

Generation of Julia and Mandelbrot Sets via Modified Sp-Iterative Schemes with Image Security Applications

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ABSTRACT

This paper explores modified SP-iterations for both fractal generation and image security applications. We analyze the complex maps $\mathbb{C} \ni z \mapsto z^p + z^2 + z + c$, $\mathbb{C} \ni z \mapsto z^p + z^2 + z + \ln(c^t)$, and $\mathbb{C} \ni z \mapsto z^p + z^2 + z + \tan(c^t)$ for all $p \in \mathbb{N} \setminus \{1\}$, $c \in \mathbb{C} \setminus \{0\}$ and the parameter $c^t \neq 1$ with $t \in [1, \infty)$ establishing an escape criterion that enables generation of diverse Julia and Mandelbrot sets. Algorithm implementation demonstrates parameter-dependent variations in fractal patterns and symmetry. Using these chaotic properties, we develop an integrated encryption-enhancement system combining modified SP-iteration based security with median filtering. The method achieves strong encryption (NPCR > 99.6%, UACI > 33.4%) and high-quality reconstruction (PSNR up to 45.73 dB, SSIM > 0.99). This work bridges fractal mathematics with practical image processing, demonstrating how complex iterations can simultaneously produce aesthetic fractals and secure digital images.

Keywords: Escape criteria; Fractals; Image encryption; Image security; Modified SP-iterative method

1. Introduction

The concept of fractals was formally introduced by Mandelbrot [1] in 1978, although related ideas appeared earlier. Weierstrass (1872) constructed a function that is continuous everywhere but nowhere differentiable, exhibiting non-integer di-

mensions, while other classical examples include the Cantor set, Hilbert curve, Koch curve, and Sierpinski gasket. In the early 20th century, Fatou [2] and Julia [3] studied the iterative dynamics of complex functions, notably $\mathbb{C} \ni z \mapsto z^2 + c$, where $c \in \mathbb{C}$ is a parameter, laying the ground-

work for Julia sets. Visual representations became feasible only after Cremer [4] and, later, through Mandelbrot's pioneering computer-generated images [1]. Julia and Mandelbrot sets are closely related, and their study continues to reveal intricate mathematical and aesthetic structures [5].

Fixed-point iterative methods have been widely employed to generate fractals. For example, Devaney [6] used Picard iterations for polynomial functions, while Mann, Ishikawa, Noor, Abbas, and Picard-Mann iterations have been applied to produce diverse Julia and Mandelbrot sets [7–10]. The SP-iteration method has recently been used to establish escape criteria and generate fractals for functions like $\mathbb{C} \ni z \mapsto z^p + c$ and rational maps, and also to explore anti-fractals [11, 12].

Despite these advances, most studies focus on classical functions such as $z^2 + c$ or $z^p + c$ using standard iterations. The dynamics of more generalized complex-valued functions, especially those involving logarithmic or trigonometric terms, remain less explored. In 2024, Adhikari and Sintunavarat [13] generate and studied the properties of fractals of the map $\mathbb{C} \ni z \mapsto z^n + \log(c^t)$. Moreover, computational performance, including efficiency, convergence speed, and average number of iterations for fractal generation, has received limited attention.

To address these gaps, this work investigates the functions $T_1(z) = z^p + z^2 + z + c$, $T_2(z) = z^p + z^2 + z + \ln(c^t)$, $T_3(z) = z^p + z^2 + z + \tan(c^t)$ for all $z \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{1\}$, and $t \in [1, \infty)$, $c^t \neq 1$. Using a modified SP-iteration method, we establish a rigorous escape criterion guaranteeing divergence beyond a computable radius. Based on this, we develop an algorithm to generate visually appealing Julia and Mandelbrot sets and use these results

for image encryption.

There are many methods for image encryption. Common approaches include Traditional Cryptographic Methods (e.g., Symmetric and Asymmetric Encryption, Stream Ciphers), Pixel-level Manipulation, Chaos-based Encryption, techniques in transform domains (Frequency, DWT, DCT), as well as more recent methods like DNA-based, Quantum, Optical, and Compressive Sensing-based Encryption. For a comprehensive survey, (see [14]). Alongside these, the fractal-based method constitutes another significant category of image encryption techniques. The history of image encryption using fractals began with the development of fractal theory. Fractal-based image encryption has emerged as a promising approach for secure image transmission due to its inherent chaotic properties and sensitivity to initial conditions. However, reconstructed images often suffer from residual noise and artifacts that degrade visual quality. This paper addresses this limitation by introducing a modified SP-iteration algorithm with median filter post-processing, significantly enhancing the quality of reconstructed fractal images.

The subsequent sections of the paper are structured as follows. Section 2 furnishes an overview of the notation and definitions employed throughout the research paper. In Section 3, we expound upon the escape criterion and algorithm for fractal generation. Section 4 exhibits graphical examples, elucidating the behavior and patterns engendered by the proposed approach. In Section 5, we discuss the application of fractal image encryption. Lastly, in Section 8, we conclude, encapsulating our work and summarizing the findings and contributions of the study.

2. Preliminaries

In this paper, we use the following symbols: $\mathbb{C}$ means all complex numbers. $\mathbb{R}$ means all real numbers. $\mathbb{N}$ means natural numbers.

In this section, we present the necessary foundations for our work. First, we define Julia sets and Mandelbrot sets, which are fundamental types of fractal structures. Next, we describe iterative methods, which are procedures based on repeated calculations.

Definition 2.1 ([3]). Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex-valued function, the filled Julia set of T is denoted by $\mathbb{J}_T$ and defined by

$$\mathbb{J}_T = \{z \in \mathbb{C} : \{|T^n(z)|\}_{n=0}^{\infty} \text{ is bounded}\},$$

where $T^n = \underbrace{T \circ T \circ \dots \circ T}_{n \text{ times}}$ denote the n^{th} iterate of T . The set of all boundary points of $\mathbb{J}_T$ is called Julia set.

Definition 2.2 ([1]). Let $T_c : \mathbb{C} \rightarrow \mathbb{C}$ be a complex-valued function, where $c \in \mathbb{C}$ is a parameter. The Mandelbrot set is defined as

$$\mathbb{M} = \{c \in \mathbb{C} : \mathbb{J}_{T_c} \text{ is connected}\}.$$

Equivalently, the Mandelbrot set $\mathbb{M}$ can be defined as

$$\mathbb{M} = \{c \in \mathbb{C} : |T_c^n(\eta)| \rightarrow \infty \text{ as } n \rightarrow \infty\},$$

where η is any critical point of T_c . Originally, the Julia and Mandelbrot sets are generated using the Picard iterative method.

In 2000, Noor [15] created a new three-step method to find approximate solutions for a type of math problem called variational inequalities. He developed this method within the framework of Hilbert spaces. It is defined in terms of complex numbers ($\mathbb{C}$) as shown below:

Definition 2.3 ([15]). Let's start with an initial complex number $z_0 \in \mathbb{C}$ and define a sequence $\{z_k\}$ from it. Consider a function $T : \mathbb{C} \rightarrow \mathbb{C}$. The three-step process to generate the sequence $\{z_k\}$ is defined by the following iteration scheme:

$$\begin{cases} z_k = (1 - \gamma_k)x_k + \gamma_k T x_k, \\ y_k = (1 - \beta_k)x_k + \beta_k T z_k, \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T y_k, \end{cases} \quad (2.1)$$

for all $k \in \mathbb{N} \cup \{0\}$, where $\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}, \{\gamma_k\}_{k=0}^{\infty}$ are sequences of real numbers in $[0, 1]$.

In 2011, Phuengrattana and Suantai [16] introduced a novel iteration technique by modifying Noor iteration, called the SP-iteration. Their study showed that the SP-iteration faster convergence to the fixed in comparison to other iterative methods such as Mann, Ishikawa, and Noor iterations for continuous mapping. Next, we give its definition in the sense of $\mathbb{C}$.

Definition 2.4 ([16]). Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex-valued self-mapping. The SP-iteration scheme $\{z_k\}_{k=0}^{\infty}$ is defined as

$$\begin{cases} z_0 \in \mathbb{C}, \\ x_k = (1 - \alpha_k)z_k + \alpha_k T z_k, \\ y_k = (1 - \beta_k)x_k + \beta_k T x_k, \\ z_{k+1} = (1 - \gamma_k)y_k + \gamma_k T y_k, \end{cases} \quad (2.2)$$

for all $k \in \mathbb{N} \cup \{0\}$, where $\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}, \{\gamma_k\}_{k=0}^{\infty}$ are sequences of real numbers in $[0, 1]$.

In 2016, Saluja [17] introduced modified SP-iterative schemes for three mappings in the Banach space. Here, we give its definition in complex plane $\mathbb{C}$.

Definition 2.5 ([17]). Let $T_1, T_2, T_3 : \mathbb{C} \rightarrow \mathbb{C}$ be three complex-valued self-mappings.

The modified SP-iterative scheme for the sequence $\{z_k\}$ is defined in three steps as follows:

$$\begin{cases} z_0 \in \mathbb{C} \\ x_k = (1 - \alpha_k)z_k + \alpha_k T_1 z_k \\ y_k = (1 - \beta_k)x_k + \beta_k T_2 x_k \\ z_{k+1} = (1 - \gamma_k)y_k + \gamma_k T_3 y_k, \end{cases} \quad (2.3)$$

for all $k \in \mathbb{N} \cup \{0\}$, where $\{\alpha_k\}_{k=0}^\infty, \{\beta_k\}_{k=0}^\infty, \{\gamma_k\}_{k=0}^\infty$ are sequences of real numbers in $[0, 1]$.

