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# The Inverse Moyal Distribution and Its Properties

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**ABSTRACT**

In this paper, we introduce a new extension of the Moyal distribution, called the inverse Moyal (IMoyal) distribution, for heavy-tailed distribution. We derive its important statistical properties, such as the probability density function, cumulative distribution function, survival function, hazard function, and cumulative hazard function. Furthermore, we provide the formula for maximum likelihood estimation and perform simulation studies to assess its performance. Finally, we apply the IMoyal distribution to two real-world datasets: an automobile insurance premium data and a commercial bank asset data of Thailand. The analyses show that the proposed distribution demonstrates a superior fit to the data compared to existing distributions.

**Keywords:** Maximum likelihood estimation; Moyal distribution; Transformation method

## 1. Introduction

Datasets in various fields, such actuarial science, finance, and economics, have exhibited special characteristics such as unimodality, positive skewness, a bathtub-shaped distribution, and a thick right tail. Therefore, classical distributions are not sufficient to handle these features. In contrast, heavy-tailed distributions align with these characteristics and are more flexible than fundamental distributions for such data. Consequently, several distributions have been studied to model real-world datasets. There are many approaches to expanding distributions, such as a transformation method, a mixture method, and a composite model. The transformation method is the simplest and most widely applied approach for constructing new distributions. The original distribution is useful for the statistical mechanics field, known as the Moyal distribution. Moyal [1] introduced the Moyal distribution as an approximation of the Landau distribution [2] for quantum

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mechanics. It was described as energy loss due to ionization in a universal form, as mentioned in Walck [3]. In a real-world application, Aryal et al. [4] compared the Moyal distribution to the normal distribution to analyze the probability behavior of airline spill data. The Moyal distribution is in a location-scale family with the standard distribution, defined for real number $y$, as

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y+e^{-y})}.$$

In general form, a random variable $Z$ is said to have a Moyal distribution, $Z \sim \text{Moyal}(\mu, \sigma)$, if its density function, defined for all real number $z$, is

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\left(\frac{z-\mu}{\sigma}\right)+e^{-\left(\frac{z-\mu}{\sigma}\right)}\right)},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The corresponding cumulative distribution function is given by

$$\begin{aligned} F_Z(z) &= 1 - \frac{\gamma\left(\frac{1}{2}, \frac{1}{2}e^{-\left(\frac{z-\mu}{\sigma}\right)}\right)}{\Gamma\left(\frac{1}{2}\right)} \\ &= 1 - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{z-\mu}{2\sigma}\right)}\right), \end{aligned}$$

where $z, \mu \in \mathbb{R}, \sigma > 0, \gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt$ and the error function is defined by $\operatorname{erf}(x) = \frac{\gamma\left(\frac{1}{2}, x^2\right)}{\Gamma\left(\frac{1}{2}\right)}$. It is characterized by positive skewness and high kurtosis. These characteristics are useful for extreme data sets and capturing rare events. The Moyal distribution has been developed and applied to various fields. Cordeiro et al. [5] introduced the beta-Moyal (BMo) distribution by incorporating the Moyal distribution into the class of beta-generalized distributions. Genc et al. [6] extended this further with the beta Moyal-slash (BMSl) distribution. Bhati et al. [7] proposed the generalized log-Moyal (GlogM) distribution by applying an exponential transformation to the standard Moyal distribution. Li et al. [8] developed the generalized log-Moyal gamma (GLMGA) distribution by combining the GlogM and Gamma distributions. Insurance loss data have also been modeled by applying both the GlogM and GLMGA distributions. Furthermore, Arslan [9] introduced the $\alpha$-monotone generalized log-Moyal ($\alpha$-GlogM) distribution as a flexible distribution for environmental data. All extensions of the Moyal distribution can cope with heavier tail data sets. In this paper, we use the transformation method to derive a new extension of the Moyal distribution, called the inverse Moyal (IMoyal) distribution. The proposed distribution provides characteristics such as a heavy-tailed distribution and flexibly responds to an insurance dataset and a financial dataset.

This paper is structured as follows. Section 2 proposes the inverse Moyal distribution and its statistical properties. Section 3 provides the procedure of parameter estimation by using maximum likelihood estimation. Section 4 presents various performance simulation studies of the IMoyal distribution. Section 5 applies the IMoyal distribution to practical datasets. Section 6 provides a conclusion of this paper.

## 2. Properties

In this section, we propose the inverse Moyal distribution and explore its properties, such as the density function, distribution function, survival function, hazard function, and cumulative hazard function.

**Definition 2.1.** A random variable $X$ is called the inverse Moyal distribution, denoted as $X \sim \text{IMoyal}(\mu, \sigma)$ where

$$X = \begin{cases} 0, & \text{if } Z = 0 \\ \frac{1}{|Z|}, & \text{if } Z \neq 0 \end{cases},$$

and $Z$ follows the Moyal distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$.

**Theorem 2.2.** *The probability density function of the IMoyal($\mu, \sigma$) distribution is given by*

$$f_X(x) = \frac{x^{-2}}{\sqrt{2\pi\sigma^2}} \left( e^{-\frac{1}{2}\left(\frac{x^{-1}-\mu}{\sigma} + e^{-\left(\frac{x^{-1}-\mu}{\sigma}\right)}\right)} + e^{-\frac{1}{2}\left(\frac{-x^{-1}-\mu}{\sigma} + e^{-\left(\frac{-x^{-1}-\mu}{\sigma}\right)}\right)} \right), \tag{2.1}$$

*where $x > 0, \mu \in \mathbb{R}$, and $\sigma > 0$.*

*Proof.* By applying the inverse transform, let $\mathcal{Z} = \mathbb{R}, \mathcal{X} = \mathbb{R}^+$ and

$$g(z) = \begin{cases} 0, & \text{if } z = 0 \\ \frac{1}{|z|}, & \text{if } z \neq 0. \end{cases}$$

On $A_1 = (0, \infty)$: $g_1(z) = z^{-1}$ is monotone decreasing. Hence, $g_1^{-1}(x) = x^{-1}$.
On $A_2 = (-\infty, 0)$: $g_2(z) = -z^{-1}$ is monotone increasing. Hence, $g_2^{-1}(x) = -x^{-1}$.
By the density function of the Moyal distribution [1], the density function of the inverse Moyal distribution is

$$\begin{aligned} f_X(x) &= f_Z\left(g_1^{-1}(x)\right)\left|\frac{d}{dx}\left(g_1^{-1}(x)\right)\right| \\ &\quad + f_Z\left(g_2^{-1}(x)\right)\left|\frac{d}{dx}\left(g_2^{-1}(x)\right)\right| \\ &= x^{-2} f_Z\left(g_1^{-1}(x)\right) + x^{-2} f_Z\left(g_2^{-1}(x)\right) \\ &= x^{-2}\left(f_Z\left(x^{-1}\right) + f_Z\left(-x^{-1}\right)\right) \\ &= \frac{x^{-2}}{\sqrt{2\pi\sigma^2}} \left( e^{-\frac{1}{2}\left(\frac{x^{-1}-\mu}{\sigma} + e^{-\left(\frac{x^{-1}-\mu}{\sigma}\right)}\right)} \right. \\ &\quad \left. + e^{-\frac{1}{2}\left(\frac{-x^{-1}-\mu}{\sigma} + e^{-\left(\frac{-x^{-1}-\mu}{\sigma}\right)}\right)} \right), \end{aligned}$$

for $x > 0$. □

Fig. 1 illustrates density plots of the IMoyal distribution for different values of parameters. The densities are right-skewed, and the skewness increases as $\mu$ remains fixed and $\sigma$ increases. However, the curve provides a heavier tail as $\sigma$ decreases. Besides this, the density curves can be unimodal or bimodal, depending on the values of both parameters.

