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An Analytical Study of Fixed Point Theorems in Intuitionistic Fuzzy b-Metric Spaces

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ABSTRACT

In this paper, the study is devoted to investigating fixed point theorems in intuitionistic fuzzy b-metric spaces (IFb-MS) and extends classical results from fuzzy b-metric spaces (Fb-MS). It introduces a broader class of sufficient conditions ensuring the Cauchy nature of sequences, thereby generalizing and simplifying contraction mappings and their proofs. The unified approach accommodates diverse contractive conditions and offers novel insights into the establishment of existence and uniqueness results for fixed points in the proposed generalized topology.

Keywords: Fixed point; IFb-MS; Cauchy sequence; Contraction conditions

1. Introduction

Zadeh [1] presented the concept of fuzzy logic, and revolutionized how uncertainty is addressed in mathematical modeling. The central thought behind fuzzy sets is that components have values between $[0, 1]$. This system has found broad applications in different disciplines, counting counterfeit insights, control frameworks and optimization in the setting of metric spaces. The concept of *fuzzy metric space* (FMS) was first proposed by Kaleva [2]. Kramosil and Michalek [3] further refined these thoughts, setting up a foundational approach. George and Veeramani [4] expanded upon previous work to develop a more comprehensive numerical system over time. Broad investigation has been devoted to the study of FMS driving a broad set of critical fixed point hypotheses. These incorporate key results con-

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tributed by different analysts (see [5–11]) which have played a crucial part in forming the hypothesis of FMS.

Park [12] expanded the concept of FMS by introducing *intuitionistic fuzzy metric spaces* (IFMS); these spaces incorporate both the belongingness and non-belongingness functions, providing a refined approach to quantifying uncertainty. Over time, research on IFMS has gained significant traction, with numerous studies focusing on fixed point theory (see [13–15]).

The further generalization led to the emergence of b-metric spaces, which was initially introduced by Bakhtin [16] and Czerwik [17]. These spaces modify the triangle inequality and facilitate the analysis of more complex structures. Research has examined the correlation between b-metric spaces and FMS as shown in Hassanzadeh et al. [18]. This investigation led to the introduction of *fuzzy b-metric spaces* (Fb-MS) as presented by Sedghi and Shobe [8]. IFb-MS that integrate properties of IFMS and b-metric spaces offer a rich framework for investigating fixed point theory.

This hypothesis explores the nearness of fixed points inside IFb-MS, extending earlier work, especially the fixed point hypotheses in Fb-MS created by Dusan et al. [19]. This research work broadens and refines these discoveries to the system of IFb-MS. It adapts the result to the framework of IFb-MS by incorporating both membership and non-membership functions. The resulting advancements yield novel contributions to the fixed point theory in the context of IFb-MS.

This study examines fixed points of mappings that adhere to generalized contraction conditions. It establishes necessary and sufficient criteria for sequence convergence within IFb-MS. The research broadens existing fixed point results, providing a unified framework that deepens the understanding of fixed point theory in IFb-MS. These findings have valuable applications in fuzzy logic, soft computing, and mathematical modeling, particularly in scenarios involving uncertainty and imprecision. Moreover, the results pave the way for future studies on IFb-MS and their relevance in image processing, optimization, and control theory.

This research begins with an exploration of foundational concepts that are important for the main result of this research.

2. Preliminaries

Definition 2.1 ([20])**.** "A binary operation $\mu : [0,1] \times [0,1] \to [0,1]$ that combines two elements from the interval $[0,1]$ is identified as a continuous t-conorm provided it fulfills the following specific criteria.

1. $\mu(a,0) = a, \quad \forall a \in [0,1]$
2. The operation is commutative and associative; that is, for all $a, b, c \in [0,1]$, it holds that $\mu(a,b) = \mu(b,a)$ and $\mu(a,\mu(b,c)) = \mu(\mu(a,b),c)$.
3. Whenever $a \leq c$ and $b \leq d$, then $\mu(a,b) \leq \mu(c,d)$ for all $a, b, c, d \in [0,1]$.
4. The function μ is continuous throughout its domain."

Definition 2.2. [21] "Suppose τ is a t-norm. Its associated dual t-conorm $\mu : [0,1]^2 \to [0,1]$ can be represented as

$$\mu(x,y) = 1 - \tau(1-x, 1-y)."$$

Example 2.3 ([21])**.** "Presented below are four prominent t-norms and their corresponding dual t-conorms.