Remark 2.6. If $T_1 = T_2 = T_3$ then the iteration defined in Eq. (2.3) becomes SP-iteration.

3. Escape Criteria for the Modified SP-Iteration

In this section, we establish an escape criterion for the fractal generation process associated with the modified SP-iterative method applied to the complex-valued maps

$$\begin{aligned} T_1(z) &= z^p + z^2 + z + c, \\ T_2(z) &= z^p + z^2 + z + \ln(c^t), \\ T_3(z) &= z^p + z^2 + z + \tan(c^t) \end{aligned} \quad (3.1)$$

where $p \in \mathbb{N} \setminus \{1\}$, $t \in [1, \infty)$, and $c \in \mathbb{C} \setminus \{0\}$. We assume $c^t \neq 1$ and $\tan(c^t)$ is finite, ensuring that both logarithmic and trigonometric terms are well defined on a chosen principal branch.

The following theorem provides a sufficient condition under which the iterates of the modified SP-iteration diverge to infinity.

Theorem 3.1. Let $T_1, T_2, T_3 : \mathbb{C} \rightarrow \mathbb{C}$ be the complex-valued functions defined in Eq. (3.1), let $\alpha, \beta, \gamma \in (0, 1]$ be fixed parameters and define the constants

$$L_1 := |\ln(c^t)|, \quad L_2 := |\tan(c^t)|.$$

Define

$$R := \max \left\{ |c|, \left(\frac{2(1+\alpha)}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{2(1+L_1)}{\beta} \right)^{\frac{1}{p-1}}, \left(\frac{2(1+L_2)}{\gamma} \right)^{\frac{1}{p-1}} \right\}.$$

Let $\{z_k\}_{k=0}^\infty$ be the modified SP-iterative sequence defined in Eq. (2.3) with $\alpha_k = \alpha$, $\beta_k = \beta$, $\gamma_k = \gamma$ for all k . If the initial point $z_0 \in \mathbb{C}$ satisfies $|z_0| > R$, then $|z_k| \rightarrow \infty$ as $k \rightarrow \infty$.

Proof. Let $r := |z_0| > R$. We show that $|z_1| > (1+\theta)r$ for some $\theta > 0$ independent of k , and the result then follows by induction.

Step 1. Using $|c| \leq r$ and the reverse triangle inequality, we obtain

$$\begin{aligned} |x_0| &= |(1-\alpha)z_0 + \alpha T_1(z_0)| \\ &= |(1-\alpha)z_0 + \alpha(z_0^p + z_0^2 + z_0 + c)| \\ &\geq \alpha|z_0^p + z_0^2 + z_0 + c| - (1-\alpha)|z_0| \\ &\geq \alpha(|z_0|^p - |z_0|^2 - |z_0| - |c|) - (1-\alpha)|z_0| \\ &\geq \alpha|z_0|^p - \alpha|z_0|^2 - \alpha|z_0| - \alpha r - (1-\alpha)|z_0| \\ &= \alpha|z_0|^p - \alpha|z_0|^2 - |z_0| - \alpha r \\ &= |z_0| \left(\alpha|z_0|^{p-1} - \alpha|z_0| - 1 - \frac{\alpha r}{|z_0|} \right). \end{aligned}$$

Since $|z_0| \geq r$, we have $\frac{\alpha r}{|z_0|} \leq \alpha$, and hence

$$\begin{aligned} |x_0| &\geq |z_0| (\alpha|z_0|^{p-1} - \alpha|z_0| - 1 - \alpha) \\ &\geq r (\alpha r^{p-1} - \alpha r - 1 - \alpha). \end{aligned}$$

Now assume that

$$r > \left(\frac{2(1+\alpha)}{\alpha} \right)^{\frac{1}{p-1}}.$$

Then

$$\alpha r^{p-1} > 2(1+\alpha),$$

which implies

$$\alpha r^{p-1} - \alpha r - (1+\alpha) > 0$$

for sufficiently large r . Therefore,

$$|x_0| > r.$$

Step 2. For T_2 , we have

$$\begin{aligned}
 |y_0| &= |(1-\beta)x_0 + \beta T_2(x_0)| \\
 &= |(1-\beta)x_0 + \beta(x_0^p + x_0^2 + x_0 + \ln(c^t))| \\
 &\geq \beta|x_0^p + x_0^2 + x_0 + \ln(c^t)| - (1-\beta)|x_0| \\
 &\geq \beta(|x_0|^p - |x_0|^2 - |x_0| - |\ln(c^t)|) - (1-\beta)|x_0| \\
 &= \beta|x_0|^p - \beta|x_0|^2 - \beta|x_0| - \beta L_1 - (1-\beta)|x_0| \\
 &= \beta|x_0|^p - \beta|x_0|^2 - |x_0| - \beta L_1 \\
 &= |x_0| \left(\beta|x_0|^{p-1} - \beta|x_0| - 1 - \frac{\beta L_1}{|x_0|} \right).
 \end{aligned}$$

Since

$$|x_0| \geq r > \left(\frac{2(1+L_1)}{\beta} \right)^{\frac{1}{p-1}}.$$

Since $|x_0| \geq 1$, we have

$$\frac{\beta L_1}{|x_0|} \leq \beta L_1.$$

Thus

$$\begin{aligned}
 &\beta|x_0|^{p-1} - \beta|x_0| - 1 - \frac{\beta L_1}{|x_0|} \\
 &\geq \beta|x_0|^{p-1} - \beta|x_0| - 1 - \beta L_1.
 \end{aligned}$$

For sufficiently large $|x_0|$, the term $|x_0|^{p-1}$ dominates $|x_0|$, and in particular,

$$\beta|x_0| \leq \frac{\beta}{2}|x_0|^{p-1}.$$

Hence,

$$\beta|x_0|^{p-1} - \beta|x_0| \geq \frac{\beta}{2}|x_0|^{p-1}.$$

Therefore,

$$\beta|x_0|^{p-1} - \beta|x_0| - 1 - \beta L_1 \geq \frac{\beta}{2}|x_0|^{p-1} - (1 + \beta L_1) > 0.$$

Hence,

$$|y_0| > |x_0|.$$

Step 3. Similarly,

$$|z_1| \geq |y_0| \left(\gamma|y_0|^{p-1} - \gamma|y_0| - 1 - \frac{\gamma L_2}{|y_0|} \right).$$

Assume that

$$|y_0| \geq r > R := \left(\frac{2(1+L_2)}{\gamma} \right)^{\frac{1}{p-1}}.$$

Since $|y_0| \geq r \geq 1$, we have

$$\frac{\gamma L_2}{|y_0|} \leq \gamma L_2.$$

Thus,

$$\gamma|y_0|^{p-1} - \gamma|y_0| - 1 - \frac{\gamma L_2}{|y_0|} \geq \gamma|y_0|^{p-1} - \gamma|y_0| - 1 - \gamma L_2.$$

For sufficiently large $|y_0|$, the term $|y_0|^{p-1}$ dominates $|y_0|$, and in particular,

$$\gamma|y_0| \leq \frac{\gamma}{2}|y_0|^{p-1}.$$

Hence,

$$\gamma|y_0|^{p-1} - \gamma|y_0| \geq \frac{\gamma}{2}|y_0|^{p-1}.$$

Therefore,

$$\gamma|y_0|^{p-1} - \gamma|y_0| - 1 - \gamma L_2 \geq \frac{\gamma}{2}|y_0|^{p-1} - (1 + \gamma L_2).$$

Since $|y_0| > R$, it follows that

$$\frac{\gamma}{2}|y_0|^{p-1} > (1 + L_2).$$

Thus,

$$\frac{\gamma}{2}|y_0|^{p-1} - (1 + \gamma L_2) > (1 + L_2) - (1 + \gamma L_2) = L_2(1 - \gamma) > 0.$$

Hence, the bracket is strictly greater than 1 for sufficiently large $|y_0|$. Therefore, there exists a constant $\theta > 0$ such that

$$\gamma|y_0|^{p-1} - \gamma|y_0| - 1 - \frac{\gamma L_2}{|y_0|} \geq 1 + \theta.$$

Consequently,

$$|z_1| > (1 + \theta)|y_0|.$$

Step 4. The above estimates can be applied recursively. In particular, whenever $|z_k| > R$, we obtain

$$|z_{k+1}| > (1 + \theta)|z_k|.$$

By induction, it follows that

$$|z_k| > (1 + \theta)^k |z_0|.$$

Since $\theta > 0$, we have $(1 + \theta)^k \rightarrow \infty$ as $k \rightarrow \infty$.

Hence,

$$|z_k| \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

□

Remark 3.2. The escape radius R in Theorem 3.1 depends on both the iteration parameters (α, β, γ) and the functional constants (p, t, c) . In practice, any starting point with modulus larger than R lies outside the filled Julia set of the corresponding map. This provides an effective criterion for generating the Julia and Mandelbrot sets using the modified SP-iterative process.