**Theorem 2.3.** *The distribution and survival functions of the IMoyal($\mu, \sigma$) are given by*

$$\begin{aligned} F_X(x) &= 1 - \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) \\ &\quad + \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right), \end{aligned}$$

*and*

$$S(x) = \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right), \tag{2.2}$$

*where $x > 0, \mu \in \mathbb{R}$, and $\sigma > 0$, respectively.*

*Proof.* By the definition of the distribution function, the distribution of the inverse Moyal can be expressed as:

$$\begin{aligned} F_X(x) &= \int_0^x \frac{u^{-2}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{u^{-1}-\mu}{\sigma} + e^{-\left(\frac{u^{-1}-\mu}{\sigma}\right)}\right)} du \\ &\quad + \int_0^x \frac{u^{-2}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{-u^{-1}-\mu}{\sigma} + e^{-\left(\frac{-u^{-1}-\mu}{\sigma}\right)}\right)} du \end{aligned}$$

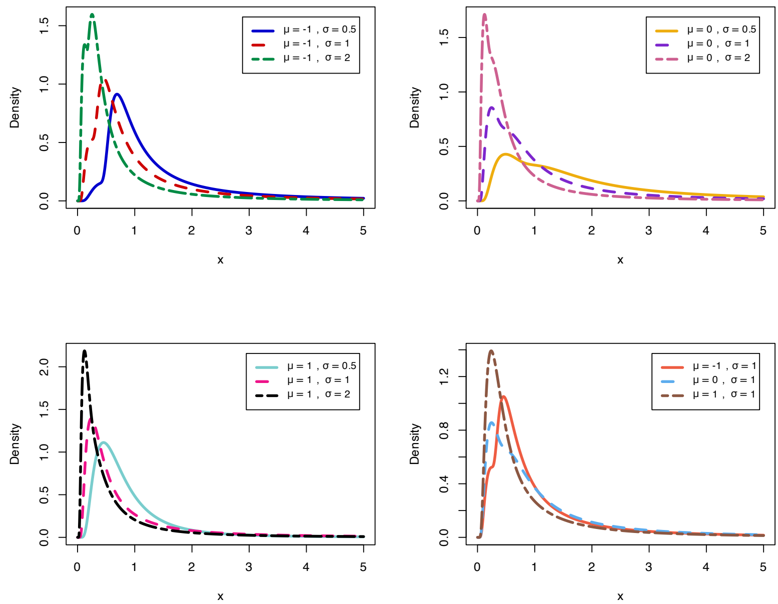


**Fig. 1.** The density functions of the inverse Moyal distribution with different parameters.

$$= \int_0^x u^{-2} f_Z\left(u^{-1}\right)\, du$$
$$+ \int_0^x u^{-2} f_Z(-u^{-1})\, du.$$

By changing variable $z = u^{-1}$, the distribution of the inverse Moyal is

$$\begin{aligned} F_X(x) &= \int_{x^{-1}}^{\infty} f_Z(z)\, dz + \int_{-\infty}^{-x^{-1}} f_Z(z)\, dz \\ &= 1 - F_Z(x^{-1}) + F_Z(-x^{-1}) \\ &= 1 - \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) \\ &\quad + \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right). \end{aligned}$$

Then the survival function is expressed as

$$S(x) = 1 - F_X(x)$$
$$= \operatorname{erf}\left(\tfrac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\tfrac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right).$$

□

**Remark 2.4.** *From the expression of the distribution function $F_x(x)$ obtained in Theorem 2.3, we can prove that $\int_0^\infty f_X(x)dx = \lim_{x\to\infty} F_X(x) = 1$. Therefore, the function $f_x(x)$ defined in Theorem 2.2 achieves the property of a density function.*

**Theorem 2.5.** *The hazard function and the cumulative hazard function of the IMoyal($\mu, \sigma$) are given by*

$$h(x)$$
$$= \frac{\frac{x^{-2}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x^{-1}-\mu}{\sigma} + e^{-\left(\frac{x^{-1}-\mu}{\sigma}\right)}\right)}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)}$$
$$+ \frac{\frac{x^{-2}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{-x^{-1}-\mu}{\sigma} + e^{-\left(\frac{-x^{-1}-\mu}{\sigma}\right)}\right)}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}} e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)}$$

*and*

$$H(x) = -\log\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)\right),$$

*where* $x > 0, \mu \in \mathbb{R}$, *and* $\sigma > 0$, *respectively.*

*Proof.* From the density function and the survival function in Eq. (2.1)-(2.2), the hazard function is

$$h(x) = \frac{f_X(x)}{S(x)}$$
$$= \frac{\frac{x^{-2}}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x^{-1}-\mu}{\sigma}+e^{-\left(\frac{x^{-1}-\mu}{\sigma}\right)}\right)}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)} + \frac{\frac{x^{-2}}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{-x^{-1}-\mu}{\sigma}+e^{-\left(\frac{-x^{-1}-\mu}{\sigma}\right)}\right)}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)}.$$

From the survival function in Eq. (2.2), the cumulative hazard function is

$$H(x) = -\log S(x)$$
$$= -\log\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)\right).$$

□

Fig. 2 exhibits the hazard functions of the IMoyal distribution for different parameter values, showing examples of constant and inverted bathtub shapes. The hazard curve remains approximately constant when $\sigma$ is close to zero. In contrast, for a larger value of values $\sigma$, the hazard rate increases significantly for lower values of $x$ and then drops rapidly as $x$ increases.

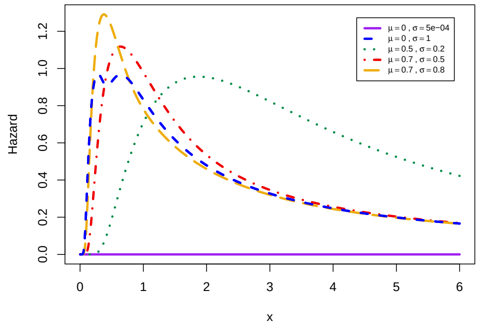


**Fig. 2.** The hazard functions of the inverse Moyal distribution with different parameters.

**Remark 2.6.** *The* $q$ *quantile (for* $0 < q < 1$*),* $x_q$*, of the inverse Moyal distribution is obtained by solving* $F_X(x_q) = q$*.*

**Theorem 2.7.** *The inverse Moyal distribution is a heavy-tailed distribution.*

*Proof.* Let $\alpha > 0$. From the survival function in Theorem 2.3, we have

$$\lim_{x\to\infty}\frac{S(x)}{e^{-\alpha x}}$$
$$= \lim_{x\to\infty}\frac{\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right) - \operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)}{e^{-\alpha x}}$$
$$= \frac{2}{\alpha\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(-\frac{\mu}{\sigma}-e^{\mu/\sigma}\right)}\lim_{x\to\infty}\frac{e^{\alpha x}}{x^2}$$
$$= \infty.$$

