (X, d) where $X = [0, 1]$ with the usual metric, the self-mapping T on X defined by $Tx = 1-x$ for all $x \in X$ is not a contraction mapping. In fact, T is a nonexpansive mapping. The mapping has a unique fixed point $x = \frac{1}{2}$ and the sequence of Picard iterates $\{x_n\}$ with $x_0 \neq \frac{1}{2}$ does not converge to the fixed point.

Thus a different technique is required for approximation of the fixed point. In this regard, many researchers have developed different iteration procedures for fixed point approximation. Some of the prominent authors in this regard are Mann [2] in 1953, Ishikawa [3] in 1974, Jungck [4] in 1976, Rhoades [5] in 1991, Noor [6] in 2000, Agarwal et al. [7] in 2007, etc. and the references therein. We note here that, for example, the Mann iterates defined by

$$x_n = (1 - \alpha_{n-1})x_{n-1} + \alpha_{n-1}Tx_{n-1},$$

***Corresponding author:** mark02mm@yahoo.co.in
doi: 10.14456/scitechasia.2025.72

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ converges to the fixed point of $Tx = 1 - x$. Since then many authors have considered the convergence and stability of new iteration schemes.

Recently, Pansuwan and Sintunavarat [1] introduced a new hybrid iterative algorithm to approximate fixed point of Suzuki's generalized nonexpansive mappings as follows. For $x_0 \in C$, a nonempty subset of a Banach space X and a self mapping T on C satisfying condition (C),

$$\left.\begin{aligned} x_{n+1} &= T^n y_n \\ y_n &= T\big((1-\alpha_n)x_n + \alpha_n z_n\big) \\ z_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned}\right\} \quad (1.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ and $n = 0, 1, 2, \ldots$. They also showed the convergence of Eq. (1.1) in uniformly convex Banach spaces.

In this paper, a new hybrid iteration scheme is considered in a uniformly convex Banach space. For $x_1 \in C$, a nonempty subset of a Banach space X and a self mapping T on C,

$$\left.\begin{aligned} x_n &= T^n\Big((1-\alpha_{n-1})y_{n-1} + \alpha_{n-1} T y_{n-1}\Big) \\ y_{n-1} &= T\Big((1-\beta_{n-1})x_{n-1} + \beta_{n-1} z_{n-1}\Big) \\ z_{n-1} &= T x_{n-1}, \end{aligned}\right\} \quad (1.2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ and $n = 0, 1, 2, \ldots$.

We put the above iteration scheme in the form

$$\begin{aligned} x_n &= \mu(T, x_{n-1}) \\ &= T^n\Big((1-\alpha_{n-1})T\big((1-\beta_{n-1})x_{n-1} \\ &\quad + \beta_{n-1}Tx_{n-1}\big) + \alpha_{n-1}T^2\big((1-\beta_{n-1}) \\ &\quad x_{n-1} + \beta_{n-1}Tx_{n-1}\big)\Big). \end{aligned}$$

2. Preliminaries

In this section, some preliminary definitions and results required for our subsequent discussion are presented.

Definition 2.1 ([8])**.** A Banach space X is said to be uniformly convex if $\forall\, \varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for $x, y \in X$,

$$\left.\begin{aligned} \|x\| &\le 1, \\ \|y\| &\le 1, \\ \|x-y\| &> \varepsilon \end{aligned}\right\} \implies \left\|\frac{x+y}{2}\right\| \le 1 - \delta.$$

Lemma 2.2 ([9])**.** *Let X be a uniformly convex Banach space and $0 < \alpha \le t_n \le \beta < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\limsup_{n\to\infty} \big\|t_n x_n - (1-t_n)y_n\big\| = r$ for some $r \ge 0$. Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Let C be a nonempty closed convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. Then, for each x in X, the asymptotic radius of $\{x_n\}$ at x is defined as

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} \|x_n - x\|,$$

and the asymptotic centre of $\{x_n\}$ relative to C is defined as

$$A\big(C, \{x_n\}\big) = \big\{x \in C : \; r\big(x, \{x_n\}\big) = r\big\}.$$

The following is the definition of T-stability of iteration schemes given by Harder and Hicks [10].

