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Pivot-Based Parameter Estimation for the Gumbel Distribution with Progressive Type-II Censoring Scheme

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ABSTRACT

In this article, we propose a parameter estimation method based on pivotal quantities for the Gumbel distribution following the progressive Type-II censoring scheme. We also conduct a simulation study to assess the efficiency of the proposed method compared to the classical maximum likelihood estimation method. Simulation results indicate that a pivot-based method performs better than the likelihood method in terms of both bias and mean squared error, particularly for the case of small sample sizes. Finally, we apply the proposed method to two real applications and evaluate its performance through different goodness-of-fit criteria. The results demonstrate competitive performance of the method across both applications.

Keywords: Parameter estimation; Pivotal quantity; Point estimation; Progressive Type-II censoring; The Gumbel distribution

1. Introduction

The Gumbel distribution, also known as the extreme value distribution, is an important distribution widely applied in different contexts such as in meteorology (Gumbel [1]), hydrology (Rao and Hamed [2]), finance (McNeil et al. [3]), and engineering (Castillo [4]). For instance, Heinz et al. [5] introduced homogeneity tests based on the extreme value distribution, Loaiciga [6] utilized the Gumbel distribution to analyze extreme hydrologic events, and Huang et al. [7] employed the Gumbel distribution to study the pit corrosion depth of concrete girder bridges.

Moreover, the Gumbel distribution has been extended to study different types of censored data usually occurring in med-

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ical data such as Type-I censoring (Raynal and Salas [8]) and Type-II censoring (Abbas et al.[9]). One widely applied censoring scheme is the Type-II progressive censoring which was first introduced in Herd [10] in terms of the multi-censored sample (Balakrishnan and Aggarwala, [11]). The concept of the progressive Type-II is that a generalization of Type-II censoring occurs when the experiment is divided into m channels. In this case, the numbers to be censored in different channels $R=(R_1, R_2, ..., R_m)$, known as the censoring scheme, are fixed, but the experimental times, $T_1, T_2, ..., T_m$, are not fixed. Specifically, in a life-testing experiment with a total of n units, only m units are fully observed until failure. When the first unit fails, R_1 of the remaining $n-1$ units are randomly removed (censored) from the test. After the next failure, R_2 of the remaining $n-2-R_1$ units are censored, and this process continues. Finally, when the m-th failure occurs, all the remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ units are censored. Since the progressive Type-II is known to be an efficient censoring scheme that can save time and budgets, it has been widely applied in different contexts. For example, Mann [12] introduced a new estimator for the Weibull distribution using the concept of order statistics and linear regression under progressive Type-II censoring. Furthermore, Balakrishnan et al. [13] developed recurrence relations for the single and product moments of progressively Type-II censored data from a logistic distribution. They also investigated the properties of these relations, including the means, variances, and covariances. Several studies of the progressive for Gumbel distribution have been conducted in literature. For instance, the censoring scheme was discussed intensively in Balakrishnan and Aggarwala [11]. Parameter estimation for the progressive Type-II censoring for the Gumbel distribution was investigated in Balakrishnan et al. [14] where the author introduced an approximation of the maximum likelihood estimation (AMLE) by using the Taylor series expansion. Their results suggest that the approximated estimates are similar to the maximum likelihood estimation (MLE). Later, Salem et al. [15] studied the maximum likelihood estimator (MLE) and the Bayes estimator for the Gumbel distribution with progressive Type-II censoring.

Many parameter estimation techniques to the maximum likelihood estimation method have been investigated in literature. One such technique is the pivot-based estimation technique which has been proposed for many aspects. For example, Kim et al. [16] proposed a new parameter estimation based pivotal quantity to obtain parameter estimation for the Pareto distribution. The authors showed that the Pivot-based method performs better than the maximum likelihood estimation technique. Therefore, the approach has been widely extended in literature. For instance, Seo et al. [17] extended the pivotal quantity method of Kim et al. [16] to estimate the parameters of the Pareto distribution with Type-II censoring scheme. Shang and Gui [18] proposed a pivotal inference for the Rayleigh distribution based on progressive first-failure censored data, combining the pivotal quantity method with the modified method of moments introduced by Kirimi et al. [19]. Hussain et al. [20] also used the pivotal quantity method with the least squares estimation and order statistics to obtain parameter estimation of the Pareto distribution.

In this study, we extend the pivot-based parameter estimation method of Shang and Gui [7] to the Gumbel distribution under the progressive Type-II censor-

ing scheme. Moreover, we evaluate its performance through a simulation study and two real applications. The remainder of the paper is organized as follows. Section 2 presents the maximum likelihood estimator (MLE), which is used as a benchmark for comparison with the proposed PQ method. Section 3 introduces the proposed parameter estimation approach. Section 4 presents simulation results comparing the method with the maximum likelihood method. Section 5 applies the method to two real datasets. Finally, a discussion is provided in Section 6.

2. Maximum Likelihood Estimation

Let m be the number of failures before the termination of study and $X_1 < X_2 < \cdots < X_m$ denote a progressively Type-II censored sample from the Gumbel distribution with the probability density function

$$f_X(x) = \frac{1}{\sigma} e^{(x-\mu)/\sigma} e^{-e^{(x-\mu)/\sigma}},$$

and the cumulative distribution function (CDF)

$$F_X(x) = 1 - e^{-e^{(x-\mu)/\sigma}},$$

for $-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$. The progressive Type-II censoring scheme is described by $(R_1, \ldots, R_m)$ where R_i $(i = 1, ..., m)$ denotes the number of units removed after i^{th} failure. The likelihood function is given by

$$L(\mu, \sigma) = C \prod_{i=1}^{m} f(x_i; \mu, \sigma) \left[1 - F(x_i; \mu, \sigma)\right]^{R_i},$$

where

$$C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots \left(n - \sum_{i=1}^{m-1} R_i - m + 1\right),$$

is a constant that depends only on the censoring scheme. The log-likelihood function is therefore

$$\ell(\mu, \sigma) = \log C + \sum_{i=1}^{m} \log f(x_i; \mu, \sigma) + \sum_{i=1}^{m} R_i \log\left[1 - F(x_i; \mu, \sigma)\right].$$

The maximum likelihood estimators are obtained by solving the score equations

$$\frac{\partial \ell(\mu, \sigma)}{\partial \mu} = 0, \qquad \frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = 0,$$

where these equations may be evaluated utilizing a numerical solver such as the R's `nleqslv` function.