(i) *Minimum* τ_M and *Maximum* μ_M can be expressed as

$$\tau_M(x, y) = \min(x, y),$$

$$\mu_M(x, y) = \max(x, y);$$

(ii) *Product* τ_P and *Probabilistic Sum* μ_P which can be expressed as

$$\tau_P(x, y) = x \cdot y,$$

$$\mu_P(x, y) = x + y - x \cdot y;$$

(iii) *Łukasiewicz* τ_L and μ_L defined as

$$\tau_L(x, y) = \max(x + y - 1, 0),$$
$$\mu_L(x, y) = \min(x + y, 1);$$

(iv) *Drastic Product* τ_W and *Strongest t-conorm* μ_W can be expressed as

$$\tau_W(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_W(x, y) = \begin{cases} \max(x, y), & \text{if } \min(x, y) = 0, \\ 1, & \text{otherwise."} \end{cases}$$

Definition 2.4 ([22])**.** "Suppose τ is a t-norm. For each integer $n \geq 1$, define the sequence of functions $\tau_n : [0, 1] \to [0, 1]$ recursively by:

$$\tau_1(x) = \tau(x, x), \quad \tau_{n+1}(x) = \tau(\tau_n(x), x).$$

The t-norm τ is said to be of H-type if the collection $\{\tau_n(x)\}_{n\in\mathbb{N}}$ maintains equicontinuity at the point $x = 1$. Notably, the minimum t-norm $\tau_{\min}$ serves as a basic example of an H-type t-norm."

Proposition 2.5 ([23])**.** *"Suppose* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* $[0, 1]$ *with the property that* $\lim_{n\to\infty} x_n = 1$*, and consider a t-norm* τ *which is of H-type. Then,*

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} x_i = \lim_{n\to\infty} \tau_{i=n}^{\infty} x_{n+i} = 1."$$

Definition 2.6 ([8])**.** "A 3-tuple $(\mathfrak{X}, \mathscr{P}, \tau)$ is designated as a Fb-MS if it satisfies the following structural criteria: $\mathfrak{X}$ must be a non-empty set, τ must represent a continuous triangular norm (t-norm), and $\mathscr{P}$ must be a fuzzy set defined over the Cartesian product $\mathfrak{X}^2 \times (0, \infty)$. The formulation is subject to specific axioms that must hold for all elements $x, y \in \mathfrak{X}$, for all positive real numbers $t, s > 0$, and with respect to a fixed constant $b \geq 1$.

$(FbM1)$ $\mathscr{P}(x, y, t) > 0$

$(FbM2)$ $\mathscr{P}(x, y, t) = 1$ iff $x = y$

$(FbM3)$ $\mathscr{P}(x, y, t) = \mathscr{P}(y, x, t)$

$(FbM4)$ $\tau\left(\mathscr{P}\left(x, y, \frac{t}{b}\right), \mathscr{P}\left(y, z, \frac{s}{b}\right)\right) \leq \mathscr{P}(x, z, t + s)$

$(FbM5)$ $\mathscr{P}(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous."

The collection of FbMS encompasses a significantly wider range than that of FMS. In fact, any FMS is transformed into a fuzzy b-metric by $b = 1$. However, as the following example demonstrates, a fuzzy b-metric on $\mathfrak{X}$ need not yield a fuzzy metric on $\mathfrak{X}$.

Definition 2.7 ([24])**.** "An operator $T : \mathbb{R} \to \mathbb{R}$ is considered b-nondecreasing if, whenever $x > by$, it follows that $T(x) \geq T(y)$ for all $x, y \in \mathbb{R}$."

Lemma 2.8 ([8])**.** *"Suppose* $\mathscr{P}(x, y, t)$ *is an Fb-MS. Then, for every* $x, y \in \mathfrak{X}$*, the operator* $\mathscr{P}(x, y, t)$ *is b-nondecreasing as a function of* t*."*

Definition 2.9 ([12])**.** "Suppose fuzzy subsets $\mathscr{P}, \mathscr{Q} \subseteq \mathfrak{X}^2 \times (0, \infty)$, and let τ and μ denote continuous t-norm and t-conorm operations, respectively. The structure $(\mathscr{P}, \mathscr{Q})$ is recognized as an IFMS on $\mathfrak{X}$ whenever the following axiomatic conditions are fulfilled:

$(IFM1)$ $\mathscr{P}(x,y,t) + \mathscr{Q}(x,y,t) \leq 1$

$(IFM2)$ $\mathscr{P}(x,y,t) > 0$

$(IFM3)$ $\mathscr{P}(x,y,t) = 1$ iff $x = y$

$(IFM4)$ $\mathscr{P}(x,y,t) = \mathscr{P}(y,x,t)$

$(IFM5)$ $\tau\left(\mathscr{P}(x,y,t), \mathscr{P}(y,z,s)\right) \leq \mathscr{P}(x,z,t+s)$

$(IFM6)$ $\mathscr{P}(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous

$(IFM7)$ $\mathscr{Q}(x,y,t) < 1$

$(IFM8)$ $\mathscr{Q}(x,y,t) = 0$ iff $x = y$

$(IFM9)$ $\mathscr{Q}(x,y,t) = \mathscr{Q}(y,x,t)$

$(IFM10)$ $\mu\left(\mathscr{Q}(x,y,t), \mathscr{Q}(y,z,s)\right) \geq \mathscr{Q}(x,z,t+s)$

$(IFM11)$ $\mathscr{Q}(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

A five-tuple $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ is called an IFMS. The functions $\mathscr{P}(x,y,t)$ and $\mathscr{Q}(x,y,t)$ represent the membership grade of closeness and the membership grade of separation between x and y with respect to t, respectively."

Definition 2.10 ([12])**.** "Suppose an IFMS $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, where $t > 0$, $r \in (0,1)$, and $x \in \mathfrak{X}$. The set $\mathscr{B}_x(r,t)$ is defined as

$$\mathscr{B}_x(r,t) = \Big\{ y \in \mathfrak{X} : \mathscr{P}(x,y,t) > 1-r, \quad \mathscr{Q}(x,y,t) < r \Big\}.$$

The collection $\mathscr{B}_x(r,t)$ denotes an open ball centered at x, characterized by a radius r and an auxiliary parameter t, which governs its fuzzy proximity structure within the underlying space."

Definition 2.11 ([12])**.** "Suppose an IFMS $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ and a sequence $\{x_n\} \subset \mathfrak{X}$.

1. The sequence $\{x_n\}$ converges to x if, for any $t > 0$ and any r with $0 < r < 1$, there exists a natural number n_0 such that

   $$\mathscr{P}(x_n,x,t) > 1-r, \quad \mathscr{Q}(x_n,x,t) < r$$

   hold for all $n \geq n_0$. As n approaches infinity, $\mathscr{P}(x_n,x,t) \to 1$ and $\mathscr{Q}(x_n,x,t) \to 0$ for each $t > 0$. Such convergence is denoted by $x_n \to x$ as $n \to \infty$.

2. The sequence $\{x_n\}$ is called Cauchy if, given any $t > 0$ and $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

   $$\mathscr{P}(x_n,x_m,t) > 1-r, \mathscr{Q}(x_n,x_m,t) < r$$

   for every $n, m \geq n_0$.

3. A space $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ is referred to as $(\mathscr{P}, \mathscr{Q})$-*complete* if each and every sequence satisfying the Cauchy criterion within this framework admits a convergent limit that resides in $\mathfrak{X}$."

Definition 2.12. Let $\mathfrak{X}$ be a non-empty set, and let $b \geq 1$ be a given real number. Let τ denote a continuous triangular norm (t-norm) and μ represent a continuous triangular conorm (t-conorm). Suppose $\mathscr{P}$ and $\mathscr{Q}$ are fuzzy sets on the Cartesian product $\mathfrak{X}^2 \times (0,\infty)$. The five-tuple $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ is said to define an Intuitionistic Fuzzy b-Metric Space (IFb-MS) if, for all $x, y, z \in \mathfrak{X}$ and for all $t, s > 0$, the following axiomatic conditions are satisfied.