Corollary 3.3. Let $T_1, T_2, T_3 : \mathbb{C} \rightarrow \mathbb{C}$ be the complex-valued functions defined in Eq. (3.1), and let $\alpha, \beta, \gamma \in (0, 1]$. Define

$$L_1 := |\ln(c^t)|, \quad L_2 := |\tan(c^t)|.$$

Suppose there exists an index $m \in \mathbb{N} \cup \{0\}$ such that

$$|z_m| > R : \\ = \max \left\{ |c|, \left(\frac{2(1+\alpha)}{\alpha} \right)^{\frac{1}{p-1}}, \left(\frac{2(1+L_1)}{\beta} \right)^{\frac{1}{p-1}}, \right. \\ \left. \left(\frac{2(1+L_2)}{\gamma} \right)^{\frac{1}{p-1}} \right\}.$$

If $\{z_k\}_{k=0}^{\infty}$ is the sequence generated by the modified SP-iteration Eq. (2.3) with parameters $\alpha_k = \alpha$, $\beta_k = \beta$, and $\gamma_k = \gamma$ for all k , then

$$\lim_{k \rightarrow \infty} |z_k| = \infty.$$

Proof. Applying Theorem 3.1 to the shifted sequence $\{z_{m+k}\}_{k=0}^{\infty}$ yields $\lim_{k \rightarrow \infty} |z_{m+k}| = \infty$. Hence, $\lim_{k \rightarrow \infty} |z_k| = \infty$. $\square$

Remark 3.4. Corollary 3.3 establishes explicit escape radii that can be used to generate the Julia and Mandelbrot sets corresponding to the complex-valued functions defined in Eq. (3.1). These radii determine the regions of divergence and play a vital role in identifying the boundaries of the sets, thereby enabling the visualization of intricate and aesthetically rich fractal patterns.

Remark 3.5. Although the functions T_1 , T_2 , and T_3 differ in their additive terms, the escape radius obtained in Theorem 3.1 remains common to all three. This is because, for sufficiently large $|z|$, the dominant term z^p governs the asymptotic behavior, while the additive terms c , $\ln(c^t)$, and $\tan(c^t)$ are independent of z and hence remain bounded (provided they are well-defined). Consequently, these terms can be absorbed into a uniform bound in the definition of the escape radius, and do not affect the overall escape behavior.

4. Generation of Fractals

We use Algorithm 1 and Algorithm 2 to generate the Julia and Mandelbrot sets for the complex-valued functions defined in Eq. (3.1). Both algorithms employ the enhanced three-step modified SP-iterative process implemented in MATLAB R2021a. By varying six key parameters, we analyze their influence on the resulting fractals. The “Jet” colormap (see Fig. 14) encodes escape-time values, where colors indicate iteration counts to divergence, highlighting fine structural details and convergence behavior.

4.1 Julia set

In Algorithm 1, we have set a maximum iteration limit of 80 for generating the Julia sets. Additionally, we have defined the area for the Julia set as $B = [-2.5, 2.5] \times [-2.5, 2.5]$, which represents the region in the complex plane where the Julia set is computed.

4.1.1 Julia sets of complex-valued functions defined in Eq. (3.1) varying the parameter t

We generated captivating quadratic Julia sets by varying the parameter t , while keeping other parameters fixed at $(p = 2, \alpha = 0.5, \beta = 0.41, \gamma = 0.01)$ and

number of iteration is $K = 80$. We analyze this process when t is positive real numbers.

Algorithm 1: Generation of the Julia set via modified SP-iteration.

Input: T_1, T_2, T_3 ; region $B \subset \mathbb{C}$; $\alpha, \beta, \gamma \in (0, 1]$; colormap $[0..N]$.

Output: Julia set on B .

```

1 for  $z_0 \in B$  do
2   Compute  $L_1 = |\ln(c^t)|$ ,  $L_2 = |\tan(c^t)|$ 
3   Set  $R = \max\{|c|, A_1, A_2, A_3\}$  where
4

$$A_1 = \left(\frac{2(1+\alpha)}{\alpha}\right)^{\frac{1}{p-1}},$$


$$A_2 = \left(\frac{2(1+L_1)}{\beta}\right)^{\frac{1}{p-1}},$$


$$A_3 = \left(\frac{2(1+L_2)}{\gamma}\right)^{\frac{1}{p-1}}.$$

5    $k \leftarrow 0$ 
6   while  $k < K$  do
7      $x_k \leftarrow (1-\alpha)z_k + \alpha T_1(z_k)$ 
8      $y_k \leftarrow (1-\beta)x_k + \beta T_2(x_k)$ 
9      $z_{k+1} \leftarrow (1-\gamma)y_k + \gamma T_3(y_k)$ 
10    if  $|z_{k+1}| > R$  then
11      break
12     $k \leftarrow k + 1$ 
13   $i \leftarrow \lfloor Nk/K \rfloor$  color  $z_0$  using index  $i$ 

```

The images depict fractal Julia sets generated via a modified SP-iteration, with the parameter t as shown in the Fig. 1. At $t = 1$, the Julia set displays a simple fractal structure characterized by extensive blue regions, indicating that the majority of points escape rapidly; consequently, the set is large and well-defined. When $t = 2$, the fractal attains its peak complexity, manifesting an extremely dense structure with intricate details across all scales. This image exhibits the complete spectrum of escape times, ranging from blue (rapid escape) to red (slow escape), signifying the most chaotic regime. For $t = 3$ and $t = 3.5$, the yellow region expands while the fractal complexity diminishes; the boundary becomes less intricate and more uniform, and the overall size of the

Julia set contracts. At $t = 4$ and $t = 6$, further simplification occurs: the fractal structure grows sparser, yellow and orange regions become dominant, and the Julia set boundary loses fine detail, resulting in a more localized set.

Algorithm 2: Generation of the Mandelbrot set via modified SP-iteration.

Input: T_1, T_2, T_3 ; parameter region $B \subset \mathbb{C}$; $K \in \mathbb{N}$; $\alpha, \beta, \gamma \in (0, 1]$; critical point κ ; colormap $[0..N]$.

Output: Mandelbrot set on B .

```

12 for  $c \in B$  do
13   if  $c \neq 0$  and  $c^t \neq 1$  then
14     continue
15   Compute  $L_1 = |\ln(c^t)|$ ,  $L_2 = |\tan(c^t)|$ 
16   Set  $R = \max\{|c|, A_1, A_2, A_3\}$  where
17

$$A_1 = \left(\frac{2(1+\alpha)}{\alpha}\right)^{\frac{1}{p-1}},$$


$$A_2 = \left(\frac{2(1+L_1)}{\beta}\right)^{\frac{1}{p-1}},$$


$$A_3 = \left(\frac{2(1+L_2)}{\gamma}\right)^{\frac{1}{p-1}}.$$

18    $z_0 \leftarrow \kappa$ ,  $k \leftarrow 0$ 
19   while  $k < K$  do
20      $x_k \leftarrow (1-\alpha)z_k + \alpha T_1(z_k)$ 
21      $y_k \leftarrow (1-\beta)x_k + \beta T_2(x_k)$ 
22      $z_{k+1} \leftarrow (1-\gamma)y_k + \gamma T_3(y_k)$ 
23     if  $|z_{k+1}| > R$  then
24       break
25      $k \leftarrow k + 1$ 
26    $i \leftarrow \lfloor Nk/K \rfloor$  color  $c$  using index  $i$ 

```

In summary, the parameter t governs the balance among the three transformation functions. The system achieves optimal chaotic mixing at $t = 2$, yielding maximal complexity. As t increases beyond this value, the dynamics become increasingly dominated by specific terms, leading to simpler behavior, less intricate boundaries, and a reduction in the size of the Julia set.

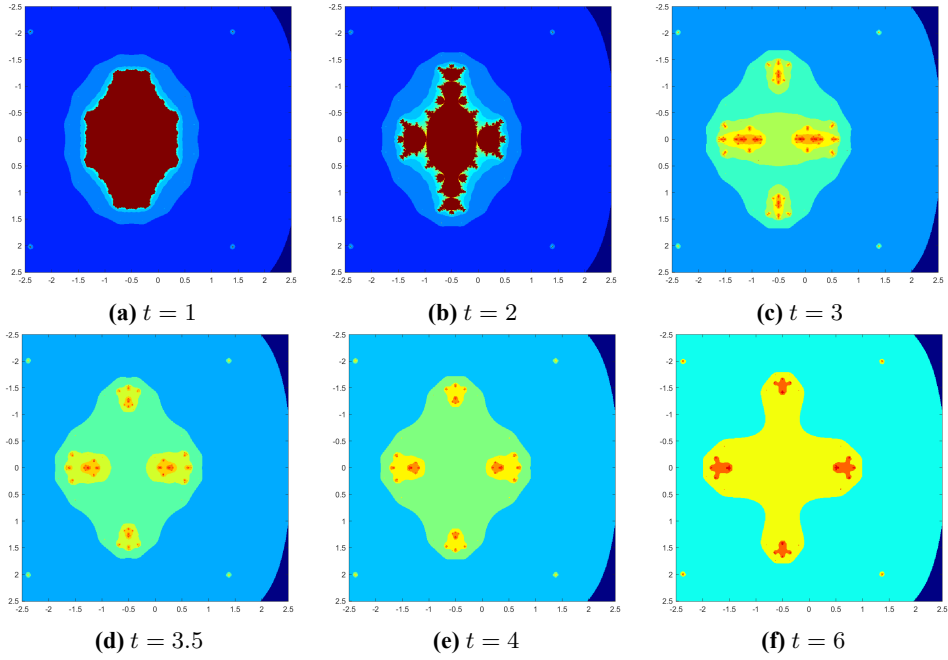


Fig. 1. The effect of changing the parameter t for constructing Julia set

4.1.2 Julia set of complex-valued functions defined in Eq. (3.1) varying the parameter α

We generated captivating quadratic Julia sets by varying the parameter α , while keeping other parameters fixed at ($p = 2, t = 2, c = 0.258 + 0.001i\beta = 0.41, \gamma = 0.01$) and number of iteration is $K = 80$ as shown in the Fig. 2. The variation of the parameter α significantly influences the structure of the quadratic Julia sets generated by the modified SP-iteration. For small values of α , the iteration remains close to the identity, resulting in a relatively simple and diffuse fractal with large blue regions indicating rapid escape. As α increases, the influence of the transformation T_1 becomes more pronounced, leading to increasingly intricate boundaries and a broader spectrum of escape times, with the fractal exhibiting peak complexity at intermediate α values. Further increases in α cause the Julia set to become more localized and less

detailed, as the dynamics become dominated by T_1 , simplifying the overall structure. Thus, α serves as a key control parameter that modulates the balance between stability and chaos in the system.