3. Parameter estimation based on pivotal quantities

This section introduces the proposed estimation method. The estimating equations are presented first, followed by the theoretical results that support the proposed approach.

3.1 The Method

To obtain the pivot-based parameter estimation, we first transform the sample to a standard exponential distribution and then apply the properties of the chi-squared distribution. Specifically, for each $i = 1, 2, ..., m$, define $Y_i = \exp\left(\frac{X_i - \mu}{\sigma}\right)$, then we obtain in Theorem 6.1 that $Y_1 < Y_2 < \cdots < Y_m$ form a progressively Type-II censored sample from a standard exponential distribution. Having obtained Y_i, we construct a new set of variables T_i, where

$$T_1 = 2nY_1,$$
$$T_i = 2\left[m - i + 1 + \sum_{j=i}^{m} R_j\right](Y_i - Y_{i-1}); (i = 2, ..., m).$$

By the same technique as in Shang and Gui [18], we prove in Theorem 6.2 that the ran-

dom variables $T_i (i = 2, ..., m)$ are independent and identically distributed as $\chi^2(2)$. Therefore, (see Lemma 6.2)

$$U_m = \sum_{i=1}^{m} T_i = 2 \sum_{i=1}^{m} (1 + R_i) \exp\left(\frac{X_i - \mu}{\sigma}\right), \tag{3.1}$$

follows a chi-squared distribution with $2m$ degrees of freedom with the expectation

$$E(U_m) = 2m, \tag{3.2}$$

and the variance

$$\mathrm{Var}(U_m) = 4m.$$

Therefore, the pivotal quantity $\frac{U_m}{2m}$, has mean and variance

$$E\left(\frac{U_m}{2m}\right) = 1 \text{ and } \mathrm{Var}\left(\frac{U_m}{2m}\right) = \frac{1}{m},$$

respectively. Integrating these results with the Chebyshev's inequality, the pivotal quantity $\frac{U_m}{2m}$ converges to one in probability as m goes to infinity.
Similarly, define

$$\begin{aligned} S_m &= \sum_{i=2}^{m} T_i \\ &= U_m - T_1 \\ &= 2 \sum_{i=1}^{m} (1 + R_i) \exp\left(\frac{X_i - \mu}{\sigma}\right) \\ &\quad - 2n \exp\left(\frac{X_1 - \mu}{\sigma}\right). \end{aligned}$$

Then, S_m follows a chi-squared distribution with $2m - 2$ degrees of freedom with the expectation

$$E(S_m) = 2(m - 1),$$

and the variance

$$\mathrm{Var}(S_m) = 4(m - 1).$$

Consequently, by the independence between T_1 and S_m, the statistic Q_m, defined as

$$Q_m = \frac{T_1/2}{S_m/[2m - 2]} \tag{3.3}$$

$$= \frac{(m-1)ne^{\frac{1}{\sigma}(X_1 - \mu)}}{\sum_{i=1}^{m} (1 + R_i) e^{\frac{1}{\sigma}(X_i - \mu)} - ne^{\frac{1}{\sigma}(X_1 - \mu)}}, \tag{3.4}$$

is a pivotal quantity following $F(2, 2(m - 1))$ distribution. Accordingly, we obtain

$$E\left(\frac{Q_m}{m-1} - \frac{1}{m-2}\right) = 0, \tag{3.5}$$

and

$$\begin{aligned} \mathrm{Var}\left(\frac{Q_m}{m-1} - \frac{1}{m-2}\right) &= \frac{\mathrm{Var}(Q_m)}{(m-1)^2} \\ &= \frac{m-1}{(m-2)^2(m-3)}, \end{aligned}$$

converging to zero as m approaches to infinity. Therefore, by Chebyshev's inequality,

$$\frac{Q_m}{m-1} - \frac{1}{m-2} \xrightarrow{P} 0.$$

Following Shang and Gui [18], the estimators of μ and σ are obtained based on the asymptotic properties of the pivotal quantities. By the Weak Law of Large Numbers, the pivotal quantities converge in probability to their corresponding expectations. Therefore, from Eqs. (3.1)-(3.2) and Eqs. (3.3)-(3.5), the estimating equations are given by

$$U_m = 2m, \tag{3.6}$$

and

$$Q_m = \frac{T_1/2}{S_m/(2(m-1))} = \frac{m-1}{m-2}, \tag{3.7}$$

respectively. Using Eqs. (3.6)-(3.7), the estimators $\hat{\mu}$ and $\hat{\sigma}$ can be obtained by solving the following two equations

$$\frac{(m-1)ne^{\frac{x_1}{\sigma}}}{\sum_{i=1}^{m}(1+R_i)e^{\frac{x_i}{\sigma}}-ne^{\frac{x_1}{\sigma}}}=\frac{m-1}{m-2}, \tag{3.8}$$

and

$$2\sum_{i=1}^{m}(1+R_i)\exp\left\{\left(\frac{X_i-\hat{\mu}}{\hat{\sigma}}\right)\right\}=2m. \tag{3.9}$$

Rearranging Eq. (3.8), we can see that $\hat{\sigma}$ can be obtained by numerically solving

$$\frac{\sum_{i=1}^{m}(1+R_i)e^{x_i/\hat{\sigma}}}{e^{x_1/\hat{\sigma}}}=n(m-1), \tag{3.10}$$

which can be done by a nonlinear equation solver such as the R's `nleqslv` function. Substituting the obtained $\hat{\sigma}$ into Eq. (3.9), we can obtain the estimator $\hat{\mu}$. In particular, Eq. (3.10) yields,

$$\hat{\mu}=\hat{\sigma}\log\left(\sum_{i=1}^{m}(1+R_i)e^{x_i/\hat{\sigma}}\right). \tag{3.11}$$

3.2 Theoretical Results

In this theorem, we present important theoretical properties of the proposed estimators $\hat{\sigma}$ and $\hat{\mu}$ constructed in Section 3.1. We begin by showing that the estimating equation for σ has a unique positive solution, ensuring the uniqueness of $\hat{\sigma}$. We then establish the consistency of $\hat{\sigma}$, followed by the consistency of $\hat{\mu}$.