Definition 2.3 ([10])**.** Let $T : X \longrightarrow X$ and w be a fixed point of T. For any $x_0 \in X$, let the sequence $\{x_n\}$ generated by the iteration scheme $x_{n+1} = \mu(T, x_n)$, $n = 0, 1, 2, \ldots$ converges to w. Let $\{u_n\}$ be an arbitrary sequence, and set $\epsilon_n = \|u_{n+1} - x_{n+1}\|$, $n = 0, 1, 2, \ldots$. Then the iterative scheme $\mu(T, x_n)$ is called T-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} u_n = w$.

Senter and Dotson [11] defined the following condition and obtained a convergence result.

Definition 2.4. [11] Let C be a convex subset of a Banach space X. A self mapping T of C with nonempty fixed point set F in C is said to satisfy Condition I if there exist a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$, such that

$$\|Tx - x\| \geq f\big(d(x, F)\big) \quad \text{for all } x \in C,$$

where $d(x, F) = \inf\big\{\|x - z\| : z \in F\big\}$.

3. Convergence result

In this section, the convergence of the iteration scheme Eq. (1.2) to a fixed point of a nonexpansive mapping in a Banach space is discussed.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the iteration scheme Eq. (1.2), then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$.*

Proof. Let $w \in F(T) \neq \emptyset$, then for $x \in C$ since T is a nonexpansive mapping, we have,

$$\|Tx - w\| \leq \|x - w\|.$$

By definition of the iteration scheme Eq. (1.2), for nonnegative positive integer $n - 1$,

$$\|z_{n-1} - w\| = \|Tx_{n-1} - w\| \leq \|x_{n-1} - w\|,$$

and

$$\begin{aligned}\|y_{n-1} - w\| &= \Big\|T\big((1 - \beta_{n-1})x_{n-1} \\ &\qquad + \beta_{n-1}z_{n-1}\big) - w\Big\| \\ &\leq \big\|(1 - \beta_{n-1})(x_{n-1} - w) + \beta_{n-1}(z_{n-1} - w)\big\| \\ &\leq (1 - \beta_{n-1})\|x_{n-1} - w\| + \beta_{n-1}\|Tx_{n-1} - w\| \\ &\leq (1 - \beta_{n-1})\|x_{n-1} - w\| + \beta_{n-1}\|x_{n-1} - w\| \\ &= \|x_{n-1} - w\|\end{aligned}$$

so that,

$$\begin{aligned}\|x_n - w\| &= \big\|T^n\big((1 - \alpha_{n-1})y_{n-1} \\ &\qquad + \alpha_{n-1}Ty_{n-1}\big) - w\big\| \\ &\leq (1 - \alpha_{n-1})\big\|y_{n-1} - w\big\| + \alpha_{n-1}\big\|Ty_{n-1} \\ &\quad - w\big\| \\ &\leq (1 - \alpha_{n-1})\big\|y_{n-1} - w)\big\| + \alpha_{n-1}\big\|y_{n-1} \\ &\quad - w\big\| = \|y_{n-1} - w\|,\end{aligned}$$

i.e.,

$$\|x_n - w\| \leq \|x_{n-1} - w\|.$$

Thus the sequence $\big\{\|x_n - w\|\big\}$ of nonnegative numbers is bounded and nonincreasing, and thus $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \longrightarrow C$ be a nonexpansive mapping. For any $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the iteration scheme Eq. (1.2), then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Let $w \in F(T) \neq \emptyset$. Then by Lemma 3.1, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$ and $\{x_n\}$ is bounded. Suppose that $\lim_{n\to\infty} \|x_n - w\| = \mu$ so that

$$\limsup_{n\to\infty} \|x_n - w\| = \mu. \tag{3.1}$$

Since $\|z_{n-1} - w\| = \|Tx_{n-1} - w\| \leq \|x_{n-1} - w\|$, it follows that

$$\limsup_{n\to\infty} \|z_{n-1} - w\| \leq \mu. \tag{3.2}$$

If $\mu = 0$, then since $\|Tx - w\| \leq \|x - w\|$, we get $\|Tx_n - x_n\| \leq 2\|x_n - w\|$ and taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