$(IFbM1)$ $\mathscr{P}(x,y,t) + \mathscr{Q}(x,y,t) \leq 1$

$(IFbM2)$ $\mathscr{P}(x,y,t) > 0$

$(IFbM3)$ $\mathscr{P}(x,y,t) = 1$ if and only if $x = y$

$(IFbM4)$ $\mathscr{P}(x,y,t) = \mathscr{P}(y,x,t)$

$(IFbM5)$ $\mathscr{P}(x,z,t+s) \geq \tau\left(\mathscr{P}\left(x,y,\frac{t}{b}\right), \mathscr{P}\left(y,z,\frac{s}{b}\right)\right)$

(*IFbM*6) $\mathscr{P}(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous

(*IFbM*7) $\mathscr{Q}(x, y, t) < 1$

(*IFbM*8) $\mathscr{Q}(x, y, t) = 0$ if and only if $x = y$

(*IFbM*9) $\mathscr{Q}(x, y, t) = \mathscr{Q}(y, x, t)$

(*IFbM*10) $\mathscr{Q}(x, z, t + s) \leq \mu\left(\mathscr{Q}\left(x, y, \frac{t}{b}\right), \mathscr{Q}\left(y, z, \frac{s}{b}\right)\right)$

(*IFbM*11) $\mathscr{Q}(x, y, \cdot) : (0, \infty) \to [0, 1)$ is continuous

An IFb-MS is defined as a five-tuple $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, where:

- The function $\mathscr{P}(x, y, t)$ quantifies the membership grade of closeness between x and y relative to the parameter t.
- The function $\mathscr{Q}(x, y, t)$ quantifies the non-membership grade of separation between x and y relative to the parameter t.

Definition 2.13. In an IFb-MS:

i A sequence $\{x_n\}$ is considered to converge to x if, for each $t > 0$ and $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\mathscr{P}(x_n, x, t) > 1 - r$ and $\mathscr{Q}(x_n, x, t) < r$ for all $n \geq n_0$. Furthermore, $\mathscr{P}(x_n, x, t) \to 1$, and $\mathscr{Q}(x_n, x, t) \to 0$ as $n \to \infty$, for every $t > 0$. This convergence is represented as $x_n \to x$ as $n \to \infty$.

ii The sequence $\{x_n\}$ is called a Cauchy sequence if, for any $t > 0$ and any $r \in (0, 1)$, there exists an $n_0 \in \mathbb{N}$ such that $\mathscr{P}(x_n, x_m, t) > 1 - r$ and $\mathscr{Q}(x_n, x_m, t) < r$ for all $n, m \geq n_0$. Equivalently, this means $\lim_{n,m\to\infty} \mathscr{P}(x_n, x_m, t) = 1$ and $\lim_{n,m\to\infty} \mathscr{Q}(x_n, x_m, t) = 0$.

iii An IFb-MS is designated as *complete* if each Cauchy sequence residing within the space possesses a limiting point that remains confined to the space under consideration.

3. Main Results

Throughout this section, we employ the IFb-MS $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, as defined in Definition 2.12, subject to the explicit imposition of the following additional limit conditions for all $x, y \in \mathfrak{X}$:

$$\begin{aligned} &\lim_{t\to\infty} \mathscr{P}(x, y, t) = 1, \\ &\lim_{t\to\infty} \mathscr{Q}(x, y, t) = 0. \end{aligned} \tag{H1}$$

These explicit assumptions are necessary to guarantee the convergence of sequence arguments evaluated at infinity.

Lemma 3.1. *Suppose* $\{x_n\}$ *is a sequence in an IFb-MS* $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$. *Assume that* $\xi \in (0, \frac{1}{b})$ *such that:*

$$\mathscr{P}(x_n, x_{n+1}, t) \geq \mathscr{P}\left(x_{n-1}, x_n, \frac{t}{\xi}\right) \tag{3.1}$$

$$\mathscr{Q}(x_n, x_{n+1}, t) \leq \mathscr{Q}\left(x_{n-1}, x_n, \frac{t}{\xi}\right) \tag{3.2}$$

for $n \in \mathbb{N}$, $t > 0$. *If there exist* $x_0, x_1 \in \mathfrak{X}$ *and* $v \in (0, 1)$ *such that*

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} \mathscr{P}\left(x_0, x_1, \frac{t}{v^i}\right) = 1, \quad t > 0 \tag{3.3}$$

$$\lim_{n\to\infty} \mu_{i=n}^{\infty} \mathscr{Q}\left(x_0, x_1, \frac{t}{v^i}\right) = 0, \quad t > 0 \tag{3.4}$$

Then $\{x_n\}$ *is a Cauchy sequence in* $\mathfrak{X}$.

Proof. Choose a constant $\delta \in (\xi b, 1)$. Because $\xi < 1/b$, it is guaranteed that $\xi b < 1$,

meaning such a δ exists. The geometric series $\sum_{i=1}^{\infty}\delta^i$ is convergent, which guarantees there exists an integer $n_0 \geq 2$ such that $\sum_{i=n}^{\infty}\delta^i < 1$ for all $n \geq n_0$.