Proposition 3.1. *For $m > 2$, the estimating equation*

$$\sum_{i=1}^{m}(1+R_i)\exp\left(\frac{x_i-x_1}{\sigma}\right)=n(m-1), \tag{3.12}$$

has the unique positive solution for σ, say $\hat{\sigma}$.

Proof. Let $h(\sigma)$ be the term on the left side defined as a function of σ,

$$h(\sigma)=\sum_{i=1}^{m}(1+R_i)\exp\left(\frac{x_i-x_1}{\sigma}\right);\sigma>0.$$

Then $h(\sigma)$ is a continuous function of σ. Therefore, to prove the main result, we will show that the function h is strictly decreasing on $[0,\infty)$. In particular, since $x_1 < x_2 < \cdots < x_m$, we obtain

$$h'(\sigma)=-\sum_{i=2}^{m}(1+R_i)\frac{x_i-x_1}{\sigma^2}\exp\left(\frac{x_i-x_1}{\sigma}\right)<0.$$

In addition,

$$\lim_{\sigma\to 0^+}h(\sigma)=\infty,$$

and

$$\lim_{\sigma\to\infty}h(\sigma)=\sum_{i=1}^{m}(1+R_i)=n.$$

Consequently, $h(\sigma)$ decreases continuously from infinity to n and crosses $n(m-1) > n$ exactly once. Therefore, the theorem is proved. □

Theorem 3.2. *The proposed estimators $\hat{\sigma}$ and $\hat{\mu}$ are consistent estimators of the parameters σ and μ, respectively.*

Proof. From Eq. (3.11), we can see that $\hat{\mu}$ can be regarded as a continuous function of $\hat{\sigma}$. Therefore, an integration of the consistency of $\hat{\sigma}$ and the continuous mapping theorem leads to the consistency of $\hat{\mu}$. As a result, it remains to prove the consistency of $\hat{\sigma}$. To achieve this, for any given parameter vector (μ,σ), we first define

$$Y_i=\exp\left(\frac{X_i-\mu}{\sigma}\right),\qquad \hat{a}=\frac{\sigma}{\hat{\sigma}}.$$

Then,

$$X_i-X_1=\sigma\log\left(\frac{Y_i}{Y_1}\right),$$

and the estimating Eq. (3.12) can be written as

$$\sum_{i=1}^{m}(1+R_i)\left(\frac{Y_i}{Y_1}\right)^{\hat{a}} = n(m-1).$$

Define $M_m(a) = \sum_{i=1}^{m} \frac{(1+R_i)}{n}\left(\frac{Y_i}{Y_1}\right)^{a}$.
Then,

$$M_m(\hat{a}) = m-1. \qquad (3.13)$$

From the definition of Q_m,

$$M_m(1) = 1+\frac{m-1}{Q_m} = \frac{m-1}{Q_m}\left(1+\frac{Q_m}{m-1}\right).$$

Therefore,

$$\log M_m(1) = \log(m-1) - \log Q_m + \log\left(1+\frac{Q_m}{m-1}\right). \qquad (3.14)$$

To prove the main result, we first define $r_m = \dfrac{\log M_m(1)}{\log(m-1)}$, and prove the following statements.

1.) r_m converges to one in probability as m approaches infinity.

2.) For the estimator $\hat{a}$, we obtain that

a) $1 \le \hat{a} \le (r_m)^{-1}$ if $\hat{a} \ge 1$,

b) $(r_m)^{-1} \le \hat{a} \le 1$ if $0 < \hat{a} < 1$.

To prove 1.), we first consider the limit in distribution of Q_m defined in Eq. (3.3). We first know that $\frac{T_1}{2} \sim \text{Exp}(1)$. In addition, the Weak Law of Large Numbers implies that

$$\frac{S_m}{2(m-1)} = \frac{1}{m-1}\sum_{i=2}^{m}\frac{T_i}{2} \xrightarrow{P} 1.$$

Applying the Slutsky's theorem, we obtain that Q_m converges in distribution to Exp(1). Therefore, $\frac{\log Q_m}{\log(m-1)}$ and $\frac{\log(1+Q_m/(m-1))}{\log(m-1)}$ converge to zero in distribution, implying the convergence in probability. Applying these limits to Eq. (3.14), we can show that r_m converges to one in probability as m approaches to infinity.
To prove 2.), we note that if $\hat{a} \ge 1$, then the function $f(x) = x^{\hat{a}}$ is convex; consequently, the Jensen's inequality and Eq. (3.13) imply that

$$m-1 = M_m(\hat{a}) \ge \{M_m(1)\}^{\hat{a}}.$$

Applying the logarithm function, we have

$$1 \le \hat{a} \le (r_m)^{-1}.$$

By the same technique, if $0 < \hat{a} \le 1$, then the function $f(x) = x^{\hat{a}}$ is concave; and the Jensen's inequality yields

$$M_m(\hat{a}) \le \{M_m(1)\}^{\hat{a}},$$

so that

$$(r_m)^{-1} \le \hat{a} \le 1.$$

Therefore, results in 1.) and 2.) imply that $\hat{a}$ converges to one in probability; consequently, we obtain from the definition of $\hat{a}$ that $\hat{\sigma}$ converges to σ in probability. In particular, $\hat{\sigma}$ in a consistent estimator of σ □

4. Simulation Results

In this section, we perform a simulation study to investigate performance of the proposed pivot-based method comparing to the well-known maximum likelihood estimation (MLE) method. The settings of parameters are $\mu = 0, 1,$ and $\sigma = 1$. The sample sizes considered are $n = 10, 30, 50$ ranging from small to large sample sizes. Different numbers of failures m are considered for each sample size to evaluate the estimation methods across a range of realistic scenarios. In particular, we consider the following three settings:

Scenario 1 : $n = 10,\ m = 4, 5, 6,$

Scenario 2 : $n = 30,\ m = 4, 15, 18,$

Scenario 3 : $n = 50,\ m = 4, 25, 30.$

In each scenario, we include two extreme censoring schemes: Scheme 1 (S1) := $(0, 0, \ldots, 0, n - m)$ and Scheme 2 (S2) := $(n - m, 0, 0, \ldots, 0)$. For simplicity, the two schemes are denoted as $((m-1)*0, n-m)$ and $(n-m, (m-1)*0)$, respectively. For example, $(4 * 0, 20)$ represents the scheme $(0, 0, 0, 0, 20)$. Each scenario is repeated for 10,000 simulation runs. To generate the dataset for each simulation run, we follow the simulation algorithm of Balakrishnan and Sandhu (1995) described as follows.