If $\mu > 0$, since $y_n = (1-\beta_n)z_n + \beta_n T z_n$, we have,

$$\|y_n - w\| \leq (1-\beta_n)\|Tx_n - w\| + \beta_n\|Tz_n - w\| \leq \|x_n - w\|,$$

and taking the limit supremum as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|y_n - w\| \leq \mu.$$

Again, since (from the proof of Lemma 3.1) $\|x_n - w\| \leq \|y_{n-1} - w\|$, taking limit infimum as $n \to \infty$, we get

$$\begin{aligned}\mu &= \liminf_{n\to\infty} \|x_n - w\| \\ &\leq \liminf_{n\to\infty} \|y_{n-1} - w\| \leq \mu,\end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|y_{n-1} - w\| = \mu.$$

This implies that

$$\begin{aligned}\mu &= \limsup_{n\to\infty} \|y_{n-1} - w\| \\ &= \limsup_{n\to\infty} \Big\|T\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}z_{n-1}\big) - w\Big\| \\ &\leq \limsup_{n\to\infty} \Big\|(1-\beta_{n-1})x_{n-1} + \beta_{n-1}z_{n-1} - w\Big\| \\ &\leq \limsup_{n\to\infty} \Big\{(1-\beta_{n-1})\|x_{n-1} - w\| + \beta_{n-1}\|Tx_{n-1} - w\|\Big\} \\ &\leq \limsup_{n\to\infty} \|x_{n-1} - w\| = \mu,\end{aligned}$$

i.e.,

$$\limsup_{n\to\infty} \Big\|(1-\beta_{n-1})(x_{n-1} - w) + \beta_{n-1}(z_{n-1} - w)\Big\| = \mu.$$

It then follows from Lemma 3.1 that $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$.

Conversely, let $\{x_n\}$ be bounded and $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $u \in A(C, \{x_n\})$. Then,

$$\begin{aligned}r(Tu, \{x_n\}) &= \limsup_{n\to\infty} \|Tu - x_n\| \\ &\leq \limsup_{n\to\infty} \Big(\|Tu - Tx_n\| + \|Tx_n - x_n\|\Big) \\ &= \limsup_{n\to\infty} \|u - x_n\| \\ &= r(u, \{x_n\}).\end{aligned}$$

This implies $Tu \in A(C, \{x_n\})$. But as X is a uniformly convex Banach space, $A(C, \{x_n\})$ is a singleton set and hence $Tu = u$, i.e., $u \in F(T)$ and $F(T) \neq \emptyset$, as required. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the iteration scheme Eq.* (1.2) *converges strongly to an element of $F(T)$ if and only if $\liminf_{n\to\infty} \|x_n - F(T)\| = 0$, where $\|x_n - F(T)\| = \inf_{w\in F(T)} \|x_n - w\|$.*

Proof. If the sequence $\{x_n\}$ defined by the iteration scheme Eq. (1.2) strongly converges to a fixed point of T, then obviously

$$\liminf_{n\to\infty} \|x_n - F(T)\| = 0.$$

To prove the converse, we first note that $F(T)$ is closed. If $\{w_k\}$ is a sequence in $F(T)$ which converges to some $w \in C$, then since T is nonexpansive,

$$\|w_n - Tw\| = \|T^2 w_n - Tw\| \leq \|Tw_n - w\| = \|w_n - w\|,$$

and thus,

$$0 = \lim_{n\to\infty} \|w_n - w\| \geq \lim_{n\to\infty} \|w_n - w\| = \left\| \lim_{n\to\infty} w_n - Tw \right\| = \|w - Tw\|$$

showing that $w \in F(T)$, and hence $F(T)$ is closed.

We know, from Lemma 3.1 that $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$ so that $\|x_{n+1} - F(T)\| \leq \|x_n - F(T)\|$, which implies the sequence $\{\|x_n - F(T)\|\}$ is non-increasing and bounded below, and therefore, $\liminf_{n\to\infty} \|x_n - F(T)\|$ exists.