Let $m > 0$ and $n > n_0$. Because $\mathscr{P}$ is continuous and b-nondecreasing with respect to the time parameter, and since $\sum_{i=n}^{\infty}\delta^i < 1$ and $b \geq 1$, it follows that $t > t\sum_{i=n}^{n+m-1}\delta^i \geq \frac{t}{b}\sum_{i=n}^{n+m-1}\delta^i$. Consequently, we can establish the initial estimate:

$$\mathscr{P}(x_n, x_{n+m}, t) \geq \mathscr{P}\left(x_n, x_{n+m}, \frac{t}{b}\sum_{i=n}^{n+m-1}\delta^i\right).$$

We explicitly derive the sequence of parameters by splitting the sum as $\sum_{i=n}^{n+m-1}\delta^i = \delta^n + \sum_{i=n+1}^{n+m-1}\delta^i$ and applying the b-triangle inequality defined in axiom ($IFbM5$):

$$\begin{aligned}&\mathscr{P}(x_n, x_{n+m}, t)\\ &\geq \tau\left(\mathscr{P}\left(x_n, x_{n+1}, \frac{t\delta^n}{b^2}\right),\right.\\ &\qquad\left.\mathscr{P}\left(x_{n+1}, x_{n+m}, \frac{t\sum_{i=n+1}^{n+m-1}\delta^i}{b^2}\right)\right).\end{aligned}$$

Repeatedly applying axiom ($IFbM5$) to expand the second term yields the nested m-step estimate:

$$\begin{aligned}&\mathscr{P}(x_n, x_{n+m}, t)\\ &\geq \tau\left(\mathscr{P}\left(x_n, x_{n+1}, \frac{t\delta^n}{b^2}\right),\right.\\ &\qquad\tau\left(\mathscr{P}\left(x_{n+1}, x_{n+2}, \frac{t\delta^{n+1}}{b^3}\right), \ldots\right.\\ &\qquad\left.\left.\ldots, \mathscr{P}\left(x_{n+m-1}, x_{n+m}, \frac{t\delta^{n+m-1}}{b^{m+1}}\right)\right)\right).\end{aligned}$$

To properly track the time parameter transformations, we use the iterative contraction from Eq. (3.1). For any arbitrary time $s > 0$, iterating (3.1) yields $\mathscr{P}(x_i, x_{i+1}, s) \geq \mathscr{P}\left(x_0, x_1, \frac{s}{\xi^i}\right)$. We systematically substitute the specific time arguments $s_i = \frac{t\delta^i}{b^{i-n+2}}$ from our nested estimate into this iterative bound, yielding:

$$\begin{aligned}&\mathscr{P}(x_n, x_{n+m}, t)\\ &\qquad\geq \tau_{i=n}^{n+m-1}\mathscr{P}\left(x_0, x_1, \frac{t\delta^i}{b^{i-n+2}\xi^i}\right).\end{aligned}$$

For $n \geq 2$ and $i \geq n$, the exponent property $b^{i-n+2} \leq b^i$ holds since $b \geq 1$. Thus, $\frac{1}{b^{i-n+2}} \geq \frac{1}{b^i}$. Because $\mathscr{P}$ is b-nondecreasing with respect to time, we simplify the parameter argument:

$$\begin{aligned}\mathscr{P}(x_n, x_{n+m}, t) &\geq \tau_{i=n}^{n+m-1}\mathscr{P}\left(x_0, x_1, \frac{t\delta^i}{b^i\xi^i}\right)\\ &\geq \tau_{i=n}^{\infty}\mathscr{P}\left(x_0, x_1, \frac{t}{\nu^i}\right),\end{aligned}$$

where $\nu = \frac{b\xi}{\delta}$.

We apply the exact same explicit unrolling procedure to the non-membership degree $\mathscr{Q}$. Because $\mathscr{Q}$ is b-nonincreasing, we apply axiom ($IFbM10$) alongside Eq. (3.2) to obtain:

$$\begin{aligned}\mathscr{Q}(x_n, x_{n+m}, t) &\leq \mathscr{Q}\left(x_n, x_{n+m}, \frac{t}{b}\sum_{i=n}^{n+m-1}\delta^i\right)\\ &\leq \mu_{i=n}^{n+m-1}\mathscr{Q}\left(x_0, x_1, \frac{t\delta^i}{b^{i-n+2}\xi^i}\right)\\ &\leq \mu_{i=n}^{\infty}\mathscr{Q}\left(x_0, x_1, \frac{t}{\nu^i}\right).\end{aligned}$$

Crucially, we must verify the behavior of the transformed time parameter $\frac{t}{\nu^i}$. Because δ was chosen from the interval $(\xi b, 1)$, the constructed parameter $\nu = \frac{b\xi}{\delta}$ strictly satisfies $0 < \nu < 1$. As a result, for any fixed $t > 0$, the time argument $\frac{t}{\nu^i} \to +\infty$ as $i \to \infty$.