From Nair et al. [10], the IMoyal is a heavy-tailed distribution. □

**Remark 2.8.** *The expected value and moments of the IMoyal distribution do not exist.*

## 3. Parameter Estimation

In this section, we provide the mathematical formulae and the procedure for performing maximum likelihood estimation for the IMoyal distribution. The log-likelihood function for the parameter $\Theta := (\mu, \sigma)$, given the observations $\mathrm{x} = (x_1, \ldots, x_n)$, is expressed as follows:

$$\ell(\Theta|\mathrm{x}) = -\frac{n}{2}\log(2\pi) - n\log\sigma - 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(N(x_i;\mu,\sigma)\right),$$

where

$$N(x_i;\mu,\sigma) = e^{-\frac{1}{2}\left(\frac{x_i^{-1}-\mu}{\sigma} + e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)} + e^{-\frac{1}{2}\left(\frac{-x_i^{-1}-\mu}{\sigma} + e^{-\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)}\right)}.$$

The partial derivatives of $\ell(\Theta|\mathrm{x})$ with respect to parameters are given by

$$\frac{\partial\ell}{\partial\mu} = \sum_{i=1}^{n}\frac{M(x_i;\mu,\sigma) + M(-x_i;\mu,\sigma)}{2\sigma N(x_i;\mu,\sigma)}$$

and

$$\frac{\partial\ell}{\partial\sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{K(x_i;\mu,\sigma)}{2\sigma^2 N(x_i;\mu,\sigma)} + \sum_{i=1}^{n}\frac{K(-x_i;\mu,\sigma)}{2\sigma^2 N(x_i;\mu,\sigma)},$$

where

$$M(x_i;\mu,\sigma) = \left(1 - e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right) \times e^{-\frac{1}{2}\left(\frac{x_i^{-1}-\mu}{\sigma} + e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)}$$

and

$$K(x_i;\mu,\sigma) = \left(x_i^{-1} - \mu\right)\left(1 - e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right) \times e^{-\frac{1}{2}\left(\frac{x_i^{-1}-\mu}{\sigma} + e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)}$$

By equating the partial derivatives to zero, we obtain the maximum likelihood estimates for $\mu$ and $\sigma$. Since a closed form solution is not available, an optimization tool such as the optim function in R can be implemented to obtain the maximum likelihood estimates. In addition, we can obtain the asymptotic variance of the ML estimates by the estimated Fisher information matrix, defined as

$$\mathbf{I}(\Theta) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = -\mathbb{E}\begin{bmatrix} \frac{\partial^2\ell}{\partial\mu^2} & \frac{\partial^2\ell}{\partial\mu\partial\sigma} \\ \frac{\partial^2\ell}{\partial\sigma\partial\mu} & \frac{\partial^2\ell}{\partial\sigma^2} \end{bmatrix},$$

where the second order partial derivatives of $\ell(\Theta|\mathrm{x})$ are given by

$$\frac{\partial^2\ell}{\partial\mu^2} = \sum_{i=1}^{n}\frac{\left(1 - e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)M(x_i;\mu,\sigma)}{4\sigma^2 N(x_i;\mu,\sigma)} + \frac{\left(1 - e^{-\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)}\right)M(-x_i;\mu,\sigma) - 2L(x_i;\mu,\sigma)}{4\sigma^2 N(x_i;\mu,\sigma)} - \sum_{i=1}^{n}\frac{\left(M(x_i;\mu,\sigma) + M(-x_i;\mu,\sigma)\right)^2}{4\sigma^2 N(x_i;\mu,\sigma)^2},$$

$$\frac{\partial^2\ell}{\partial\mu\partial\sigma} = \frac{\partial^2\ell}{\partial\sigma\partial\mu} = \sum_{i=1}^{n}\frac{\left(1 - e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)K(x_i;\mu,\sigma)}{4\sigma^3 N(x_i;\mu,\sigma)} - \sum_{i=1}^{n}\frac{\left(1 - e^{-\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)}\right)K(-x_i;\mu,\sigma)}{4\sigma^3 N(x_i;\mu,\sigma)} - \sum_{i=1}^{n}\frac{P(x_i;\mu,\sigma) + P(-x_i;\mu,\sigma)}{2\sigma^3 N(x_i;\mu,\sigma)} - \sum_{i=1}^{n}\frac{\left(M(x_i;\mu,\sigma) + M(-x_i;\mu,\sigma)\right)K(x_i;\mu,\sigma)}{4\sigma^3 N(x_i;\mu,\sigma)^2} - \sum_{i=1}^{n}\frac{\left(\left(M(x_i;\mu,\sigma) + M(-x_i;\mu,\sigma)\right)K(-x_i;\mu,\sigma)\right)}{4\sigma^3 N(x_i;\mu,\sigma)^2}$$

$$-\sum_{i=1}^{n}\frac{M(x_i;\mu,\sigma)+M(-x_i;\mu,\sigma)}{2\sigma^2 N(x_i;\mu,\sigma)},$$

and

$$\frac{\partial^2\ell}{\partial\sigma^2}=\sum_{i=1}^{n}\frac{(x_i^{-1}-\mu)\left(1-e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)K(x_i;\mu,\sigma)}{4\sigma^4 N(x_i;\mu,\sigma)}$$
$$+\sum_{i=1}^{n}\frac{(-x_i^{-1}-\mu)\left(1-e^{-\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)}\right)K(-x_i;\mu,\sigma)}{4\sigma^4 N(x_i;\mu,\sigma)}$$
$$-\sum_{i=1}^{n}\frac{(x_i^{-1}-\mu)P(x_i;\mu,\sigma)}{2\sigma^4 N(x_i;\mu,\sigma)}$$
$$-\sum_{i=1}^{n}\frac{(-x_i^{-1}-\mu)P(-x_i;\mu,\sigma)}{2\sigma^4 N(x_i;\mu,\sigma)}$$
$$-\sum_{i=1}^{n}\frac{(K(x_i;\mu,\sigma)+K(-x_i;\mu,\sigma))^2}{2\sigma^4 N(x_i;\mu,\sigma)^2}$$
$$-\sum_{i=1}^{n}\frac{K(x_i;\mu,\sigma)+K(-x_i;\mu,\sigma)}{\sigma^4 N(x_i;\mu,\sigma)}+\frac{n}{\sigma^2},$$

where

$$L(x_i;\mu,\sigma)=e^{-\frac{1}{2}\left(3\left(\frac{x_i^{-1}-\mu}{\sigma}\right)+e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)}$$
$$+e^{-\frac{1}{2}\left(3\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)+e^{-\left(\frac{-x_i^{-1}-\mu}{\sigma}\right)}\right)}$$

and

$$P(x_i;\mu,\sigma)=(x_i^{-1}-\mu)e^{-\frac{1}{2}\left(3\left(\frac{x_i^{-1}-\mu}{\sigma}\right)+e^{-\left(\frac{x_i^{-1}-\mu}{\sigma}\right)}\right)}.$$

Under the regularity conditions, the asymptotic distribution of $\widehat{\Theta}$ for $\Theta$ is approximated by

$$\sqrt{n}(\widehat{\Theta}-\Theta)\to N_2(\mathbf{0},\mathbf{I}^{-1}(\Theta)),$$

where $\mathbf{I}^{-1}(\Theta)$ is the inverse of the Fisher information matrix. Consequently, the asymptotic $100(1-\alpha)\%$ confidence intervals for $\mu$ and $\sigma$ can be obtained as

$$\widehat{\mu}\pm z_{\alpha/2}\sqrt{\mathbf{I}_{11}^{-1}(\widehat{\Theta})}\text{ and }\widehat{\sigma}\pm z_{\alpha/2}\sqrt{\mathbf{I}_{22}^{-1}(\widehat{\Theta})},$$

respectively.