1. Generate m independent observations $W_1, W_2, \ldots, W_m$ from the Uniform$(0, 1)$ distribution.

2. For each $i = 1, 2, \ldots, m$, define

$$V_i = W_i^{1/(i+R_m+R_{m-1}+\cdots+R_{m-i+1})}.$$

3. For each $i = 1, 2, \ldots, m$, set

$$U_i = 1 - V_m V_{m-1} \ldots V_{m-i+1}.$$

Therefore, the resulting observations $U_1, U_2, \ldots, U_m$ form a progressively Type-II censored sample from the Uniform$(0, 1)$ distribution.

4. For each $i = 1, 2, \ldots, m$, set

$$X_i = F^{-1}(U_i),$$

where $F^{-1}(\cdot)$ is the inverse cumulative distribution function of the Gumbel distribution. Therefore, $X_1, X_2, \ldots, X_m$ form a progressively Type-II censored sample from the Gumbel distribution.

5. Obtain the PQ estimate of σ by numerically solve

$$\frac{\sum_{i=1}^{m}(1 + R_i)\exp(X_i/\hat{\sigma})}{\exp(X_1/\hat{\sigma})} = n(m - 1).$$

The nonlinear equation is solved using the `nleqslv` function in R.

6. After obtaining $\hat{\sigma}$, compute the PQ estimate $\hat{\mu}$ from

$$\hat{\mu} = \hat{\sigma} \log\left[\frac{1}{m}\sum_{i=1}^{m}(1 + R_i)\exp\left(\frac{X_i}{\hat{\sigma}}\right)\right].$$

7. Obtain the maximum likelihood estimates $(\hat{\mu}_{\mathrm{MLE}}, \hat{\sigma}_{\mathrm{MLE}})$ by solving the corresponding likelihood equations

$$\frac{\partial \ell(\mu, \sigma)}{\partial \mu} = 0, \qquad \frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = 0,$$

using the `nleqslv` function in R.

8. Repeat Steps 1–7 for N simulation replications. For each estimation method, calculate the average bias and mean squared error of the estimators. Convergence is declared when the default convergence criterion of `nleqslv` is satisfied.

To investigate the sensitivity of the numerical procedure to the choice of initial values, the average estimates of $\hat{\mu}$ and $\hat{\sigma}$ were computed using initial values perturbed by 10%, 50%, and 100% around the true parameter values (μ, σ) = (0,1). The same simulated data set was used for all runs. The results, presented in Table 3 show that the average estimates remain nearly unchanged across the three perturbation levels for both the PQ and MLE methods under both censoring schemes. This suggests that the numerical solutions are stable and not sensitive to the choice of initial values.

Table 1. The average bias (AB) and the mean squared error (MSE) of PQs and MLEs for $(\mu, \sigma) = (0, 1)$.

n	m	Scheme	μ				σ			
			PQ		MLE		PQ		MLE	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	4	S1: (3*0,6)	-0.0447	0.1058	-0.3238	0.4969	0.0870	0.0802	-0.2333	0.2341
		S2: (6,3*0)	-0.0698	0.2210	-0.1807	0.3099	0.2221	0.1429	-0.1192	0.1239
	5	S1: (4*0,5)	-0.0317	0.1192	-0.2104	0.3095	0.1023	0.0813	-0.1836	0.1768
		S2: (5,4*0)	-0.0780	0.2078	-0.1297	0.2451	0.2349	0.1461	-0.1102	0.1065
	6	S1: (5*0,4)	-0.0237	0.1289	-0.1413	0.2137	0.1378	0.0785	-0.1492	0.1401
		S2: (4,5*0)	-0.0848	0.1816	-0.0941	0.1961	0.2318	0.1184	-0.1009	0.0935
30	4	S1: (3*0,26)	-0.0124	0.0386	-0.6339	1.3937	0.0647	0.0342	-0.2461	0.2389
		S2: (26,3*0)	-0.0351	0.1570	-0.2363	0.3439	0.1475	0.0938	-0.0615	0.0784
	15	S1: (14*0,15)	-0.0062	0.0302	-0.0704	0.0903	0.0449	0.0299	-0.0594	0.0583
		S2: (15,14*0)	-0.0537	0.0746	-0.0390	0.0747	0.1481	0.0445	-0.0355	0.0353
	18	S1: (17*0,12)	-0.0050	0.0352	-0.0484	0.0630	0.0589	0.0305	-0.0495	0.0472
		S2: (12,17*0)	-0.0539	0.0633	-0.0286	0.0611	0.1321	0.0360	-0.0342	0.0315
50	4	S1: (3*0,46)	-0.0114	0.0268	-0.7512	1.9668	0.0465	0.0237	-0.2475	0.2418
		S2: (46,3*0)	-0.0163	0.1379	-0.2345	0.3470	0.1230	0.0797	-0.0434	0.0656
	25	S1: (24*0,25)	0.0033	0.0174	-0.0411	0.0523	0.0387	0.0188	-0.0346	0.0333
		S2: (25,24*0)	-0.0441	0.0447	-0.0214	0.0436	0.1124	0.0243	-0.0208	0.0206
	30	S1: (29*0,20)	0.0024	0.0203	-0.0264	0.0381	0.0452	0.0208	-0.0273	0.0271
		S2: (20,29*0)	-0.0418	0.0383	-0.0149	0.0365	0.0992	0.0187	-0.0190	0.0183

Having obtained the 10,000 samples following the progressive Type-II censoring scheme, we compute parameter estimates by the proposed pivot-based method and the maximum likelihood estimation method. The results of the two methods are compared by using the average bias (AB) and the mean squared error (MSE). In particular, for each estimator $\hat{\theta}$ of a parameter θ, the average bias (AB) and the mean squared error (MSE) are respectively defined as

$$\mathrm{AB}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right), \qquad (4.1)$$

$$\mathrm{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2, \qquad (4.2)$$

where N is the number of simulation runs and $\hat{\theta}_i$ is the estimate from the i-th sample.