Taking the limit as $n \to \infty$, the conditions provided in Eqs. (3.3)-(3.4) dictate that these infinite operations evaluate to 1

and 0, respectively. Because the parameter transformations legitimately tend to $+\infty$, it follows that $\lim_{n\to\infty} \mathscr{P}(x_n, x_{n+m}, t) = 1$ and $\lim_{n\to\infty} \mathscr{Q}(x_n, x_{n+m}, t) = 0$. Consequently, for any $t > 0$ and any $r \in (0,1)$, there exists an $n_0 \in \mathbb{N}$ such that $\mathscr{P}(x_n, x_{n+m}, t) > 1 - r$ and $\mathscr{Q}(x_n, x_{n+m}, t) < r$ for all $n, n+m \geq n_0$. This precisely fulfills the formal criteria of Definition 2.13, confirming that $\{x_n\}$ is a Cauchy sequence. □

Corollary 3.2. *Consider the sequence $\{x_n\}$ in the IFb-MS $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, where both τ and μ satisfy the conditions for H-type. Suppose there exists a constant $\xi \in \left(0, \frac{1}{b}\right)$ such that*

$$\mathscr{P}(x_n, x_{n+1}, t) \geq \mathscr{P}\left(x_{n-1}, x_n, \frac{t}{\xi}\right),$$
$$\mathscr{Q}(x_n, x_{n+1}, t) \leq \mathscr{Q}\left(x_{n-1}, x_n, \frac{t}{\xi}\right),$$

for all $n \in \mathbb{N}$ and $t > 0$. Then $\{x_n\}$ is a Cauchy sequence.

Lemma 3.3. *If there exists ξ in $(0,1)$ and elements $y, x \in \mathfrak{X}$ such that*

$$\mathscr{P}(x, y, t) \geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right) \tag{3.5}$$

$$\mathscr{Q}(x, y, t) \leq \mathscr{Q}\left(x, y, \frac{t}{\xi}\right) \tag{3.6}$$

for $t > 0$, then $x = y$.

Proof. By repeatedly iterating conditions (3.5) and (3.6) n times, we obtain the following complete iteration argument:

$$\begin{aligned}\mathscr{P}(x, y, t) &\geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right) \\ &\geq \mathscr{P}\left(x, y, \frac{t}{\xi^2}\right) \\ &\geq \cdots \geq \mathscr{P}\left(x, y, \frac{t}{\xi^n}\right),\end{aligned}$$

$$\begin{aligned}\mathscr{Q}(x, y, t) &\leq \mathscr{Q}\left(x, y, \frac{t}{\xi}\right) \\ &\leq \mathscr{Q}\left(x, y, \frac{t}{\xi^2}\right) \\ &\leq \cdots \leq \mathscr{Q}\left(x, y, \frac{t}{\xi^n}\right),\end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. Because $\xi \in (0,1)$, the time parameter $\frac{t}{\xi^n} \to +\infty$ as $n \to \infty$.

Taking the limit as $n \to \infty$ and applying the limiting assumptions provided in Hypothesis Eq. (H1), we establish the following bounds:

$$\mathscr{P}(x, y, t) \geq \lim_{n\to\infty} \mathscr{P}\left(x, y, \frac{t}{\xi^n}\right) = 1,$$
$$\mathscr{Q}(x, y, t) \leq \lim_{n\to\infty} \mathscr{Q}\left(x, y, \frac{t}{\xi^n}\right) = 0.$$

By the fundamental definition of fuzzy sets, the degrees of membership and non-membership are strictly bounded such that $\mathscr{P}(x, y, t) \leq 1$ and $\mathscr{Q}(x, y, t) \geq 0$. Consequently, the inequalities above force exact equalities:

$$\mathscr{P}(x, y, t) = 1,$$
$$\mathscr{Q}(x, y, t) = 0.$$

Invoking axioms (*IFbM*3) and (*IFbM*8) from Definition 2.12, these exact values definitively conclude that $x = y$. □