## 4. Simulation

In this section, we present simulation studies to evaluate the performance of the parameter estimates for different parameter values, as specified in Table 1. The sample sizes considered range from small to large, specifically 25, 50, 500, and 5000. For each sample size, we perform Monte Carlo simulations of size $N = 10000$.

**Table 1.** Scenarios of the simulation based on different values of parameters.

| Scenario | $\mu$ | $\sigma$ |
|---|---|---|
| 1 | 0.25 | 1 |
| 2 | 0.25 | 2.5 |
| 3 | 0.25 | 5 |
| 4 | 0.5 | 0.5 |
| 5 | 1 | 0.5 |
| 6 | 2 | 0.5 |

The data are generated by the inverse transform method described in Algorithm 1 as follows.

**Algorithm 1** Sample generation of the inverse Moyal distribution

**Input:** The parameters $\mu$ and $\sigma$ of the distribution

**Output:** The $x$ value

1: Generate $u \sim U(0,1)$

2: Solve $x$ by $F_X(x) = u$, where
$$F_X(x)=1-\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{-x^{-1}-\mu}{2\sigma}\right)}\right)+\operatorname{erf}\left(\frac{1}{\sqrt{2}}e^{-\left(\frac{x^{-1}-\mu}{2\sigma}\right)}\right)$$

3: **Return** $x$

The criteria used to evaluate the performance of parameter estimation are the average bias, the root mean squared errors (RMSEs), and the average asymptotic standard deviation (ASD), defined as

$$\text{Bias}(\widehat{w})=\frac{1}{N}\sum_{i=1}^{N}(\widehat{w_i}-w),$$

$$\mathrm{RMSE}(\widehat{w}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{w_i} - w)^2},$$

and

$$\mathrm{ASD}(\widehat{w}) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{R(\widehat{w_i})},$$

respectively, where $w \in \{\mu, \sigma\}$, $\widehat{w}$ is the corresponding estimator, and $R(\widehat{w_i})$ represents the asymptotic variance of the estimate of $w$ at iteration $i$. In addition, the estimators are evaluated using the coverage probability (CP), which corresponds to a nominal level of $(1-\alpha)\%$, and the estimated confidence interval, defined by the lower confidence interval (LCI) and the upper confidence interval (UCI). The confidence interval is given by

$$\overline{\widehat{w}} \pm z_{\alpha/2}\widehat{\mathrm{SE}(w)},$$

where

$$\overline{\widehat{w}} = \frac{1}{N}\sum_{i=1}^{N}\widehat{w_i}$$

and

$$\widehat{\mathrm{SE}(w)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}R(\widehat{w_i})}.$$

The maximum likelihood estimates for $\mu$ and $\sigma$, along with the corresponding criteria at the significance level $\alpha = 0.05$, for scenarios 1-6 are presented in Table 2.

From Table 2, we can see that the estimates are close to the true parameters, and the biases are also close to zero, for all sample sizes. Furthermore, as the sample size increases, both the RMSE and ASD values decrease and the coverage probabilities converge to 0.95. Across different sample sizes, the true parameters fall within the estimated confidence intervals for all scenarios. Hence, analytical findings support the validity and accuracy of the maximum likelihood estimation method for the IMoyal distribution.

## 5. Application

In this section, we apply the IMoyal distribution to two real-world datasets, especially an automobile insurance premium data and a commercial bank asset data of Thailand. Its performance is evaluated in comparison with several distributions from three main classifications, specifically the well-known heavy-tailed distributions, the Moyal family, and the right-tailed distributions. To begin with, the heavy-tailed distributions include the Lomax, Paralogistic, and Inverse Burr (IBurr) distributions. The Moyal family is analyzed by the generalized-log Moyal (GlogM) and the generalized-log Moyal gamma (GLMGA) distributions, and these distributions are heavy-tailed as well. Finally, the right-tailed distributions consist of the Gamma, the Weibull, and the Pareto distributions.

As evaluation criteria, we consider some important information criteria and goodness-of-fit statistics to assess the performance of the IMoyal distribution. The information criteria considered are Akaike Information Criterion (AIC), Akaike Information Corrected Criterion (AICC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn information (HQIC) given by

$$\begin{aligned}
\mathrm{AIC} &= 2k - 2\ell,\\
\mathrm{AICC} &= \mathrm{AIC} + \frac{2k(k+1)}{n-k-1},\\
\mathrm{BIC} &= k\log(n) - 2\ell,\\
\mathrm{CAIC} &= \mathrm{BIC} + p,\\
\mathrm{HQIC} &= 2k\log(\log(n)) - 2\ell,
\end{aligned}$$