Since $\liminf_{n\to\infty} \|x_n - F(T)\| = 0$, it follows that $\lim_{n\to\infty} \|x_n - F(T)\| = 0$. Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - w_k\| < \frac{1}{2^k}$ for all $k \geq 1$ and $\{w_k\} \subseteq F(T)$. Then $\|x_{n_{k+1}} - w_k\| \leq \|x_{n_k} - w_k\| < \frac{1}{2^k}$, which implies

$$\|w_{k+1} - w_k\| \leq \|w_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - w_k\| < \frac{1}{2^{k-1}},$$

showing that $\{w_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{w_k\}$ converges in $F(T)$, say, $\lim_{k\to\infty} w_k = w \in F(T)$. Then as $k \to \infty$,

$$\|x_{n_k} - w\| \leq \|x_{n_k} - w_k\| + \|w_k - w\| \longrightarrow 0,$$

showing that $\lim_{k\to\infty} \|x_{n_k} - w\| = 0$. Now, since $\lim_{n\to\infty} \|x_n - w\|$ exists, we must have

$$\lim_{n\to\infty} \|x_n - w\| = 0,$$

as required. □

Example 3.4. Consider the closed convex subset $C = [0,1]$ of the Banach space of real numbers $\mathbb{R}$ which is uniformly convex. Let $T : C \longrightarrow C$ be defined by $Tx = \frac{3}{4}x$. Then T is a nonexpansive mapping and $F(T) = \{0\} \neq \emptyset$.

Taking $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{n^2+1}$, let $\{x_n\}$ be the sequence generated by the iteration scheme Eq. (1.2). Taking the initial points $x_1 = 0.95$, 0.55 and 0.25, we compute the sequence generated by Eq. (1.2) (using `Sagemath`[1]) and the results are shown in Table 1.1 and Fig. 1 below.

From Table 1 and Fig. 1, it is seen that $\liminf_{n\to\infty} x_n = 0$ and hence by Theorem 3.3, the sequence $\{x_n\}$ generated by the iteration scheme Eq. (1.2) converges strongly to the fixed point of T.

Next, we prove a strong convergence result using the definition of condition (I) given by Senter and Dotson [11] for metric spaces.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If T satisfies Condition(I), then the sequence defined by the iteration scheme Eq.* (1.2) *converges strongly to some fixed point of T.*

Proof. As in the proof of Theorem 3.3, $F(T)$ is closed. By Theorem 3.2, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since T satisfies Condition (I), we have

$$\lim_{n\to\infty} f\left(\|x_n - F(T)\|\right) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Since f is a non-decreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$,

$$\lim_{n\to\infty} f\left(\|x_n - F(T)\|\right) = 0.$$

[1] https://www.sagemath.org/

Table 1. Sequence generated by Eq. (1.2).

n	$x_0 = 0.95$	$x_0 = 0.55$	$x_0 = 0.25$
1	0.950000000000000	0.550000000000000	0.250000000000000
2	0.306848144531250	0.177648925781250	0.080749511718750
3	0.084548050761222	0.048948871493339	0.022249487042427
4	0.018339428948916	0.010617564128319	0.004826165512872
5	0.003055222784582	0.001768813191074	0.000804005995942
6	0.000387072402088	0.000224094548577	0.000101861158444
7	0.000037114504506	0.000021487344714	$9.7669748701\times10^{-6}$
8	$2.6861473578\times10^{-6}$	$1.5551379440\times10^{-6}$	$7.0688088365\times10^{-7}$
9	$1.4649892143\times10^{-7}$	$8.4815165043\times10^{-8}$	$3.8552347747\times10^{-8}$
10	$6.0143243992\times10^{-9}$	$3.4819772837\times10^{-9}$	$1.5827169471\times10^{-9}$