Before presenting our main fixed-point result, we explicitly highlight its generalization over existing literature. In standard fuzzy b-metric spaces (Fb-MS), the foundational result (Theorem 2.4 by Rakić et al. [19]) relies solely on maximizing the membership degree:

$$\mathscr{P}(Tx, Ty, t) \geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right).$$

The fundamental difference in our upcoming Theorem 3.4 is the simultaneous minimization of the non-membership degree (using a t-conorm μ):

$$\mathscr{Q}(Tx, Ty, t) \leq \mathscr{Q}\left(x, y, \frac{t}{\xi}\right).$$

This dual formulation resolves hesitation margins and non-belongingness uncertainties absent in Fb-MS. Furthermore, unlike classical Intuitionistic Fuzzy Metric Spaces (where $b = 1$ and the contraction parameter is $\xi < 1$), our IFb-MS accommodates a relaxed triangle inequality ($b \geq 1$), which necessitates the tighter bound $\xi < 1/b$. Consequently, the following theorem addresses generalized uncertainties that cannot be solved by known fuzzy b-metric or intuitionistic fuzzy results alone.

Theorem 3.4. *Assume $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ denotes a complete IFb-MS and consider a self-mapping $T : \mathfrak{X} \to \mathfrak{X}$. Assume the existence of a constant $\xi \in \left(0, \frac{1}{b}\right)$ satisfying the required contraction conditions:*

$$\mathscr{P}(Tx, Ty, t) \geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right), \qquad (3.7)$$

$$\mathscr{Q}(Tx, Ty, t) \leq \mathscr{Q}\left(x, y, \frac{t}{\xi}\right), \qquad (3.8)$$

for all $x, y \in \mathfrak{X}$ and $t > 0$.
Additionally, let there be $x_0 \in \mathfrak{X}$ and $\nu \in (0, 1)$ such that

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} \mathscr{P}\left(x_0, Tx_0, \frac{t}{\nu^i}\right) = 1, \quad t > 0,$$
$$\lim_{n\to\infty} \tau_{i=n}^{\infty} \mathscr{Q}\left(x_0, Tx_0, \frac{t}{\nu^i}\right) = 0, \quad t > 0.$$

Then the function T admits a unique fixed point within the space $\mathfrak{X}$.

Proof. Consider $x_0 \in \mathfrak{X}$ and define the Picard sequence $x_{n+1} = Tx_n$, for $n \in \mathbb{N}$.

Setting $x = x_n$ and $y = x_{n-1}$ in Eqs. (3.7)-(3.8), it subsequently follows that

$$\mathscr{P}(x_n, x_{n+1}, t) \geq \mathscr{P}\left(x_{n-1}, x_n, \frac{t}{\xi}\right),$$
$$\mathscr{Q}(x_n, x_{n+1}, t) \leq \mathscr{Q}\left(x_{n-1}, x_n, \frac{t}{\xi}\right),$$

for all $n \in \mathbb{N}$ and $t > 0$. Lemma 3.1 ensures that the iterative sequence $\{x_n\}$ satisfies the Cauchy condition.

By the completeness of the space $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, there exists an element $x \in \mathfrak{X}$ which serves as the limit of the sequence. Thus, $x_n \to x$ as $n \to \infty$:

$$\lim_{n\to\infty} \mathscr{P}(x, x_n, t) = 1, \qquad (3.9)$$
$$\lim_{n\to\infty} \mathscr{Q}(x, x_n, t) = 0, \qquad (3.10)$$

for all $t > 0$.

To rigorously establish that $Tx = x$, we utilize the b-triangle inequality provided in axiom (*IFbM*5). By splitting the time parameter $t = \frac{t}{2} + \frac{t}{2}$ and introducing the Picard sequence element $x_{n+1} = Tx_n$ as the intermediate point, we derive the following exact estimate:

$$\mathscr{P}(Tx, x, t) \geq \tau\left(\mathscr{P}\left(Tx, x_{n+1}, \frac{t}{2b}\right), \mathscr{P}\left(x_{n+1}, x, \frac{t}{2b}\right)\right).$$