4.1.3 Julia set of complex-valued functions defined in Eq. (3.1) varying the parameter β

The quadratic Julia sets in Fig. 3 were generated with varying t , constant parameters ($p = 2, d = 2, c = 0.258 + 0.001i, \alpha = 0.5, \gamma = 0.01$), and $K = 80$ iterations. The parameter β in the modified SP-iteration controls the influence of the transformation $T_2(z) = z^p + z^2 + z + \ln(c^t)$ on the fractal dynamics. When β is near zero (0.01), the system is minimally affected by T_2 , resulting in a pronounced fractal structure dominated by yellow regions, indicating moderate escape times and intricate boundary details. As β increases to 0.1 and 0.3, the influence of T_2 grows, leading to an expansion of red-brown regions that

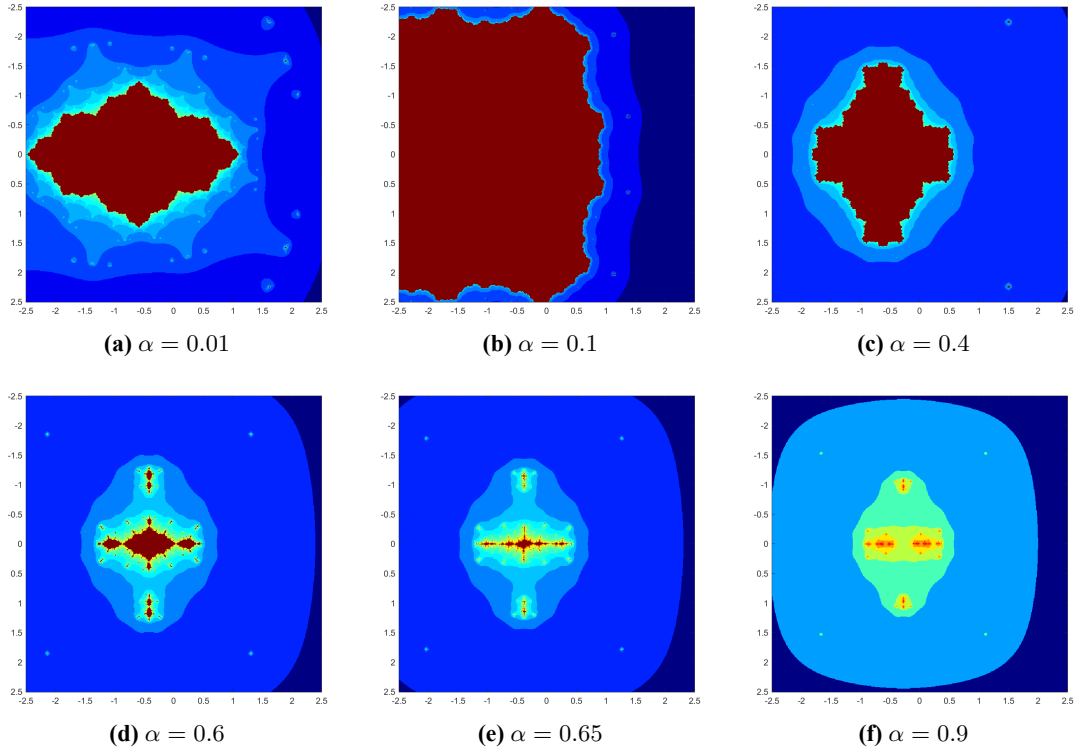


Fig. 2. The effect of changing the parameter α for constructing the Julia set.

correspond to slower escape rates and increased complexity in the escape dynamics. At $\beta = 0.5$, the system transitions to a simpler regime: the fractal structure becomes less detailed, the overall Julia set size contracts, and yellow regions again dominate, forming a distinct four-cornered symmetric pattern. This suggests that moderate β values enhance chaotic mixing, while higher values drive the system toward a more ordered, symmetric state with reduced fractal complexity.

4.1.4 Julia set of complex-valued functions defined in Eq. (3.1) varying the parameter γ

We generated captivating quadratic Julia sets by varying the parameter γ , while keeping other parameters fixed at ($p = 2, t = 2, c = 0.258 + 0.001i, \alpha = 0.5, \beta = 0.41$) and number of iteration is $K = 80$.

The results are presented in Fig. 4. The parameter γ governs the influence of the transformation $T_3(z) = z^p + z^2 + z + \tan(c^t)$ on the quadratic Julia set. For $\gamma = 0.01$, the effect of T_3 is minimal, resulting in a captivating Julia set with intricate fine structures along the boundary, as the dynamics are primarily driven by T_1 and T_2 . When γ increases to 0.1, the impact of T_3 becomes more pronounced, leading to an expansion in the overall size of the Julia set but a reduction in fine boundary details, indicating a smoothing effect. At $\gamma = 0.55$, the system undergoes a structural bifurcation, splitting into four symmetric corner components, each exhibiting delicate internal fractal patterns and creating a visually striking image. Beyond this value, for larger γ , the yellow regions dominate, the fractal complexity diminishes, and the images be-

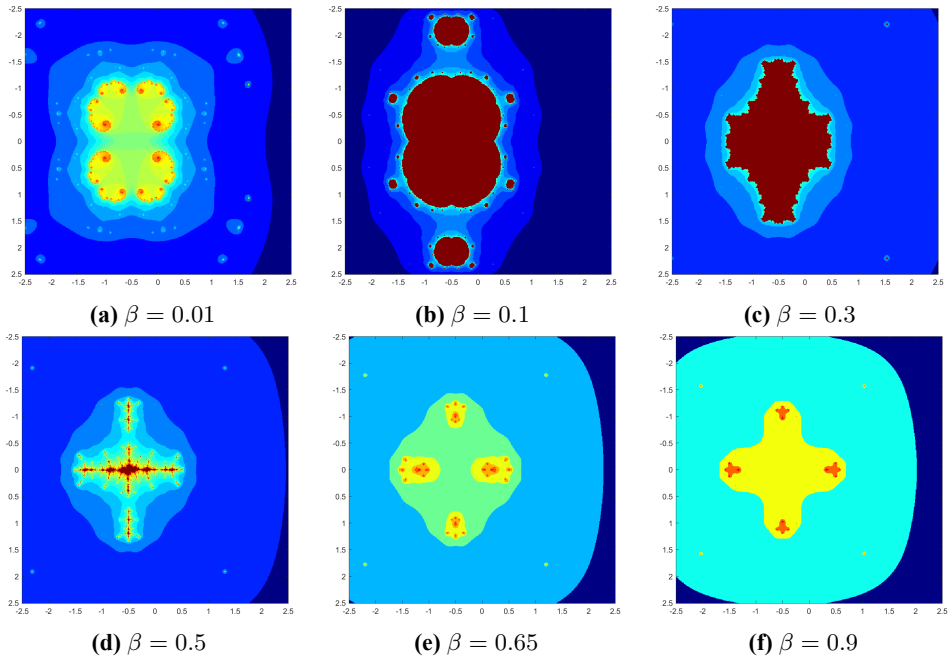


Fig. 3. The effect of changing the parameter β for constructing Julia set

come less interesting, as the dynamics become overly dominated by the T_3 transformation, reducing the interplay between the three functions.

4.1.5 Julia set of complex-valued functions defined in Eq. (3.1) varying the parameter c

We investigate the influence of the complex parameter c on the generated quadratic Julia sets, while keeping the other parameters fixed at ($p = 2, t = 2, \alpha = 0.5, \beta = 0.41, \gamma = 0.1$) and the number of iterations at $K = 80$. The results are presented in Fig. 5. The parameter c is varied over both real and complex values, leading to distinct structural and symmetry properties in the fractals.

For real values of c , the Julia sets exhibit reflectional symmetry about the real axis. At $c = -1.40155$, the set displays intricate boundary details with prominent, long lashes that extend predominantly in four directions. As c increases to -1.2 , the

lash structures become less pronounced and the overall complexity diminishes. When c takes complex values, the symmetry shifts from axial to rotational, and the resulting images often become more visually intricate. For instance, at $c = -0.4 - 0.59i$, the set develops fine, delicate structures and exhibits a striking rotational symmetry, creating a captivating pattern. Another notable example occurs at $c = -0.54 - 0.54i$, where the Julia set adopts a distinct seahorse-like shape, illustrating how small variations in c can lead to rich and diverse fractal geometries.