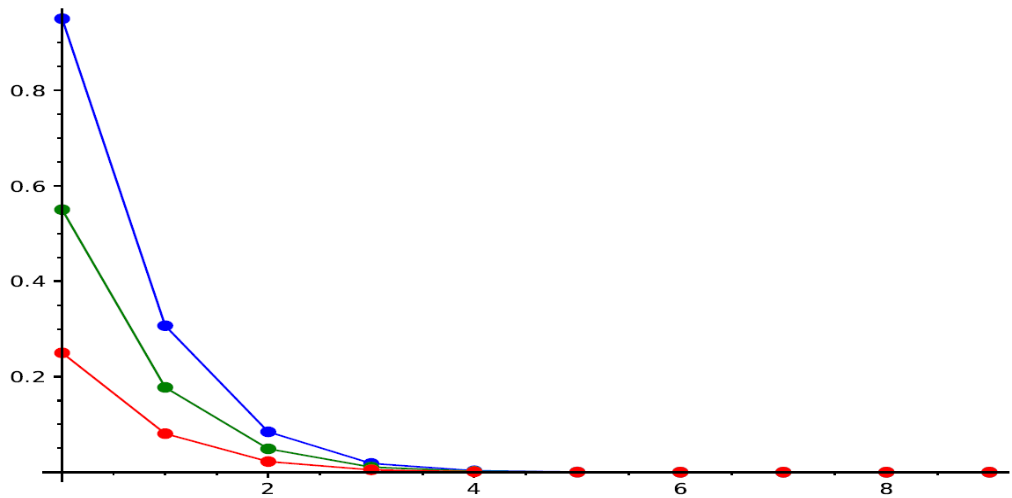

Fig. 1. Hybrid iteration Eq. (1.2) with different initial points.

The conclusion of the proof follows as in the proof of Theorem 3.3. □

Now, we obtain a stability result for the iteration scheme Eq. (1.2) in the next theorem.

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ satisfying Condition (I) or,* $\liminf_{n\to\infty} \|x_n - F(T)\| = 0$*. Then, for an arbitrary point $x_1 \in C$, the sequence $\{x_n\}$ generated by Eq. (1.2) is T-stable.*

Proof. Let $\{r_n\}$ be an arbitrary sequence in C. Let $\{x_n\}$ where $x_n = f(T, x_{n-1})$ be the the sequence generated by the iteration scheme Eq. (1.2), which converges to some $w \in F(T)$ (by Theorem 3.3 & 3.5), and let $\varepsilon_n = \|r_n - f(T, x_{n-1})\|$.

Then using Eq. (1.2), we have,

$$\|r_n - w\| \leq \|r_n - f(T, x_{n-1})\| + \|f(T, x_{n-1}) - w\| = \varepsilon_n + \|f(T, r_{n-1}) - w\|$$

$$= \varepsilon_n + \Big\| T^n\Big((1-\alpha_{n-1})T\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big) + \alpha_{n-1}T^2\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big)\Big) - w\Big\|$$
$$\leq \varepsilon_n + \Big\|(1-\alpha_{n-1})T\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big) + \alpha_{n-1}T^2\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big) - w\Big\|$$
$$\leq \varepsilon_n + (1-\alpha_{n-1})\big\|T\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big) - w\big\| + \alpha_{n-1}\big\|T^2\big((1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1}\big) - w\big\|$$
$$\leq \varepsilon_n + (1-\alpha_{n-1})\big\|(1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1} - w\big\| + \alpha_{n-1}\big\|(1-\beta_{n-1})x_{n-1} + \beta_{n-1}Tx_{n-1} - w\big\|$$
$$\leq \varepsilon_n + (1-\alpha_{n-1})(1-\beta_{n-1})\|x_{n-1} - w\| + (1-\alpha_{n-1})\beta_{n-1}\|Tx_{n-1} - w\| + \alpha_{n-1}(1-\beta_{n-1})\|x_{n-1} - w\| + \alpha_{n-1}\beta_{n-1}\|Tx_{n-1} - w\|$$
$$\leq \varepsilon_n + (1-\alpha_{n-1})(1-\beta_{n-1})\|x_{n-1} - w\| + (1-\alpha_{n-1})\beta_{n-1}\|x_{n-1} - w\| + \alpha_{n-1}(1-\beta_{n-1})\|x_{n-1} - w\| + \alpha_{n-1}\beta_{n-1}\|x_{n-1} - w\|$$
$$\leq \varepsilon_n + \Big\{1 - \alpha_{n-1} - \beta_{n-1} + \alpha_{n-1}\beta_{n-1} + \beta_{n-1} - \alpha_{n-1}\beta_{n-1} + \alpha_{n-1} - \alpha_{n-1}\beta_{n-1} + \alpha_{n-1}\beta_{n-1}\Big\}\|x_{n-1} - w\|$$
$$= \varepsilon_n + \|x_{n-1} - w\|.$$