where $\ell$ is the log-likelihood function, $n$ is

**Table 2.** Parameter estimation of IMoyal with different parameters.

| setting | $n$ | $\mu$ | | | | | | | $\sigma$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | MLE | Bias | RMSE | ASD | CP | LCI | UCI | MLE | Bias | RMSE | ASD | CP | LCI | UCI |
| $\mu = 0.25$, $\sigma = 1$ | 25 | 0.2489 | -0.0011 | 0.5686 | 0.4708 | 0.8139 | -0.7644 | 1.2622 | 0.9366 | -0.0634 | 0.2508 | 0.1981 | 0.8237 | 0.5318 | 1.3415 |
| | 50 | 0.2416 | -0.0084 | 0.4666 | 0.3846 | 0.8107 | -0.5741 | 1.0573 | 0.9585 | -0.0415 | 0.1881 | 0.1519 | 0.8479 | 0.6510 | 1.2661 |
| | 500 | 0.2432 | -0.0068 | 0.2044 | 0.1777 | 0.8858 | -0.1187 | 0.6051 | 0.9940 | -0.0060 | 0.0695 | 0.0621 | 0.9102 | 0.8697 | 1.1182 |
| | 5000 | 0.2496 | -0.0004 | 0.0619 | 0.0611 | 0.9434 | 0.1292 | 0.3701 | 0.9994 | -0.0006 | 0.0212 | 0.0209 | 0.9458 | 0.9583 | 1.0404 |
| $\mu = 0.25$, $\sigma = 2.5$ | 25 | 0.1455 | -0.1045 | 1.5397 | 1.2265 | 0.8016 | -2.4625 | 2.7534 | 2.3440 | -0.1560 | 0.5928 | 0.4866 | 0.8424 | 1.3552 | 3.3328 |
| | 50 | 0.1905 | -0.0595 | 1.2429 | 0.9942 | 0.8020 | -1.8925 | 2.2736 | 2.3989 | -0.1011 | 0.4454 | 0.3691 | 0.8636 | 1.6543 | 3.1435 |
| | 500 | 0.2520 | 0.0020 | 0.5134 | 0.4513 | 0.8879 | -0.6601 | 1.1642 | 2.4780 | -0.0220 | 0.1632 | 0.1438 | 0.9185 | 2.1896 | 2.7664 |
| | 5000 | 0.2506 | 0.0006 | 0.1525 | 0.1520 | 0.9454 | -0.0484 | 0.5496 | 2.4985 | -0.0015 | 0.0469 | 0.0464 | 0.9491 | 2.4072 | 2.5897 |
| $\mu = 0.25$, $\sigma = 5$ | 25 | 0.0265 | -0.2235 | 3.1237 | 2.4591 | 0.7964 | -5.2106 | 5.2637 | 4.6658 | -0.3342 | 1.1869 | 0.9668 | 0.8425 | 2.6966 | 6.6350 |
| | 50 | 0.1245 | -0.1255 | 2.4864 | 2.0094 | 0.8152 | -4.0884 | 4.3375 | 4.7846 | -0.2154 | 0.8791 | 0.7301 | 0.8653 | 3.3123 | 6.2569 |
| | 500 | 0.2707 | 0.0207 | 1.0240 | 0.9037 | 0.8955 | -1.5579 | 2.0992 | 4.9502 | -0.0498 | 0.3147 | 0.2799 | 0.9232 | 4.3876 | 5.5127 |
| | 5000 | 0.2484 | -0.0016 | 0.3065 | 0.3003 | 0.9437 | -0.3422 | 0.8390 | 4.9960 | -0.0040 | 0.0890 | 0.0888 | 0.9515 | 4.8214 | 5.1705 |
| $\mu = 0.5$, $\sigma = 0.5$ | 25 | 0.4652 | -0.0348 | 0.2485 | 0.2077 | 0.9199 | 0.0109 | 0.9195 | 0.4969 | -0.0031 | 0.1208 | 0.1073 | 0.8773 | 0.2755 | 0.7183 |
| | 50 | 0.4664 | -0.0336 | 0.2035 | 0.1572 | 0.9218 | 0.1234 | 0.8094 | 0.5041 | 0.0041 | 0.0943 | 0.0821 | 0.8874 | 0.3365 | 0.6717 |
| | 500 | 0.4959 | -0.0041 | 0.0561 | 0.0484 | 0.9527 | 0.3977 | 0.5942 | 0.5006 | 0.0006 | 0.0300 | 0.0282 | 0.9448 | 0.4446 | 0.5567 |
| | 5000 | 0.4998 | -0.0002 | 0.0147 | 0.0147 | 0.9537 | 0.4710 | 0.5287 | 0.5001 | 0.0001 | 0.0088 | 0.0088 | 0.9492 | 0.4829 | 0.5173 |
| $\mu = 1$, $\sigma = 0.5$ | 25 | 1.0057 | 0.0057 | 0.1722 | 0.1554 | 0.9312 | 0.6900 | 1.3214 | 0.4917 | -0.0083 | 0.0940 | 0.0880 | 0.9085 | 0.3132 | 0.6702 |
| | 50 | 1.0030 | 0.0030 | 0.1103 | 0.1089 | 0.9424 | 0.7876 | 1.2184 | 0.4951 | -0.0049 | 0.0627 | 0.0617 | 0.9314 | 0.3725 | 0.6178 |
| | 500 | 1.0005 | 0.0005 | 0.0342 | 0.0345 | 0.9529 | 0.9328 | 1.0682 | 0.4994 | -0.0006 | 0.0194 | 0.0194 | 0.9504 | 0.4613 | 0.5375 |
| | 5000 | 1.0002 | 0.0002 | 0.0108 | 0.0109 | 0.9525 | 0.9788 | 1.0216 | 0.4999 | -0.0001 | 0.0061 | 0.0061 | 0.9515 | 0.4879 | 0.5120 |
| $\mu = 2$, $\sigma = 0.5$ | 25 | 2.0168 | 0.0168 | 0.1555 | 0.1490 | 0.9340 | 1.7205 | 2.3131 | 0.4848 | -0.0152 | 0.0838 | 0.0812 | 0.9123 | 0.3233 | 0.6464 |
| | 50 | 2.0080 | 0.0080 | 0.1093 | 0.1071 | 0.9426 | 1.7966 | 2.2194 | 0.4927 | -0.0073 | 0.0583 | 0.0579 | 0.9311 | 0.3783 | 0.6071 |
| | 500 | 2.0005 | 0.0005 | 0.0339 | 0.0343 | 0.9498 | 1.9332 | 2.0678 | 0.4992 | -0.0008 | 0.0185 | 0.0184 | 0.9494 | 0.4630 | 0.5353 |
| | 5000 | 2.0002 | 0.0002 | 0.0108 | 0.0109 | 0.9509 | 1.9789 | 2.0215 | 0.4999 | -0.0001 | 0.0058 | 0.0058 | 0.9501 | 0.4885 | 0.5113 |

the sample size, and $k$ is the number of estimated parameters.

The goodness-of-fit statistics [11] are the Kolmogorov-Smirnov (KS) statistic, the Anderson-Darling statistic (AD) and the Cramer-von Mises statistic (CvM), presented as follows.

$$\mathrm{KS} = \max_i \left( \frac{i}{n} - F(z_i;\widehat{\Theta}),\ F(z_i;\widehat{\Theta}) - \frac{i-1}{n} \right),$$

$$\mathrm{AD} = -n - \sum_{i=1}^{n} \left(\frac{2i-1}{n}\right) \left(\log\left(F(z_{(i)};\widehat{\Theta})\right) + \log\left(1 - F(z_{(n+1-i)};\widehat{\Theta})\right)\right),$$

$$\mathrm{CvM} = \frac{1}{12n} + \sum_{i=1}^{n} \left( F(z_{(i)};\widehat{\Theta}) - \frac{2i-1}{2n} \right)^2,$$

where $z_{(\cdot)}$ and $\widehat{\Theta}$ are ordered ascending data and the estimated parameter vector, respectively.

## 5.1 The Automobile Insurance Premium Data

The first dataset considered is the automobile insurance premium (in millions dollars of insurers) in 2016, studied in Alsadat et al. [12]. The data are: 204.173, 84.769, 65.335, 62.505, 46.735, 43.693, 35.072, 32.511, 30.867, 30.397, 29.776, 26.249, 23.911, 23.263, 22.796, 20.956, 19.667, 18.093, 17.419, 16.934, 16.023, 15.572, 15.374, 14.056, 14.044, 14.028, 12.493, 11.733, 11.331, 11.177, 11.008, 10.772, 9.918, 9.881, 9.421, 9.399, 9.336, 9.141, 8.919, 8.253, 8.193, 7.647, 7.589, 7.493, 7.094, 6.635, 6.365, 6.259, 6.217, 5.416, 5.323, 4.405, 4.19, 4.109, 3.764, 3.613, 3.384, 3.106, 3.019, 2.979, 2.89, 2.843, 2.841, 2.77, 2.749, 2.643, 2.616, 2.55, 2.546, 2.433, 2.425, 2.403, 2.392, 2.087, 1.945, 1.833, 1.805, 1.798, 1.741, 1.719, 1.618, 1.584, 1.416, 1.407, 1.364, 1.358, 1.238, 1.199, and 1.048.

The automobile data consist of 89 observations, ranging from 1.048 to 204.173. The mean and standard deviation of the data are 14.079 and 25.266, respectively. The sample skewness and kurtosis are 5.312 and 37.969, respectively, which are consistent with the positive skewness and high kurtosis observed in the graphical behavior of the IMoyal curve. Numerical results of the information criteria and the goodness-of-fit statistics for the automobile insurance are presented in Table 3.