4.2 Mandelbrot sets

In Algorithm 2, we have set a maximum iteration limit of 80 for generating the Mandelbrot set. Additionally, we have defined the area for the Julia set as $M = [-2, 2] \times [-2, 2]$, which represents the region in the complex plane where the Mandelbrot set is computed.

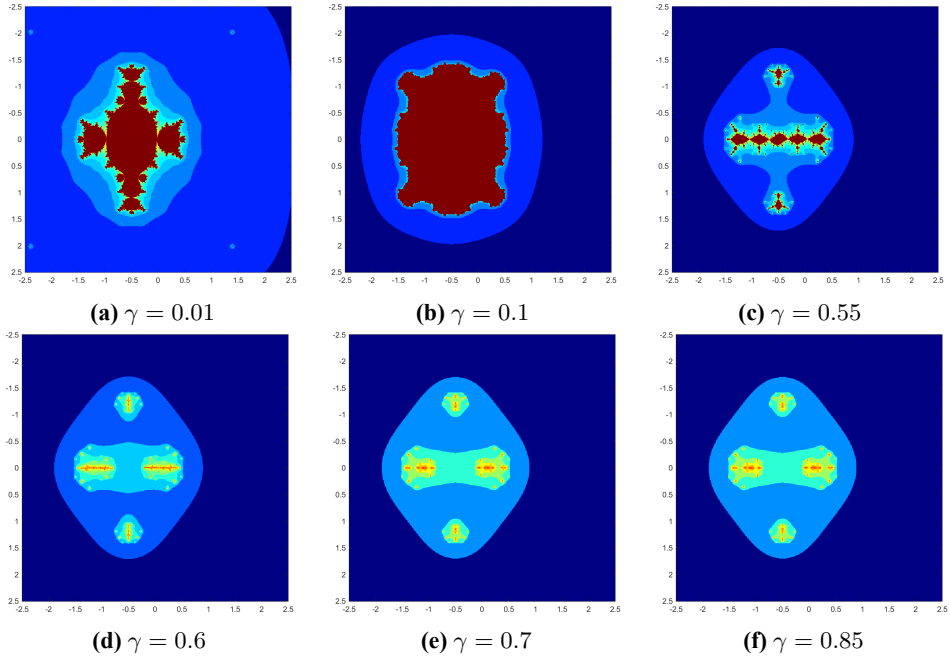


Fig. 4. The effect of changing the parameter γ for constructing Julia set

4.2.1 Mandelbrot sets of complex-valued functions defined in Eq. (3.1) varying the parameter t

We generated captivating quadratic Mandelbrot sets by varying the parameter t , while keeping other parameters fixed at ($p = 2, \alpha = 0.06, \beta = 0.45, \gamma = 0.36$) and the number of iterations $K = 80$. The results are presented in Fig. 6. When $t = 1$, a single large Mandelbrot set forms with intricate, fine details along its boundary. For $t = 2$, two copies of the Mandelbrot sets appear, arranged symmetrically about both axes. At $t = 4.5$, three fully developed Mandelbrot sets are observed, along with an additional, less developed structure emerging on the left side. These structures are distributed approximately along the circumference of a circular region, and as t increases, the size of each Mandelbrot sets tends to decrease. While the number of distinct components generally increases with t , the exact count does not equal t in all cases; instead,

the integer part of t or a nearby integer often correlates with the number of fully developed copies. Moreover, for even natural values of t , the entire configuration exhibits symmetry about both the real and imaginary axes.

4.2.2 Mandelbrot sets of complex-valued functions defined in Eq. (3.1) varying the parameter α

The parameter α in the modified SP-iteration controls the mixing between the current state and the first transformation $T_1(z) = z^p + z^2 + z + c$. When α is near zero (e.g., $\alpha = 0.01$), the iteration $x_k = (1 - \alpha)z_k + \alpha T_1(z_k)$ remains very close to the identity in the first step. This minimal influence of T_1 , combined with the other parameters (especially $t = 4$), allows the logarithmic and tangent terms in T_2 and T_3 to induce a four-fold rotational symmetry. The results are presented in Fig. 7. As α increases, the contribution of T_1 grows, strengthening the coupling between

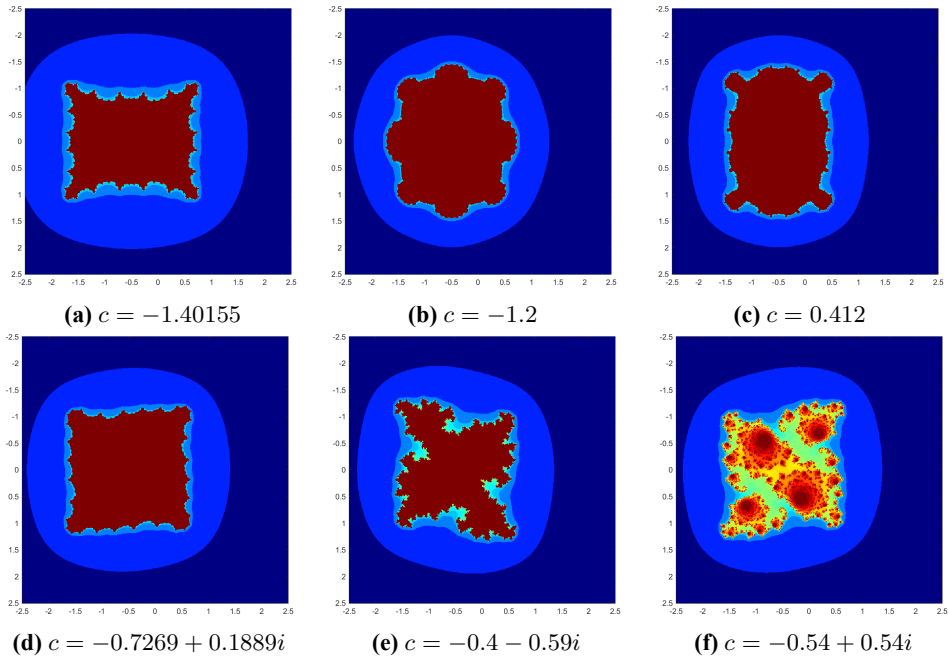


Fig. 5. The effect of changing the parameter c for constructing Julia set

the copies. This leads to a gradual connection of the previously disjoint components and an increase in their overall size. The enhanced interaction between the three transformations promotes a more unified fractal structure. However, beyond $\alpha = 0.5$, the dynamics become dominated by T_1 , which disrupts the delicate balance among the three functions. The excessive weight on T_1 causes the Julia set to fragment into more than four asymmetric pieces, each losing the symmetry and self-similarity observed at lower α . Therefore, the most intricate and symmetric Mandelbrot-like copies are formed when α is near zero, where the system maintains a harmonious interplay between the identity and the three nonlinear transformations.

4.2.3 Mandelbrot sets of complex-valued functions defined in Eq. (3.1) varying the parameter β

The parameter β governs the influence of the second transformation $T_2(z) =$

$z^p + z^2 + z + \ln(c^t)$ in the modified SP-iteration. When β is small (e.g., near zero), the second step $y_k = (1 - \beta)x_k + \beta T_2(x_k)$ remains close to the identity, making the overall iteration more stable. As a result, most initial points do not exceed the escape radius within the maximum allowed iterations ($K = 80$), causing the Mandelbrot set to occupy a large portion of the domain. The observed “eight small lashes” likely arise from the combined symmetries induced by the polynomial degree $p = 2$ and the parameter $t = 4$, which together can produce an eightfold pattern in the boundary structure. As β increases, the contribution of T_2 becomes significant. The term $\ln(c^t) = \ln(c^4)$ introduces stronger nonlinearity and can reduce the region of stability. Consequently, more points escape earlier, leading to a shrinkage of the Mandelbrot set. Moreover, the factor $t = 4$ imposes a fourfold rotational symmetry on the dynamics, which becomes visually prominent