The calculated AB and MSE are presented in Tables 1-2 for the cases of $(\mu, \sigma) = (0, 1)$ and $(\mu, \sigma) = (1, 1)$, respectively. The boxplots of parameter estimates for the setting $(\mu, \sigma) = (0, 1)$ and the cases of $m = 4$ and $m = 0.6n$, representing low and high levels of censoring, are shown in Figs. 1-2, respectively.

From Tables 1-2, we observe that the average biases (ABs) of the PQ method are negative for the parameter μ and positive for the parameter σ, whereas the ABs of the MLE are negative for both parameters. Considering the absolute average bias (AAB) and the mean squared error (MSE), the PQ method generally provides smaller values than the MLE for estimating the location parameter μ. For the scale parameter σ, however, the relative performance depends on the censoring scheme and sample size. In particular, the MLE produces smaller AAB and MSE in some of Scheme 2 settings, whereas the PQ method performs comparably in many other cases.

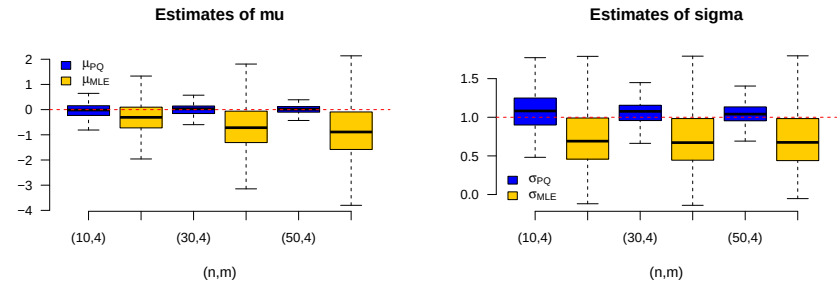
Table 2. The average bias (AB) and the mean squared error (MSE) of PQs and MLEs for $(\mu, \sigma) = (1, 1)$.

n	m	Scheme	μ				σ			
			PQ		MLE		PQ		MLE	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	4	S1: (3*0,6)	-0.0437	0.1119	-0.3275	0.5067	0.0905	0.0819	-0.2353	0.2312
		S2: (6,3*0)	-0.0717	0.2199	-0.1849	0.3173	0.2226	0.1366	-0.1208	0.1224
	5	S1: (4*0,5)	-0.0374	0.1165	-0.2180	0.3112	0.1077	0.0795	-0.1812	0.1762
		S2: (5,4*0)	-0.0868	0.1956	-0.1383	0.2403	0.2340	0.1248	-0.1081	0.1061
	6	S1: (5*0,4)	-0.0298	0.1289	-0.1509	0.2179	0.1414	0.0796	-0.1543	0.1409
		S2: (4,5*0)	-0.0912	0.1845	-0.1010	0.1976	0.2346	0.1235	-0.1050	0.0939
30	4	S1: (3*0,26)	-0.0104	0.0386	-0.6157	1.3907	0.0614	0.0339	-0.2411	0.2449
		S2: (26,3*0)	-0.0310	0.1589	-0.2296	0.3399	0.1444	0.0945	-0.0588	0.0813
	15	S1: (14*0,15)	-0.0064	0.0308	-0.0718	0.0915	0.0464	0.0300	-0.0597	0.0590
		S2: (15,14*0)	-0.0546	0.0744	-0.0400	0.0753	0.1492	0.0447	-0.0358	0.0357
	18	S1: (17*0,12)	-0.0055	0.0367	-0.0492	0.0644	0.0623	0.0308	-0.0472	0.0454
		S2: (12,17*0)	-0.0559	0.0643	-0.0305	0.0616	0.1328	0.0366	-0.0320	0.0302
50	4	S1: (3*0,46)	-0.0100	0.0271	-0.7420	2.0052	0.0452	0.0238	-0.2456	0.2431
		S2: (46,3*0)	-0.0103	0.1354	-0.2267	0.3406	0.1183	0.0783	-0.0422	0.0664
	25	S1: (24*0,25)	0.0026	0.0174	-0.0426	0.0526	0.0399	0.0189	-0.0348	0.0346
		S2: (25,24*0)	-0.0453	0.0449	-0.0228	0.0438	0.1133	0.0250	-0.0217	0.0215
	30	S1: (29*0,20)	0.0012	0.0203	-0.0297	0.0379	0.0472	0.0207	-0.0319	0.0275
		S2: (20,29*0)	-0.0433	0.0382	-0.0155	0.0366	0.0995	0.0187	-0.0227	0.0186

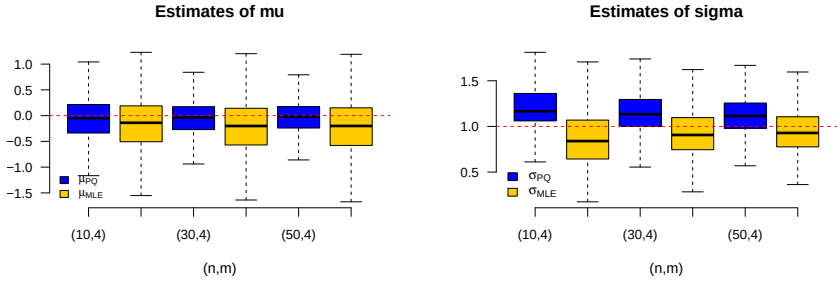
For the PQ method, the AAB values remain relatively stable across different sample sizes and censoring schemes, while the MSE generally decreases as the sample size increases. In contrast, the MLE exhibits relatively large AAB and MSE when the number of observed failures is small (e.g., $m = 4$). For example, when $n = 50$ and $m = 4$ under Scheme 1 (Scheme $(3^*0, 46)$) in Table 1, the AAB and MSE of the MLE are 0.7512 and 1.9668 for μ, and 0.2475 and 0.2418 for σ, respectively, which are substantially larger than those of the PQ method. Similar patterns can also be observed in Table 2 for the case $(\mu, \sigma) = (1, 1)$. From Figs. 1-2, the boxplots show that the PQ estimates are generally centered around the true values for both μ and σ. For the location parameter μ, the medians of the PQ estimates are consistently close to the true values across all simulation settings. For the scale parameter σ, the relative performance of the PQ and MLE methods depends on the censoring scheme and sample size. In particular, the MLE produces estimates closer to the true value in some of Scheme 2 settings, whereas the PQ method performs comparably in many other cases. The boxplots also indicate that the variability of the PQ estimates is generally comparable to that of the MLE. In several settings, particularly under Scheme 1 and for small sample sizes, the PQ estimates exhibit slightly smaller dispersion. Overall, the graphical results are consistent with the numerical summaries reported in Tables 1-2.