Table 3 presents the maximum likelihood estimates and numerical results of all distributions applied to the automobile data.The IMoyal distribution has the highest log-likelihood value (-305.840) and the lowest values of AIC, AICC, BIC, CAIC, and HQIC, which are 615.681, 615.820, 620.658, 622.658, and 617.687, respectively. In terms of the goodness-of-fit, the IMoyal distribution provides the highest p-values for the KS, AD, and CvM, compared to other distributions, particularly, the corresponding p-values are 0.462, 0.559, and 0.503, respectively, which are greater than the significance level $\alpha = 0.05$, indicating that the automobile insurance premium follows the proposed distribution.

In addition, we evaluate the performance of these distributions in fitting the automobile data by analyzing the fitted density function and P-P plot, which are presented in Figs. 3-4.

Fig. 3 shows the fitted density functions of various distributions over the range of 1.048 to 50. The IMoyal distribution provides the best-fitted density function, effectively capturing the uni-modal peak in the automobile data. In contrast, the IBurr, GlogM, and Pareto distributions exhibit fitted densities that do not align well with the density of the automobile data. From Fig. 4, the P-P plot of the fitted automobile data of the IMoyal distribution is closer to the 45-degree line of the expected probabilities and the theoretical probabilities than other distributions. The empirical evidence is in agreement with the numerical distribution values of the automobile data for the IMoyal, IBurr, GlogM and Pareto distributions are presented in Table 4.

The fitted cumulative distribution values of the IMoyal distribution closely match the observed probabilities, as shown in Table 4. To assess the flexibility of distributions for the datasets, we focus on the mean absolute error (MAE) of the probability, which is given by

$$\text{MAE}(\widehat{F}) = \frac{1}{n}\sum_{i=1}^{n}|F(z_{(i)};\widehat{\Theta}) - F(z_{(i)})|,$$

where $n$ is the sample size, $z_{(\cdot)}$ represents the ascending ordered data and $\widehat{F}$ is the estimated distribution function. In Table 5, we present the MAEs of the IMoyal, corresponding with the MAEs of the best representatives of the three competing families consisting of IBurr, GlogM, and Pareto distributions.

**Table 5.** MAE of the cumulative distribution function for the automobile insurance premium data.

| IMoyal | IBurr | GlogM | Pareto |
|---|---|---|---|
| **0.031** | 0.039 | 0.038 | 0.086 |

Table 5 indicates that the IMoyal distribution exhibits the lowest MAE, which is 0.031, compared to the IBurr, GlogM, and Pareto distributions. These findings confirm that the IMoyal distribution offers the best performance when applied to the automobile data compared to other well-known distributions used in insurance data.

## 5.2 The Commercial Bank Asset Data

The second dataset represents the commercial bank assets for the transferable deposits (in hundreds millions of baht) in Thailand from February 2003 to September 2019, available in `https://app.bot.or.th/BTWS_STAT/statistics/BOTWEBSTAT.aspx?reportID=14&language=ENG`. The data consist of 200 observations, with values ranging from 1622.280 to 6395.270. The

**Table 3.** The values of the selected criteria of the fitted distributions for the automobile insurance premium data.

| Parameters | Distributions | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Lomax | Paralogistic | Inverse Burr | GlogM | GLMGA | Inverse Moyal | Gamma | Weibull | Pareto |
| $\widehat{\alpha}$ | 2.648 | 1.329 | - | - | - | - | 0.816 | - | 1.048 |
| $\widehat{\lambda}$ | 22.938 | - | - | - | - | - | - | - | - |
| $\widehat{\theta}$ | - | 8.950 | 0.136 | - | - | - | - | - | - |
| $\widehat{\tau}$ | - | - | 36.041 | - | - | - | - | - | - |
| $\widehat{\gamma}$ | - | - | 1.06 | - | - | - | - | - | - |
| $\widehat{\mu}$ | - | - | - | 3.219 | - | 0.063 | - | - | - |
| $\widehat{\sigma}$ | - | - | - | 0.607 | 0.609 | 0.122 | 17.267 | 12.298 | - |
| $\widehat{a}$ | - | - | - | - | 177.905 | - | - | 0.822 | - |
| $\widehat{b}$ | - | - | - | - | 26.245 | - | - | - | - |
| $\widehat{k}$ | - | - | - | - | - | - | - | - | 0.534 |
| $\ell$ | -314.736 | -311.766 | -306.894 | -307.563 | -307.578 | **-305.840** | -323.065 | -320.412 | -315.547 |
| AIC | 633.471 | 627.532 | 619.788 | 619.125 | 621.155 | **615.681** | 650.130 | 644.823 | 635.095 |
| AICC | 633.611 | 627.672 | 620.070 | 619.265 | 621.438 | **615.820** | 650.270 | 644.963 | 635.234 |
| BIC | 638.448 | 632.510 | 622.765 | 624.102 | 624.132 | **620.658** | 655.107 | 649.800 | 640.072 |
| CAIC | 640.448 | 634.510 | 625.765 | 626.102 | 627.132 | **622.658** | 657.107 | 651.800 | 642.072 |
| HQIC | 635.477 | 629.539 | 622.797 | 621.131 | 624.164 | **617.687** | 652.136 | 646.829 | 637.101 |
| KS | 0.115 | 0.107 | 0.094 | 0.113 | 0.113 | 0.089 | 0.116 | 0.126 | 0.177 |
| p-value | 0.176 | 0.243 | 0.382 | 0.194 | 0.193 | **0.462** | 0.170 | 0.110 | 0.007 |
| AD | 1.255 | 0.972 | 0.954 | 1.105 | 1.091 | 0.699 | 2.652 | 1.973 | Inf |
| p-value | 0.247 | 0.372 | 0.382 | 0.306 | 0.313 | **0.559** | 0.041 | 0.095 | $<0.005$ |
| CvM | 0.138 | 0.135 | 0.182 | 0.210 | 0.207 | 0.118 | 0.437 | 0.264 | 0.860 |
| p-value | 0.428 | 0.438 | 0.305 | 0.250 | 0.254 | **0.503** | 0.058 | 0.171 | 0.005 |

**Table 4.** Cumulative distribution values at each tenth percentile for the automobile insurance premium data.

| $i$ | $z_{(i)}$ | $F(z_{(i)})$ | $F(z_{(i)};\widehat{\Theta})$ | | | |
|---|---|---|---|---|---|---|
| | | | IMoyal | IBurr | GlogM | Pareto |
| 9 | 1.618 | 0.1 | 0.082 | 0.080 | 0.078 | 0.207 |
| 18 | 2.403 | 0.2 | 0.186 | 0.187 | 0.203 | 0.358 |
| 27 | 2.841 | 0.3 | 0.241 | 0.244 | 0.268 | 0.413 |
| 36 | 4.109 | 0.4 | 0.368 | 0.383 | 0.414 | 0.518 |
| 45 | 7.094 | 0.5 | 0.554 | 0.582 | 0.602 | 0.640 |
| 54 | 9.399 | 0.6 | 0.643 | 0.669 | 0.679 | 0.690 |
| 63 | 12.493 | 0.7 | 0.722 | 0.742 | 0.744 | 0.734 |
| 72 | 18.093 | 0.8 | 0.803 | 0.818 | 0.810 | 0.782 |
| 81 | 30.867 | 0.9 | 0.882 | 0.892 | 0.877 | 0.836 |
| 89 | 204.173 | 1 | 0.982 | 0.985 | 0.974 | 0.940 |

mean and standard deviation are 3613.242 and 999.194, respectively. In addition, the sample skewness and kurtosis are 0.240 and 2.422, respectively. These data are right-skewed with low kurtosis; however, the IMoyal distribution, which is the heavier-tailed distribution, achieves satisfactory performance on this dataset. Numerical results of the information criteria and the goodness-of-fit statistics for the commercial bank asset data are presented in Table 6.