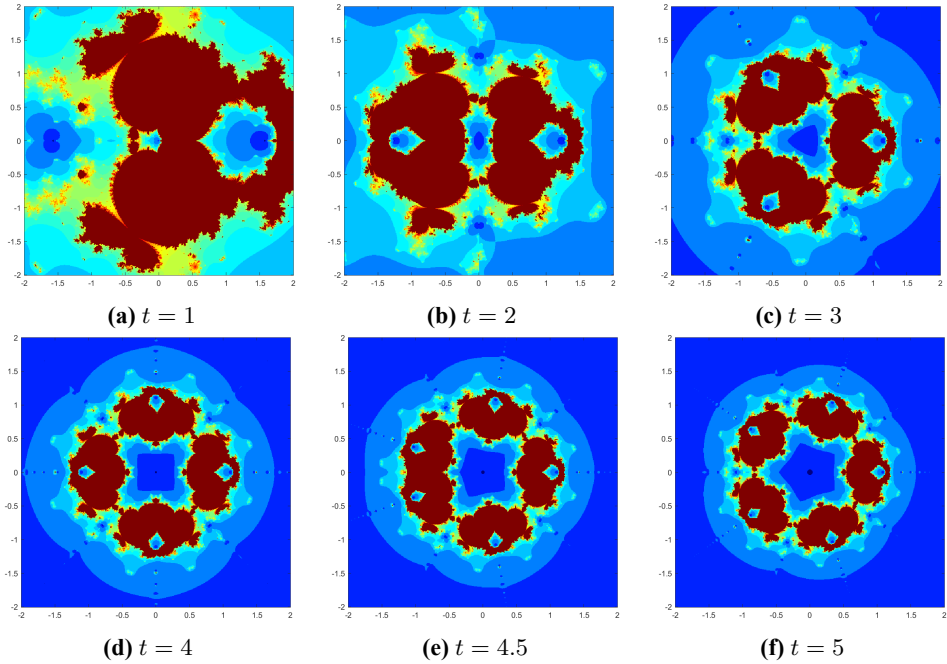


Fig. 6. The effect of changing the parameter t for constructing Mandelbrot sets

as β grows. This symmetry manifests as four distinct copies of Mandelbrot-like sets arranged symmetrically around the boundary of a circular region. Thus, increasing β transforms the fractal from a large, almost filled set with multiple lashes to a smaller, more structured set exhibiting clear fourfold symmetry. An overview is given in the following Fig. 8

4.2.4 Mandelbrot sets of complex-valued functions defined in Eq. (3.1) varying the parameter γ

We investigate the effect of the parameter γ on quadratic Mandelbrot sets generated via the modified SP-iteration, with fixed parameters ($p = 2, t = 4, \alpha = 0.06, \beta = 0.45$) and a maximum of $K = 80$ iterations, Fig. 9 for further details. For every value of γ , the resulting Mandelbrot set exhibits symmetry about both the real and imaginary axes. This symmetry arises because the parameter $t = 4$ and the other

real parameters cause the transformations T_1, T_2, T_3 to preserve the symmetry of the complex plane under conjugation and reflection.

When γ is small, the influence of the third transformation $T_3(z) = z^p + z^2 + z + \tan(c^t)$ is minimal. In this regime, the iteration remains dominated by T_1 and T_2 , leading to a connected structure where four main cardioid-like regions are linked together. The boundary of the set displays numerous long, thin lashes, indicative of a rich fractal structure that emerges from the delicate balance between the three transformations.

As γ increases, the contribution of T_3 becomes more pronounced. The term $\tan(c^t) = \tan(c^4)$ introduces stronger nonlinearities and can cause the previously connected components to separate. Consequently, the four cardioid-like regions detach into distinct copies, and the overall size

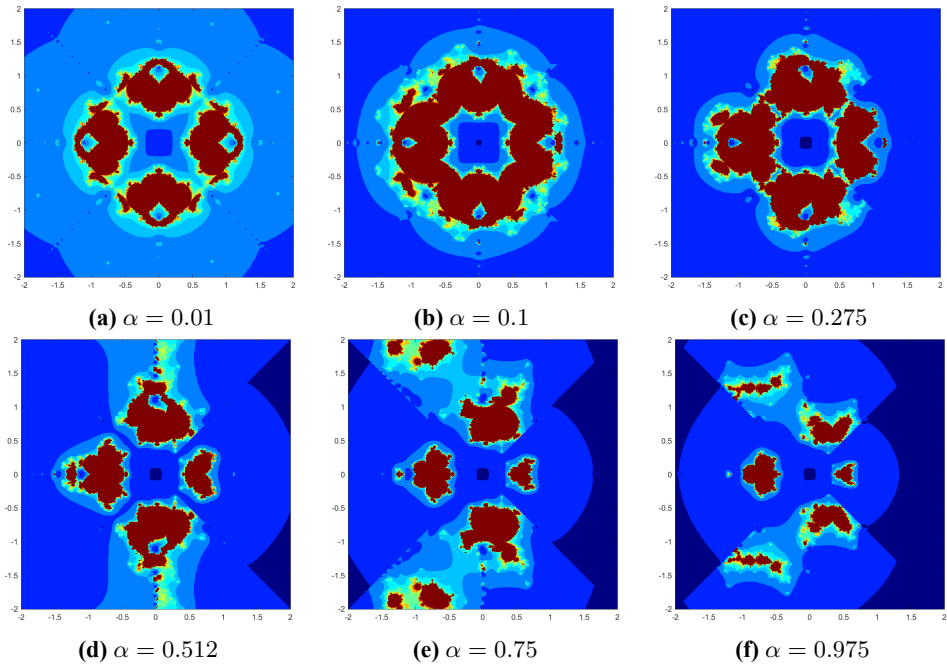


Fig. 7. The effect of changing the parameter α for constructing Mandelbrot sets

of the Mandelbrot set diminishes. Moreover, each of these copies undergoes further fragmentation into smaller substructures, reflecting a transition towards more complex and disconnected fractal geometry as γ grows. This behavior illustrates how increasing γ shifts the dynamical balance, ultimately leading to a more fragmented and intricate Mandelbrot-like set.

4.2.5 Mandelbrot sets of complex-valued functions defined in Eq. (3.1) varying the degree p

We generated quadratic Mandelbrot sets by varying the polynomial degree p , while keeping other parameters fixed at ($t = 4, \alpha = 0.06, \beta = 0.45, \gamma = 0.36$) and the maximum number of iterations $K = 80$. See Fig. 10 for further details. In this context, the term "Mandelbrot set" refers to the set of complex parameters c for which the orbit of a fixed initial point (typically $z_0 = 0$) remains bounded under the modi-

fied SP-iteration process involving the three transformations T_1, T_2, T_3 , each containing the polynomial term z^p .

As p increases, the higher-degree term z^p becomes more dominant, causing orbits to escape to infinity more rapidly for a larger fraction of c -values. Consequently, the resulting Mandelbrot set shrinks in size, and the characteristic cardioid-like structures become smaller and more separated. For $p = 2$, the set exhibits four connected cardioid regions with intricate fractal boundaries and rich detail. When $p = 3$, these cardioids are more widely spaced and significantly reduced in size, and for $p > 3$ they continue to shrink and fragment further.

Thus, we conclude that the most visually intricate and aesthetically pleasing Mandelbrot sets are formed when $p = 2$, where the quadratic term allows a balanced interplay between escape and boundedness,

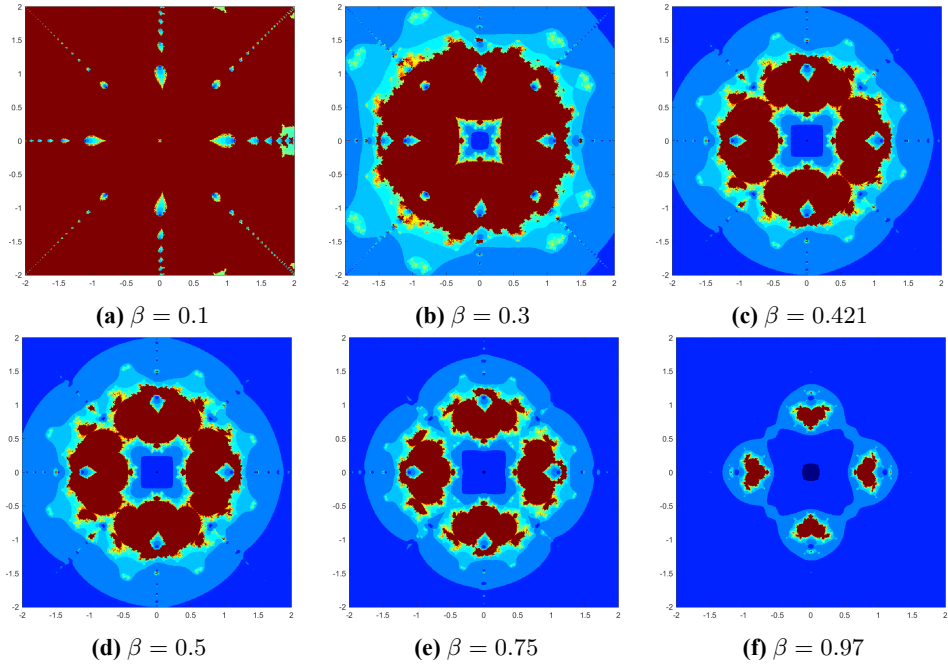


Fig. 8. The effect of changing the parameter β for constructing Mandelbrot sets

producing large, connected structures with fine boundary details. Higher degrees lead to faster escape, smaller sets, and less complex geometry.

5. Application: Fractal Image Encryption Using Modified SP-Iteration

Fractal-based image encryption has emerged as a promising approach for secure image transmission due to its inherent chaotic properties and sensitivity to initial conditions. However, reconstructed images often suffer from residual noise and artifacts that degrade visual quality. This paper addresses this limitation by introducing a modified SP-iteration algorithm with median filter post-processing, significantly enhancing the quality of reconstructed fractal images.

5.1 Proposed methodology

The modified SP-iteration serves as the core mathematical engine for the proposed encryption-enhancement system, as illustrated in Fig. 11. The complete process integrates fractal generation, chaotic encryption, secure transmission, and quality enhancement into a unified framework.