5. Applications

In this section, we apply the proposed estimation method to two real datasets. The first dataset is the "Breaking down of an insulating fluid" dataset in Viveros and Balakrishnan [21] involving the log-times to

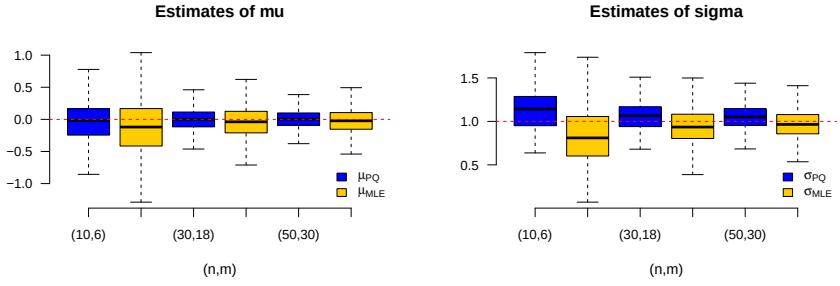


Scheme 1

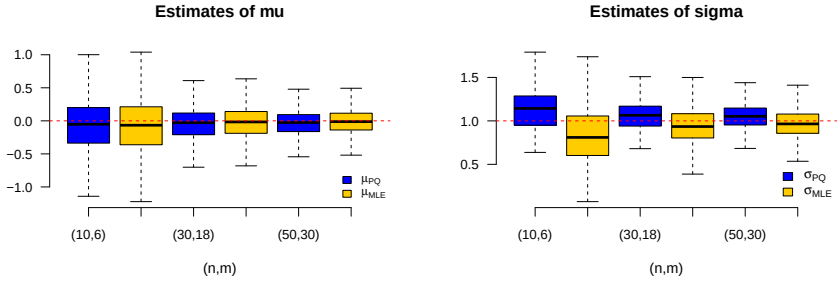


Scheme 2

Fig. 1. Boxplots of the estimates of μ and σ under Scheme 1 (top) and Scheme 2 (bottom) for $\mu = 0$, $\sigma = 1$, with $(n, m) = (10, 4), (30, 4), (50, 4)$.



Scheme 1



Scheme 2

Fig. 2. Boxplots of the estimates of μ and σ under Scheme 1 (top) and Scheme 2 (bottom) for $\mu = 0$, $\sigma = 1$, with $(n, m) = (10, 6), (30, 18), (50, 30)$.

Table 3. Average estimates of μ and σ under 10%, 50%, and 100% perturbations of the initial values.

n	m	Estimator	Scheme	Initial perturbation	Average $\hat{\mu}$	Average $\hat{\sigma}$
30	14	PQ	1	10%	0.0986	1.2274
				50%	0.1101	1.2353
				100%	0.1176	1.2478
			2	10%	-0.0697	1.1707
				50%	-0.0729	1.1643
				100%	-0.0750	1.1689
		MLE	1	10%	-0.0705	0.9400
				50%	-0.0882	0.9249
				100%	-0.1639	0.8599
			2	10%	-0.0405	0.9633
				50%	-0.3243	0.7405
				100%	-0.6694	1.0801
	18	PQ	1	10%	0.0457	1.1997
				50%	0.0524	1.2046
				100%	0.0579	1.2198
			2	10%	-0.0611	1.1675
				50%	-0.0693	1.1543
				100%	-0.0700	1.1622
		MLE	1	10%	-0.0492	0.9496
				50%	-0.0782	0.9207
				100%	-0.2197	0.7784
			2	10%	-0.0313	0.9644
				50%	-0.2449	0.7690
				100%	-0.4089	1.1851

breakdown of an insulating fluid tested at 34 kilovolts. The second dataset is the "Averaging wind speed values" dataset presented in Zhang and Gui [22] involving 30 observed monthly average wind speed values from 2006 to 2008. For each of the datasets, we compute parameter estimates corresponding with their bootstrap standard errors ($\mathrm{SE}_{\mathrm{boot}}(\hat{\theta})$) and coefficient of variation (CV). A parametric bootstrap procedure with $B = 1000$ replications was used to estimate the bootstrap standard errors and coefficients of variation. For each bootstrap replication, a complete random sample of size n was generated from the fitted Gumbel distribution using the estimated parameters. The original progressive Type-II censoring scheme was then imposed to obtain a bootstrap progressively Type-II censored sample, from which the parameter estimates were recalculated. The bootstrap procedure is summarized in Algorithm.

1. Estimate $\hat{\mu}$ and $\hat{\sigma}$ from the observed progressively Type-II censored sample.
2. Generate a complete random sample of size n from the fitted Gumbel distribution with parameters $(\hat{\mu}, \hat{\sigma})$.
3. Apply the original progressive Type-II censoring scheme $(R_1, \ldots, R_m)$ to obtain a bootstrap progressively Type-II censored sample.
4. Compute the bootstrap estimates $\hat{\mu}_b$ and $\hat{\sigma}_b$ from the bootstrap sample.
5. Repeat Steps 2–4 for $B = 1000$ bootstrap replications.