If $\lim_{n\to\infty} \varepsilon_n = 0$, then since $\lim_{n\to\infty} x_n = w$ it follows that $\lim_{n\to\infty} r_n = w$. On the other hand, if $\lim_{n\to\infty} r_n = w$, then $\lim_{n\to\infty} \varepsilon_n = 0$. Thus the sequence $\{x_n\}$ generated by the iteration scheme Eq. (1.2) is $T-$ stable. □

In the following example, we give a numerical comparison on the rate of convergence to a fixed point of the iteration schemes Eq. (1.1)-(1.2) using `Sagemath`.

Example 3.7. Consider the closed convex subset $C = [0, 1]$ of the Banach space of real numbers $\mathbb{R}$ which is uniformly convex. Let $T : C \longrightarrow C$ be defined by $Tx = 1 - x$ for all $x \in C$. Then T is a nonexpansive mapping and $F(T) = \{\frac{1}{2}\} \neq \emptyset$. Taking $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{n^2+3}$, the sequences $\{x_n\}$ and $\{y_n\}$ respectively generated by the iteration schemes Eq. (1.1)-(1.2) with initial point $x_0 = 0.75 = y_0$ are calculated and shown in Table 2 and Fig. 2 below.

Table 2. Sequence generated by Eq. (1.1)-(1.2).

n	x_n	y_n
1	0.75	0.75
2	0.68	0.50
3	0.50	0.50

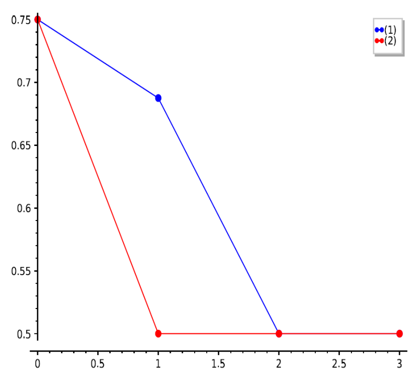

Fig. 2. Iterations Eq. (1.1)-(1.2) with initial point 0.75.

From Table 2 and Fig. 2, it is seen that the sequence $\{y_n\}$ generated by Eq. (1.2) converges to the fixed point $x = \frac{1}{2}$ faster than the sequence $\{x_n\}$ generated by the iteration scheme Eq. (1.1). Taking the initial point $x_0 = 0.25 = y_0$ or taking $\alpha_n = \frac{2}{n+2}$ and $\beta_n = \frac{3}{n^2+3}$, it is seen that the sequence $\{y_n\}$ generated by Eq. (1.2) converges to the fixed point $x = \frac{1}{2}$ faster than the sequence $\{x_n\}$ generated by the iteration scheme Eq. (1.1).

Table 3. Sequence generated by Eq. (1.1)-(1.2) with initial point 0.97.

n	x_n	y_n
1	0.970000000000000	0.970000000000000
2	0.558720000000000	0.365858133333333
3	0.179291391512381	0.123309661505829
4	0.049497484605731	0.035314940037069
5	0.011450189029753	0.008367570763303
6	0.002180078826480	0.001619077843863
7	0.000338239290634	0.000254084310369
8	0.000042518970307	0.000032210943675
9	$4.3155364893\times10^{-6}$	$3.2905449519\times10^{-6}$
10	$3.5286071727\times10^{-7}$	$2.7043171954\times10^{-7}$