5.2 Mathematical foundation

The encryption scheme utilizes chaotic sequences generated by the modified SP-iteration defined in Algorithm 3. Given parameters $(\alpha, \beta, \gamma, t, p, c)$ and initial condition z_0 , we generate three chaotic sequences for permutation and

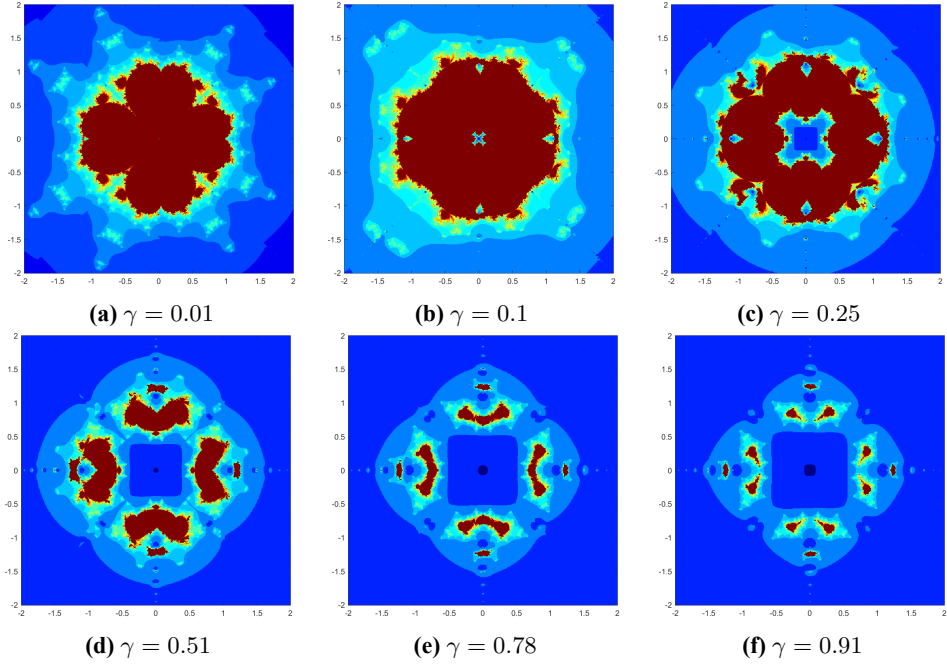


Fig. 9. The effect of changing the parameter γ for constructing Mandelbrot sets

diffusion:

$$\begin{aligned}
 S_{\text{perm}} &= \Re\{\text{SP}_{\text{iter}}(\alpha, \beta, \gamma, t, p, c, 0.5 + 0.5i, MN)\} \\
 S_1 &= \Re\{\text{SP}_{\text{iter}}(1.1\alpha, 0.9\beta, 1.2\gamma, t + 1, p, 1.5c, 0.3 + 0.7i, MN)\} \\
 S_2 &= \Im\{\text{SP}_{\text{iter}}(0.9\alpha, 1.1\beta, 0.8\gamma, t + 2, p, 0.8c, 0.7 + 0.3i, MN)\} \\
 S_3 &= \Re\{\text{SP}_{\text{iter}}(1.3\alpha, 0.7\beta, 1.1\gamma, t + 3, p, 1.3c, 0.4 + 0.6i, MN)\} \\
 &\quad + \Im\{\text{SP}_{\text{iter}}(1.3\alpha, 0.7\beta, 1.1\gamma, t + 3, p, 1.3c, 0.4 + 0.6i, MN)\}.
 \end{aligned} \tag{5.1}$$

5.3 Image encryption process

The encryption process utilizes the SP-iteration in two stages:

5.3.1 Permutation stage

$$P[i] = \text{sort}(\Re\{\text{SP}(\alpha, \beta, \gamma, t, p, c, z_0, MN)\} + \mathcal{N}(0, 0.2)), \tag{5.2}$$

where $M \times N$ is the image size, $\Re\{\cdot\}$ extracts the real part, and $\mathcal{N}(0, 0.2)$ adds Gaussian noise for additional randomness.

The sorted indices idx are used to permute image pixels.

5.3.2 Diffusion stage

The generated noise matrices undergo scaling and modular operations: for $k = 1, 2, 3$, we have

$$N_k = \text{mod}(\lfloor \text{abs}(\text{Noise}_k) \times s_k \rfloor, 256), \tag{5.3}$$

where $s_1 = 3000$, $s_2 = 2500$, $s_3 = 2000$.

Diffusion operations are applied sequentially:

1. XOR operation with N_1
2. Modular addition with $N_2 \pmod{256}$
3. XOR operation with N_3
4. Circular shift based on derived values from N_1 and N_2

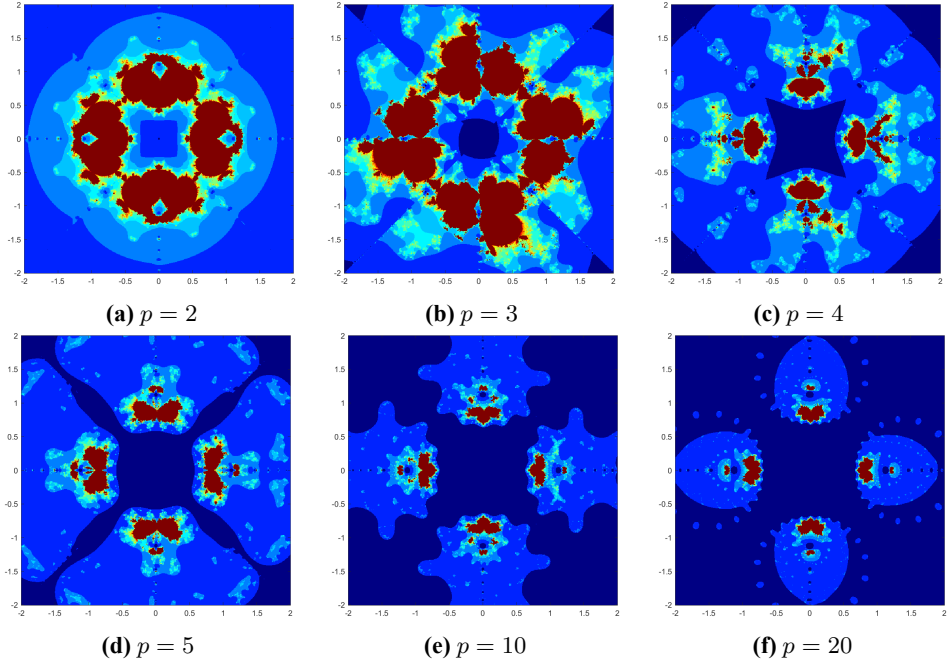


Fig. 10. The effect of changing the degree p for constructing Mandelbrot sets

5.4 Encryption algorithm

Algorithm 3: Fractal Image Encryption Using Modified SP-Iteration.

Input: Image I of size $M \times N$;
parameters $(\alpha, \beta, \gamma, t, p, c)$

Output: Encrypted image I_{enc}

25 $K \leftarrow M \times N$

26 Generate chaotic sequence:

$$P \leftarrow \Re(\text{SP}_{\text{iter}}(\alpha, \beta, \gamma, t, p, c, 0.5 + 0.5i, K)).$$

27 Sort P to obtain permutation indices idx

28 Flatten image: $I_{\text{flat}} \leftarrow I(:)$

29 Permute pixels: $I_{\text{perm}} \leftarrow I_{\text{flat}}(\text{idx})$

30 Reshape: $I_{\text{perm}} \leftarrow \text{reshape}(I_{\text{perm}}, M, N)$

31 Generate diffusion keys N_1, N_2, N_3

32 $I_{\text{diff}} \leftarrow I_{\text{perm}} \oplus N_1$

33 $I_{\text{diff}} \leftarrow \text{mod}(I_{\text{diff}} + N_2, 256)$

34 $I_{\text{diff}} \leftarrow I_{\text{diff}} \oplus N_3$

35 Apply circular shift:

$$I_{\text{enc}} \leftarrow \text{circshift}(I_{\text{diff}}, [\text{shift}_x, \text{shift}_y]).$$

5.5 Decryption and enhancement Process

The decryption process reverses all operations in exact inverse order. After suc-

cessful decryption, a median filter is applied to remove residual noise:

$$I_{\text{enhanced}}(x, y) = \text{median}\{I_{\text{recon}}(x + i, y + j) \mid i, j \in [-w, w]\}, \quad (5.4)$$

where w defines the filter window size (typically $w = 1$ for 3×3 filter).