6. Calculate the bootstrap mean, bootstrap standard error, and coefficient of variation.

For each of the datasets, we compute parameter estimates corresponding with their bootstrap standard errors ($\mathrm{SE}_{\mathrm{boot}}(\hat{\theta})$) and coefficient of variation (CV). The $\mathrm{SE}_{\mathrm{boot}}$ and CV of any estimator $\hat{\theta}$ are calculated as

$$\mathrm{SE}_{\mathrm{boot}}(\hat{\theta}) = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}_b - \bar{\theta}_{\mathrm{boot}})^2},$$

$$\mathrm{CV} = \frac{\mathrm{SE}_{\mathrm{boot}}}{\bar{\theta}_{\mathrm{boot}}},$$

where

$$\bar{\theta}_{\mathrm{boot}} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_b,$$

B is the number of bootstrap replications and $\hat{\theta}_b$ $(b = 1, 2, \ldots, B)$ is the estimate from the bth bootstrap sample. Moreover, we apply different goodness-of-fits to the fitted distribution. These criteria are Anderson-Darling (AD), Cramer-von Mises (CM), and Kolmogorov-Smirnov (KS) tests (Zhu [23]), respectively defined as

$$\mathrm{AD} = \Big[1 + \frac{0.75}{n} + \frac{2.25}{n^2}\Big] \times \Big[-n - \frac{1}{n}\sum_{j=1}^{n}\{(2j-1) \times \log(U_j) + (2n+1-2j)\log(1-U_j)\}\Big],$$

$$\mathrm{CM} = \Big[1 + \frac{0.75}{n} + \frac{2.25}{n^2}\Big] \times \Big[\sum_{j=1}^{n}\{\frac{j-0.5}{n} - U_j\}^2 + \frac{1}{12n}\Big],$$

$$\mathrm{KS} = \Big[\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\Big]\max(D^+, D^-),$$

where $D^+ = \max\limits_{j=1,2,\ldots,n}\left\{\frac{j}{n} - U_j\right\}$,

$D^- = \max\limits_{j=1,2,\ldots,n}\left\{U_j - \frac{j-1}{n}\right\}$, and

$$U_j = F(x_{(j)}; \hat{\mu}, \hat{\sigma}), \qquad j = 1, \ldots, n,$$

with $F(\cdot; \hat{\mu}, \hat{\sigma})$ is the fitted Gumbel cumulative distribution function evaluated at the ordered observations. The AD, CM, and KS statistics are computed from these transformed values following Zhu [23]. The corresponding p-values are obtained using our implementation of the procedure described in Zhu [23], with the parameters are replaced by their estimates.

5.1 Data Set 1 : Breaking down of an insulating fluid

The first dataset contains 19 observations of the logarithm of breakdown times for an insulating fluid tested at 34 kilovolts. A progressively Type-II censored sample from this dataset is shown in Table 4. The parameter estimates, bootstrap standard errors, coefficients of variation, and goodness-of-fit (GOF) statistics are summarized in Table 5.

From Table 5, we can see that the PQ method provides larger values of the bootstrap standard errors and coefficients of variation. However, considering the p-values of the three goodness-of-fit statistics, we can see that the p-values corresponding to the PQ method are higher than those of the MLE method. These results suggest that the two models are comparable, and the advancement between the two methods depends on the evaluation criteria.

5.2 Data Set 2 : Averaging wind speed values

The progressive Type-II censored sample for this dataset, generated by Zhang and Gui [18], is shown in Table 6. The parameter estimates, bootstrap standard er-

Table 4. Progressively Censored Sample Generated From the Log-Times to Breakdown Data on Insulating Fluid Tested at 34 kilovolts.

i	1	2	3	4	5	6	7	8
$x_{i:18:13}$	-1.6608	-0.2485	-0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
R_i	0	0	3	0	3	0	0	5

Table 5. Result for Log-Times to Breakdown consist of Point estimates, Bootstrap standard errors ($\mathrm{SE}_{\mathrm{boot}}(\hat{\theta})$), Coefficients of Variation (CV), Anderson-Darling (AD), Cramer-von Mises (CM), and Kolmogorov-Smirnov (KS), for Progressive Type-II censored sample of parameter estimation methods of data set 1.

Method	μ			σ			Goodness-of-fit statistics (p-values)		
	$\hat{\mu}$	$\mathrm{SE}_{\mathrm{boot}}(\hat{\mu})$	CV	$\hat{\sigma}$	$\mathrm{SE}_{\mathrm{boot}}(\hat{\sigma})$	CV	AD	CM	KS
PQ	2.192	0.4464	0.1895	0.974	0.4425	0.3424	2.7944 (0.745)	0.5509 (0.745)	1.4761 (0.738)
MLE	2.222	0.1923	0.0924	1.026	0.0999	0.1091	2.9034 (0.659)	0.5739 (0.668)	1.4868 (0.685)

Table 6. Progressive Type-II censored sample generated by Zhang and Gui [18].

i	1	2	3	4	5	6	7	8	9
$x_{i:18:13}$	2	2.7	2.8	2.9	3.2	3.3	3.3	3.4	3.5
R_i	1	0	1	1	0	1	1	0	1
i	10	11	12	13	14	15	16	17	18
$x_{i:18:13}$	3.9	4	4.7	5.3	5.4	5.5	5.7	6.7	6.9
R_i	1	0	1	1	0	1	1	0	1

rors, coefficients of variation along with the corresponding goodness-of-fit statistics, are provided in Table 7.

Similar to the first dataset, we can see that the PQ method provides slightly larger values of the bootstrap standard errors and coefficients of variation. However, considering the p-values of the three the goodness-of-fit statistics, we can see that the p-values corresponding to the PQ method are higher than those of the MLE method. These results suggest that the two models are comparable.

6. Discussion

In this study, we proposed a pivot-based parameter estimation method for the Gumbel distribution under a progressive Type-II censoring scheme. The performance of the proposed method was investigated through a simulation study. The simulation results indicate that the proposed method generally provides more accurate estimation of the location parameter μ than the maximum likelihood method in terms of bias and mean squared error. For the scale parameter σ, the relative performance depends on the censoring scheme and sample size, and the proposed method performs comparably to the maximum likelihood method in most simulation settings. Finally, the proposed method was applied to two real datasets. The results show that the proposed method yields parameter estimates and goodness-of-fit comparable to those obtained by the maximum likelihood method. Several directions for future research may be pursued, including extending the proposed method to other censoring schemes and distributions, as well as invest-

Table 7. Result for Averaging wind speed values consist of Point estimates, Bootstrap standard errors ($SE_{boot}(\hat{\theta})$), Coefficients of Variation (CV), Anderson-Darling (AD), Cramer-von Mises (CM), and Kolmogorov-Smirnov (KS), for Progressive Type-II censored sample of parameter estimation methods of data set 2.