The maximum likelihood estimates of the IMoyal distribution produce higher

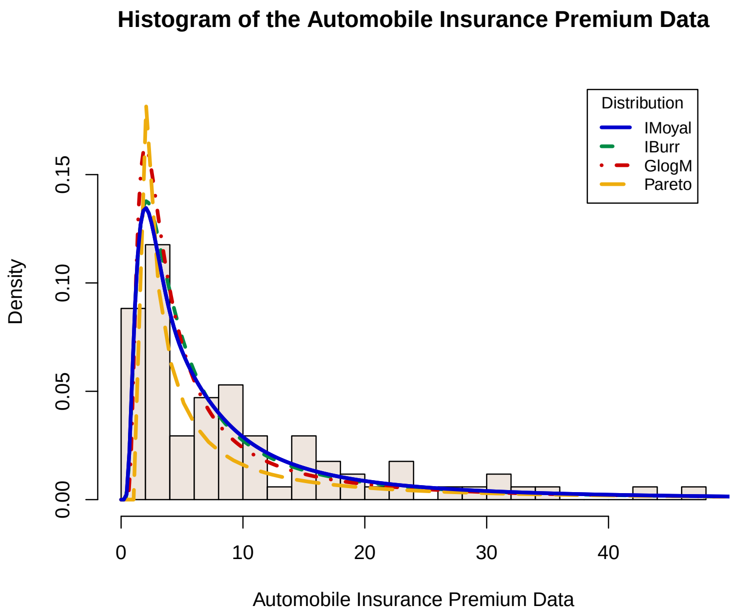


**Fig. 3.** The histogram of the automobile insurance premium data overlaid with fitted density functions of the IMoyal, IBurr, GlogM, and Pareto distributions.

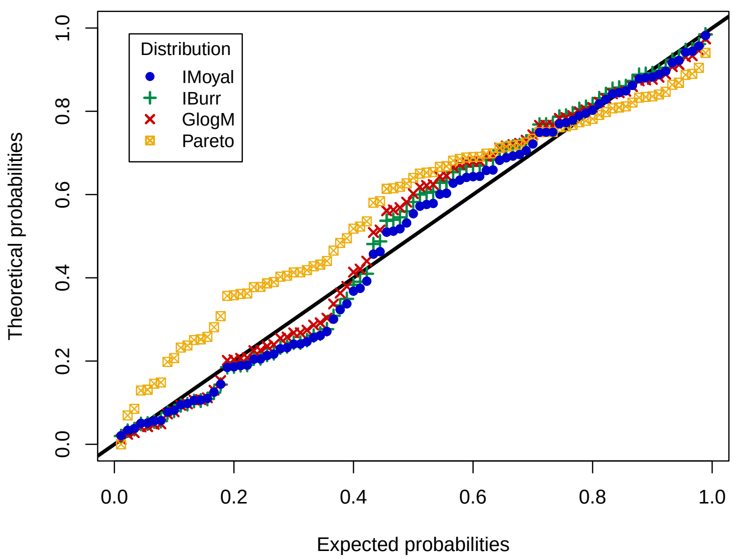


**Fig. 4.** P-P plot of the automobile insurance premium data for the IMoyal, IBurr, GlogM, and Pareto distributions.

**Table 6.** The values of the selected criteria of the fitted distributions for the commercial bank asset data.

| Parameters | Distributions | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Lomax | Paralogistic | Inverse Burr | GlogM | GLMGA | Inverse Moyal | Gamma | Weibull | Pareto |
| $\widehat{\alpha}$ | $5.339 \times 10^{6}$ | 4.500 | - | - | - | - | 12.705 | - | 1622.280 |
| $\widehat{\lambda}$ | $1.931 \times 10^{10}$ | - | - | - | - | - | - | - | - |
| $\widehat{\theta}$ | - | 5364.088 | 396.398 | - | - | - | - | - | - |
| $\widehat{\tau}$ | - | - | 1117.759 | - | - | - | - | - | - |
| $\widehat{\gamma}$ | - | - | 3.469 | - | - | - | - | - | - |
| $\widehat{\mu}$ | - | - | - | 2816.719 | - | $2.455 \times 10^{-4}$ | - | - | - |
| $\widehat{\sigma}$ | - | - | - | 0.197 | 0.167 | $4.269 \times 10^{-5}$ | 284.230 | 3989.058 | - |
| $\widehat{a}$ | - | - | - | - | 2.962 | - | - | 3.989 | - |
| $\widehat{b}$ | - | - | - | - | $4.871 \times 10^{-21}$ | - | - | - | - |
| $\widehat{k}$ | - | - | - | - | - | - | - | - | 1.316 |
| $\ell$ | -1838.472 | -1664.264 | -1683.733 | -1695.518 | -1687.160 | **-1660.948** | -1662.551 | -1664.940 | -1775.880 |
| AIC | 3680.945 | 3332.528 | 3373.466 | 3395.037 | 3380.320 | **3325.896** | 3329.101 | 3333.880 | 3555.759 |
| AICC | 3681.005 | 3332.588 | 3373.589 | 3395.098 | 3380.442 | **3325.957** | 3329.162 | 3333.941 | 3555.820 |
| BIC | 3687.541 | 3339.124 | 3378.063 | 3401.633 | 3384.916 | **3332.493** | 3335.698 | 3340.477 | 3562.356 |
| CAIC | 3689.541 | 3341.124 | 3381.063 | 3403.633 | 3387.916 | **3334.493** | 3337.698 | 3342.477 | 3564.356 |
| HQIC | 3683.614 | 3335.197 | 3377.470 | 3397.706 | 3384.324 | **3328.566** | 3331.771 | 3336.550 | 3558.429 |
| KS | 0.404 | 0.066 | 0.110 | 0.123 | 0.107 | 0.057 | 0.064 | 0.070 | 0.291 |
| p-value | <0.005 | 0.347 | 0.016 | 0.005 | 0.021 | **0.535** | 0.389 | 0.279 | <0.005 |
| AD | 47.974 | 0.934 | 4.007 | 5.797 | 4.201 | 0.750 | 1.051 | 0.867 | Inf |
| p-value | <0.005 | 0.394 | 0.009 | 0.001 | 0.007 | **0.518** | 0.332 | 0.435 | <0.005 |
| CvM | 9.847 | 0.152 | 0.634 | 0.925 | 0.616 | 0.129 | 0.186 | 0.122 | 6.644 |
| p-value | <0.005 | 0.384 | 0.018 | 0.004 | 0.020 | 0.462 | 0.297 | **0.488** | <0.005 |