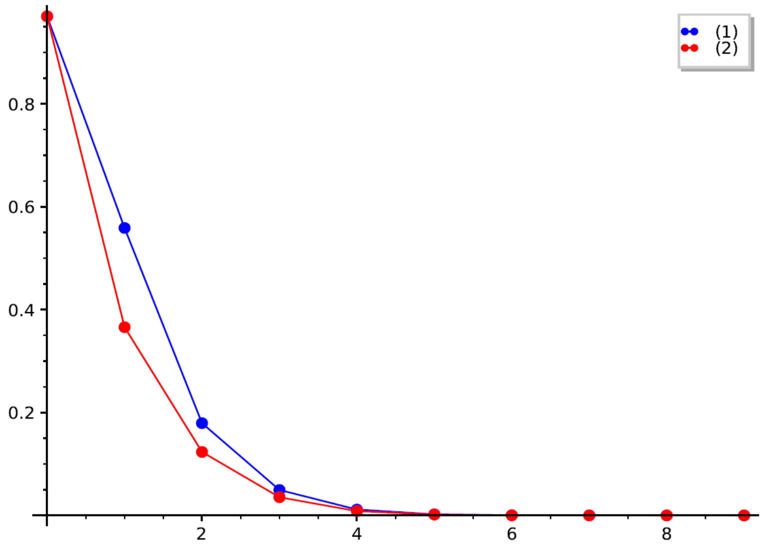

Fig. 3. Iterations Eq. (1.1)-(1.2) with initial point 0.97.

Now, let $T : C \longrightarrow C$ be defined by $Tx = \frac{4}{5}x$ for all $x \in C$. Then T is a non-expansive mapping, in fact, a contraction mapping with a unique fixed point $x = 0$. Taking $\alpha_n = \frac{2}{n+2}$ and $\beta_n = \frac{3}{n^2+3}$, the sequences $\{x_n\}$ and $\{y_n\}$ respectively generated by the iteration schemes Eq. (1.1)-(1.2) with initial point $x_0 = 0.97 = y_0$ are calculated and shown in Table 3 and Fig. 3 below.

From Table 3 and Fig. 3, it is seen that the sequence $\{y_n\}$ generated by Eq. (1.2) converges to the fixed point $x = 0$ faster than the sequence $\{x_n\}$ generated by the iteration scheme Eq. (1.1).

A similar result is also seen with initial point $x_0 = 0.85 = y_0$, $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{n^2+3}$ for all $n \in \mathbb{N}$.

Conclusion

In this paper, a new hybrid iteration scheme is introduced and its convergence as well as T-stability are discussed in a uniformly convex Banach space. Some numerical examples are also presented for a possible faster convergence rate of the sequence generated by the introduced iteration scheme to that of the sequence generated by the iteration scheme Eq. (1.1). Further study may be done to prove that the iteration scheme Eq. (1.2) converges faster to a (or the) fixed point than that of Eq. (1.1), for example by using Berinde's method.

References

[1] A. Pansuwan and W. Sintunavarat. The New Hybrid Iterative Algorithm for Numerical Reckoning Fixed Points of Suzuki's Generalized Nonexpansive Mappings with Numerical Experiments. Thai J. Math. 2021; 19 (1): 157-68.

[2] W. R. Mann. Mean value methods in iteration. Proc. Amer. Math. Soc. 1953; 44: 506-10.

[3] S. Ishikawa. Fixed Point by a New Iteration Method. Proc. Amer. Math. Soc. 1974; 44 (1): 147-50.

[4] G. Jungck. Commuting mappings and fixed points. Amer. Math. Monthly. 1976; 83: 261-3.

[5] B. E. Rhoades. Some fixed point iteration procedures. Int. J. Math. Math. Sci. 1991; 14 (1): 1-16.

[6] M. A. Noor. New approximation schemes for general variational inequalities. J. Math. Anal. Appl. 2000; 251 (1): 217-29.

[7] R. P. Agarwal, D. O'Regan, D. R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. 2007; 8: 61-79.

[8] G. Pisier, Martingales in Banach spaces (Vol. 155). Cambridge University Press; 2016.

[9] T. Suzuki. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. 2008; 340 (2): 1088-95.

[10] A. M. Harder and T. L. Hicks. A stable iteration procedure for nonexpansive mappings. Math. Japon. 1988; 33 (5): 687-92.

[11] H. F. Senter and W. G. Dotson Jr. Approximating fixed points of nonexpansive mappings. Proc. Amer. Math. Soc. 1974; 44 (2): 375-80.

[12] A. Abkar and M. Eslamian. Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces. Fixed Point Theory Appl. 2010; Article ID 457935.

[13] N. Haokip. Convergence of an iteration scheme in convex metric spaces. Proyecciones (Antofagasta, On line). 2022; 41 (3): 777-90.