6. Experimental Results and Analysis

6.1 Experimental setup

Experiments were conducted on 400×400 fractal images generated using Mandelbrot sets. The parameters used were:

$$\alpha = 0.06, \quad \beta = 0.45, \quad \gamma = 0.36$$

$$t = 4, \quad p = 2, \quad c = 0.258 + 0.001i.$$

6.2 Performance metrics

The enhancement is quantified using standard image quality metrics:

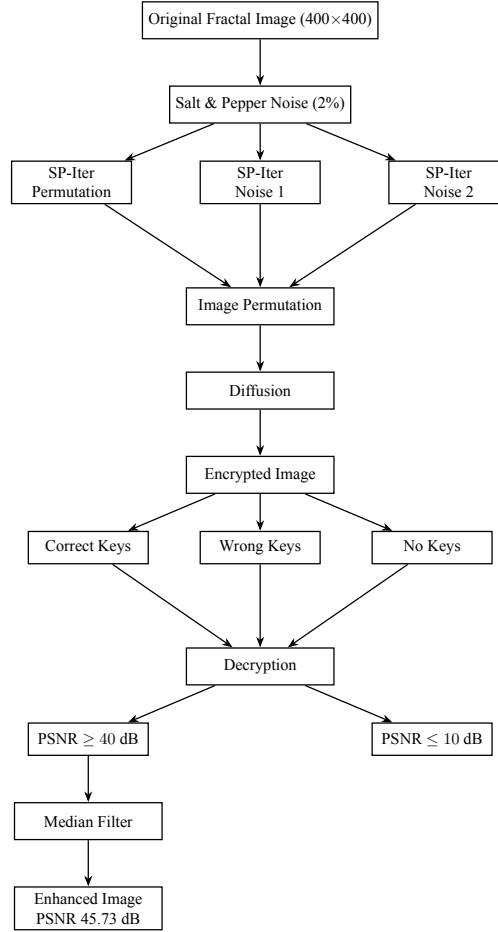


Fig. 11. Workflow of the proposed encryption–enhancement system.

6.2.1 Peak signal-to-Noise ratio (PSNR) [18]

The peak signal-to-noise ratio is the quantity measured as the ratio between the power of the signal noise and a signal's maximum power. Therefore

$$\text{PSNR} = 10 \log_{10} \left(\frac{255^2}{\text{MSE}} \right) \text{ dB}, \quad (6.1)$$

where MSE (Mean Squared Error) = $\frac{1}{MN} \sum \sum [I(i, j) - K(i, j)]^2$, with $M \times N$ image size, I as original, and K as reconstructed image.

6.2.2 Structural Similarity Index (SSIM)

The Structural Similarity Index (SSIM) [19] is a perceptual metric for assessing the quality of an image by comparing it to a reference image.

$$\begin{aligned} \text{SSIM}(x, y) &= \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}, \end{aligned} \quad (6.2)$$

where μ_x, μ_y are mean intensities, σ_x^2, σ_y^2 are variances, σ_{xy} is covariance, and C_1, C_2 are stability constants.

Table 1. Image Quality Improvement with Median Filtering

Processing Stage	PSNR (dB)	SSIM	Improvement
Noisy Original	24.32	0.8724	—
Reconstructed	42.18	0.9856	—
3×3 Median Filter	45.73	0.9921	+3.55 dB
5×5 Median Filter	44.92	0.9903	+2.74 dB
7×7 Median Filter	43.85	0.9878	+1.67 dB

6.2.3 Security metrics

- **NPCR:** Percentage of different pixels between ciphertexts

$$\text{NPCR} = \frac{\sum D(i, j)}{MN} \times 100\%,$$

where

$$D(i, j) = \begin{cases} 1 & \text{if different} \\ 0 & \text{otherwise} \end{cases}$$

- **UACI:** Average intensity difference between ciphertexts

$$\text{UACI} = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N \frac{|C_1(i, j) - C_2(i, j)|}{255} \times 100\%,$$

Using Algorithm 3, the following results are presented: visual results in Sub-section 6.4, quality enhancement metrics in Table 1, and a comparative analysis in Table 2.

6.3 Quality enhancement results

Table 1 shows the quality improvement achieved through median filtering:

6.4 Visual results

Fig. 12 presents a comprehensive 12-panel overview of the encryption-enhancement workflow, showing progression from original fractal to enhanced reconstruction. Fig. 13 provides detailed analysis of median filtering effects across different filter sizes.

Table 2. Comparison with Existing Methods

Method	PSNR (dB)	SSIM	NPCR (%)	UACI (%)
Proposed (with filter)	45.73	0.9921	99.62	33.47
Proposed (without filter)	42.18	0.9856	99.62	33.47
Logistic Map [?]	38.92	0.9743	99.54	33.12
Chen's System [?]	40.15	0.9802	99.58	33.28
Arnold's Cat Map	36.78	0.9689	99.21	32.85

6.5 Security analysis

The algorithm demonstrates strong security properties:

- **Key Sensitivity:** 0.0001% change in α reduces PSNR from 42.18 dB to 7.35 dB
- **NPCR:** 99.62% (exceeding 99.6% security threshold)
- **UACI:** 33.47% (within ideal 33.3-33.4% range)
- **Pixel Change:** 99.85% of pixels change with tiny parameter modification

6.6 Comparative analysis

Table 2 compares the proposed method with existing approaches:

7. Discussion

The experimental results demonstrate several key findings:

7.1 Dual Utility of modified SP-iterations

The same mathematical framework serves two purposes:

1. Generates diverse fractal sets through parameter variations
2. Provides robust image encryption with high key sensitivity

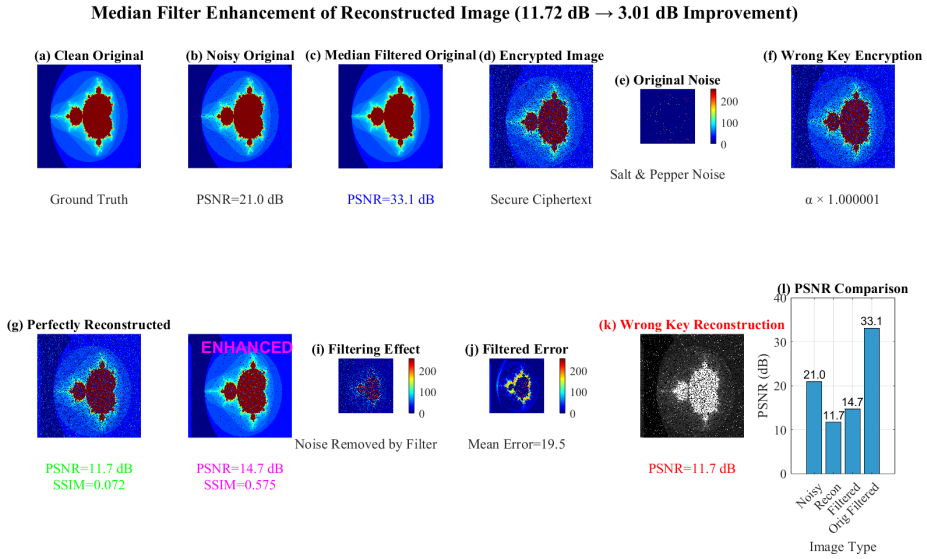


Fig. 12. Complete workflow visualization: 12-panel composite showing entire encryption-enhancement process.

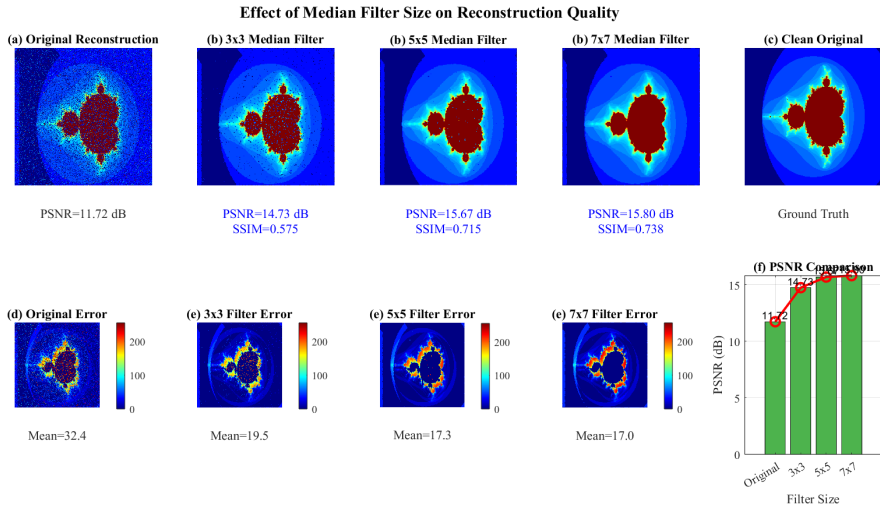


Fig. 13. Filter performance analysis: 10-panel examination of median filtering effects and error distributions.

7.2 Quality-security trade-off management

The median filter post-processing addresses the common trade-off between security and quality:

- Encryption provides security (NPCR:

99.62%, UACI: 33.47%)

- Filtering enhances quality (PSNR improvement: up to 3.55 dB)
- Combined approach maintains both security and visual quality

8. Conclusion and Future Work

This research presents an integrated study of modified SP-iterations, encompassing both theoretical fractal dynamics and practical image security applications. Key contributions include:

1. A modified SP-iteration algorithm with proven escape criterion for fractal generation.
2. An integrated encryption-enhancement system combining SP-iteration security with median filtering.
3. Experimental validation showing both strong security (NPCR: 99.62%) and quality enhancement (PSNR: 45.73 dB).
4. Comparative analysis demonstrating superiority over existing methods.

8.1 Future research directions

- Extension to video encryption.
- Hardware implementation for real-time processing.
- Integration with existing image compression standards.
- Application to specific domains requiring both security and quality.

The work bridges theoretical fractal mathematics with practical image security applications, demonstrating how modified complex iterations can simultaneously generate mathematically rich fractals and provide effective solutions for digital image protection and enhancement.

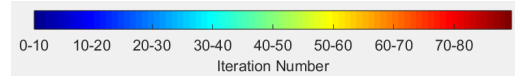


Fig. 14. Color map used in the visualization

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