log-likelihood values (-1660.948) and lower AIC, AICC, BIC, CAIC, and HQIC values, which are 3325.896, 3325.957, 3332.493, 3334.493, and 3328.566, respectively, compared to other distributions as shown in Table 6. For the goodness-of-fit statistics, the IMoyal distribution provides the hightest p-values for the KS and AD, which are 0.535 and 0.518, respectively. Considering the CvM goodness-of-fit, we can see that the IMoyal distribution provides higher p-values than other competing distributions except the Weibull distribution, whereas the two distributions give similar values which are higher than other distributions. However, considering all criteria, the IMoyal distribution achieves the best overall performances for modelling the commercial bank asset data. Among other classifications, the Paralogistic, GLMGA, and Gamma distributions are strong competitors to the IMoyal distribution for the commercial bank data. Subsequently, the IMoyal distribution is also analyzed through visualizations to assess its flexibility compared to these distributions, using the fitted density function and P-P plot, presented in Figs. 5-6. Fig. 5 shows that the fitted density curve of the IMoyal distribution is closest to the observed data and aligns with the right-skewness of the commercial bank asset data. Despite the small kurtosis of the data, the IMoyal distribution provides a flexible fit. While, other heavy-tailed or right-skewed distributions do not perform as well. In addition, from the observed-versus-fitted probabilities (P-P plot) in Figure 6, we can see that the IMoyal distribution is the most suitable for modeling the commercial bank asset data, as its points correspond to the 45 degree line compared to other distributions, as shown in Figure 6. The numerical cumulative distribution values of the commercial bank asset data for the IMoyal, Paralogistic, GLMGA, and Gamma distributions are presented in Table 7.

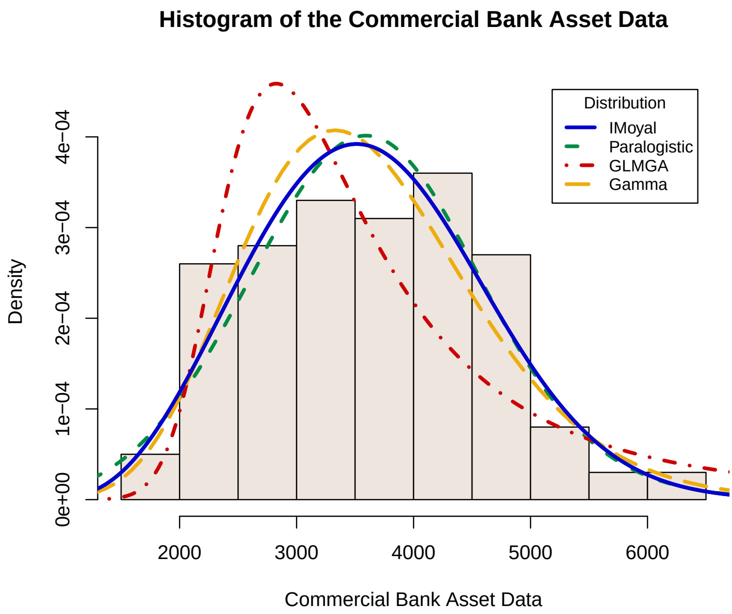


**Fig. 5.** The histogram of the commercial bank asset data overlaid with fitted density functions of the IMoyal, Paralogistic, GLMGA, and Gamma distributions.

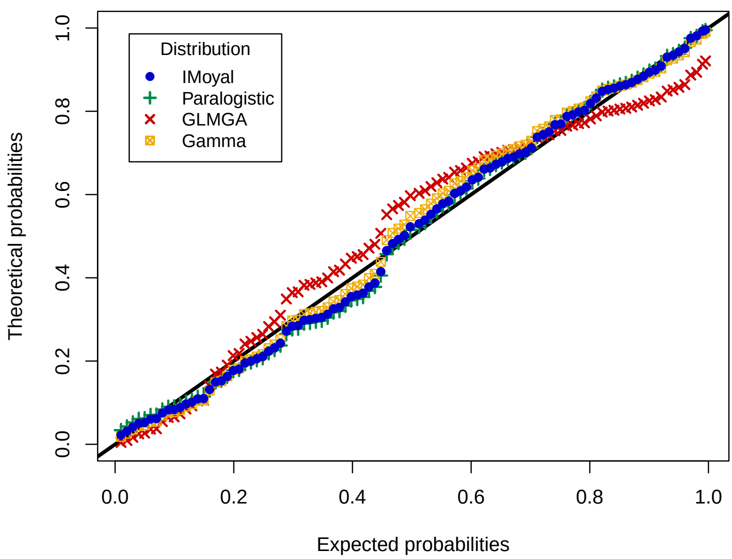


**Fig. 6.** P-P plot of the commercial bank asset data for the IMoyal, Paralogistic, GLMGA, and Gamma distributions.

**Table 7.** Cumulative distribution values at each tenth percentile for the commercial bank asset data.

| $i$ | $z_{(i)}$ | $F(z_{(i)})$ | $F(z_{(i)};\widehat{\Theta})$ | | | |
|---|---|---|---|---|---|---|
| | | | IMoyal | Paralogistic | GLMGA | Gamma |
| 20 | 2282.05 | 0.1 | 0.083 | 0.091 | 0.066 | 0.077 |
| 40 | 2678.67 | 0.2 | 0.177 | 0.176 | 0.213 | 0.179 |
| 60 | 3012.50 | 0.3 | 0.284 | 0.276 | 0.364 | 0.297 |
| 80 | 3207.82 | 0.4 | 0.355 | 0.346 | 0.447 | 0.375 |
| 100 | 3639.71 | 0.5 | 0.522 | 0.515 | 0.597 | 0.548 |
| 121 | 3938.41 | 0.6 | 0.635 | 0.632 | 0.675 | 0.658 |
| 141 | 4160.86 | 0.7 | 0.712 | 0.712 | 0.721 | 0.729 |
| 161 | 4529.31 | 0.8 | 0.819 | 0.822 | 0.782 | 0.824 |
| 181 | 4888.48 | 0.9 | 0.894 | 0.897 | 0.825 | 0.890 |
| 200 | 6395.27 | 1 | 0.996 | 0.995 | 0.921 | 0.990 |

From Table 7, we can see that the theoretical probabilities of the IMoyal distribution closely match the expected probabilities across different percentiles. Finally, the performance of the distributions for the commercial bank asset data is evaluated using the MAE, as presented in Table 8.

**Table 8.** MAE of the cumulative distribution function for the commercial bank asset data.

| IMoyal | Paralogistic | GLMGA | Gamma |
|---|---|---|---|
| **0.022** | 0.023 | 0.047 | 0.027 |

The IMoyal distribution yields the lowest Mean Absolute Error (MAE) of 0.022 among all competing distributions applied to the commercial bank asset data, as shown in Table 8. Therefore, the results are consistent with all previous results that the IMoyal distribution achieves the best fit to commercial data.

## 6. Conclusion

This paper proposed a new heavy-tailed distribution, which is an extension of the Moyal distribution, known as the IMoyal distribution. The model broadens the applicability of the Moyal distribution to heavy-tailed data frequently encountered in real-world data. In the study, we provided the mathematical properties of the IMoyal distribution and obtained its maximum likelihood parameter estimation. Moreover, we investigated the performance of the maximum likelihood estimation through simulation studies. As part of the application evaluation, we demonstrated the advantages of the proposed distribution through two applications in actuarial science and finance. Empirical findings indicate that the IMoyal distribution exhibits outstanding performance in comparison to existing alternatives. Several extensions of this study may be explored in future research, including the application of the distribution to heavy-tailed data in other fields beyond actuarial science and finance. Moreover, one could consider an extension of the distribution using a generalized inverse transform to overcome the limitation posed by the nonexistence of moments in the IMoyal distribution.

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