Method	μ			σ			Goodness-of-fit statistics (p-values)		
	$\hat{\mu}$	$SE_{boot}(\hat{\mu})$	CV	$\hat{\sigma}$	$SE_{boot}(\hat{\sigma})$	CV	AD	CM	KS
PQ	6.333	0.3483	0.0549	1.835	0.4917	0.2267	5.1037 (1.00)	1.0179 (1.00)	2.0634 (0.997)
MLE	6.358	0.2715	0.0433	1.877	0.1567	0.0867	5.1608 (1.00)	1.0314 (1.00)	2.0783 (0.938)

ing the performance of the proposed method to other datasets.

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Appendices

In this section, we provide detailed proofs of the the properties of the transformed variables Y_i and T_i discussed in Section 3.1.

Theorem 6.1. *Let $X_1, \ldots, X_m$ be a progressively Type-II censored sample from the Gumbel distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. Define*

$$Y_i = \exp\left(\frac{X_i - \mu}{\sigma}\right), \qquad i = 1, \ldots, m.$$

Then

$$Y_1, \ldots, Y_m$$

constitute a progressively Type-II censored sample from the standard exponential distribution.

Proof. Let

$$Y = \exp\left(\frac{X-\mu}{\sigma}\right).$$

Then

$$X = \mu + \sigma \ln Y, \qquad \left|\frac{dX}{dY}\right| = \frac{\sigma}{Y}.$$

Since

$$f_X(x) = \frac{1}{\sigma}\exp\left\{\frac{x-\mu}{\sigma}\right\}\exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\},$$

the density of Y is

$$\begin{aligned} f_Y(y) &= f_X(\mu+\sigma\ln y)\left|\frac{dX}{dY}\right| \\ &= \frac{1}{\sigma}\exp(\ln y)\exp(-y)\cdot\frac{\sigma}{y} \\ &= e^{-y}, \qquad y>0. \end{aligned}$$

Hence,

$$Y \sim \mathrm{Exp}(1).$$

Since the transformation

$$Y = \exp\left(\frac{X-\mu}{\sigma}\right)$$

is strictly monotone, the progressive Type-II censoring scheme is preserved. Therefore, the transformed observations form a progressively Type-II censored sample from the standard exponential distribution. □

Theorem 6.2. *For the transformed progressively Type-II censored exponential sample, define*

$$T_1 = 2nY_1,$$

and

$$T_i = 2\left(m-i+1+\sum_{j=i}^{m} R_j\right)(Y_i - Y_{i-1}), \qquad i = 2,\ldots,m.$$

Then

$$T_1, T_2, \ldots, T_m$$

are independent random variables with

$$T_i \sim \chi^2_2, \qquad i = 1,\ldots,m.$$

Proof. The joint density of the progressively Type-II censored exponential sample is

$$f(y_1,\ldots,y_m) = C\prod_{i=1}^{m} e^{-(1+R_i)y_i},$$

where C is a normalizing constant. Applying the above transformation, the Jacobian is

$$|J| = \frac{1}{2^m C}.$$

Hence,

$$\begin{aligned} f(t_1,\ldots,t_m) &= f(y_1,\ldots,y_m)|J| \\ &= \prod_{i=1}^{m}\left(\frac{1}{2}e^{-t_i/2}\right). \end{aligned}$$

Since the joint density factors into the product of the marginal densities,

$$f_{T_i}(t) = \frac{1}{2}e^{-t/2}, \qquad t>0,$$

the variables

$$T_1,\ldots,T_m$$

are mutually independent and each follows a chi-square distribution with two degrees of freedom. □

Lemma 6.3. *Let*

$$T_1 = 2nY_1$$

and, for $i = 2,\ldots,m$,

$$T_i = 2\left(m-i+1+\sum_{j=i}^{m} R_j\right)(Y_i - Y_{i-1}).$$

Then

$$\sum_{i=1}^{m} T_i = 2\sum_{i=1}^{m}(1+R_i)Y_i.$$

Proof. Define

$$c_i = m - i + 1 + \sum_{j=i}^{m} R_j, \qquad i = 2, \ldots, m.$$

Using

$$n = m + \sum_{j=1}^{m} R_j,$$

we have

$$c_2 = m - 1 + \sum_{j=2}^{m} R_j = n - 1 - R_1.$$

Moreover, for $i = 2, \ldots, m-1$,

$$\begin{aligned} c_i - c_{i+1} &= \left(m - i + 1 + \sum_{j=i}^{m} R_j\right) \\ &\quad - \left(m - i + \sum_{j=i+1}^{m} R_j\right) \\ &= 1 + R_i, \end{aligned}$$

and

$$c_m = 1 + R_m.$$

Therefore,

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{m} T_i &= nY_1 + \sum_{i=2}^{m} c_i(Y_i - Y_{i-1}) \\ &= nY_1 + \sum_{i=2}^{m} c_i Y_i - \sum_{i=2}^{m} c_i Y_{i-1}. \end{aligned}$$

Reindexing the last sum gives

$$\sum_{i=2}^{m} c_i Y_{i-1} = \sum_{i=1}^{m-1} c_{i+1} Y_i.$$

Hence,

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{m} T_i &= (n - c_2)Y_1 \\ &\quad + \sum_{i=2}^{m-1} (c_i - c_{i+1})Y_i + c_m Y_m \\ &= (1 + R_1)Y_1 \\ &\quad + \sum_{i=2}^{m-1} (1 + R_i)Y_i + (1 + R_m)Y_m \\ &= \sum_{i=1}^{m} (1 + R_i)Y_i. \end{aligned}$$

Multiplying both sides by 2 yields

$$\sum_{i=1}^{m} T_i = 2\sum_{i=1}^{m} (1 + R_i)Y_i.$$

□