Applying the contractive condition (3.7) to the first term, where $x_{n+1} = Tx_n$, yields:

$$\mathscr{P}(Tx, x, t) \geq \tau\left(\mathscr{P}\left(x, x_n, \frac{t}{2b\xi}\right), \mathscr{P}\left(x_{n+1}, x, \frac{t}{2b}\right)\right).$$

Crucially, because the t-norm τ is continuous by definition, we are mathematically justified in passing the limit as $n \to \infty$ directly into the arguments of τ. Substituting the convergence limits from Eq. (3.9) results in:

$$\begin{aligned}\mathscr{P}(Tx, x, t) \geq \tau \Bigg(& \lim_{n\to\infty} \mathscr{P}\left(x, x_n, \frac{t}{2b\xi}\right), \\ & \lim_{n\to\infty} \mathscr{P}\left(x_{n+1}, x, \frac{t}{2b}\right)\Bigg) \\ & = \tau(1,1) = 1.\end{aligned}$$

By identical logic, applying axiom (*IFbM*10), the contractive condition (3.8), and the continuity of the t-conorm μ, the separation degree $\mathscr{Q}$ is estimated as:

$$\begin{aligned}\mathscr{Q}(Tx, x, t) \leq \mu \Bigg(& \mathscr{Q}\left(Tx, x_{n+1}, \frac{t}{2b}\right), \\ & \mathscr{Q}\left(x_{n+1}, x, \frac{t}{2b}\right)\Bigg) \\ & \leq \mu\Bigg(\mathscr{Q}\left(x, x_n, \frac{t}{2b\xi}\right), \\ & \mathscr{Q}\left(x_{n+1}, x, \frac{t}{2b}\right)\Bigg).\end{aligned}$$

Passing the limit inside via the continuity of μ and applying Eq. (3.10) yields:

$$\mathscr{Q}(Tx, x, t) \leq \mu(0,0) = 0.$$

Since $\mathscr{P}$ is bounded above by 1 and $\mathscr{Q}$ is bounded below by 0, we conclude $\mathscr{P}(Tx, x, t) = 1$ and $\mathscr{Q}(Tx, x, t) = 0$. By axioms (*IFbM*3) and (*IFbM*8), this definitively proves $Tx = x$.

Suppose that x and y are both fixed points associated with the mapping T (i.e., $Tx = x$ and $Ty = y$). Then, Eqs. (3.7)-(3.8) yield the following relation:

$$\begin{aligned}\mathscr{P}(x, y, t) &= \mathscr{P}(Tx, Ty, t) \\ &\geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right), \quad t > 0,\end{aligned}$$

$$\begin{aligned}\mathscr{Q}(x, y, t) &= \mathscr{Q}(Tx, Ty, t) \\ &\leq \mathscr{Q}\left(x, y, \frac{t}{\xi}\right), \quad t > 0.\end{aligned}$$

By applying Lemma 3.3, it is definitively inferred that $x = y$, proving uniqueness. □

Example 3.5. Assume $\mathfrak{X} = [0, 1]$. For $t > 0$ and $x, y \in \mathfrak{X}$, define the fuzzy mappings

$$\mathscr{P}(x, y, t) = e^{-\frac{|x-y|}{t}},$$

$$\mathscr{Q}(x, y, t) = 1 - e^{-\frac{|x-y|}{t}}.$$

With the parameter $\xi = \frac{2}{5}$, consider the self-map $T(x) = \frac{2}{5}x + \frac{1}{5}$.

For $x, y \in \mathfrak{X}$,

$$|T(x) - T(y)| = \frac{2}{5}|x - y| = \xi|x - y|.$$

For $t > 0$,

$$\begin{aligned}\mathscr{P}(Tx, Ty, t) &= e^{-\frac{|Tx-Ty|}{t}} \\ &= e^{-\frac{\left|\frac{2x}{5}+\frac{1}{5}-\frac{2y}{5}-\frac{1}{5}\right|}{t}} \\ &= e^{-\frac{\frac{2}{5}|x-y|}{t}} \\ &= e^{-\frac{\xi|x-y|}{t}} \\ &= \mathscr{P}\left(x, y, \tfrac{t}{\xi}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathscr{Q}(Tx, Ty, t) &= 1 - e^{-\frac{|Tx-Ty|}{t}} \\ &= 1 - e^{-\frac{\left|\frac{2x}{5}+\frac{1}{5}-\frac{2y}{5}-\frac{1}{5}\right|}{t}}\end{aligned}$$

$$= 1 - e^{-\frac{\frac{2}{5}|x-y|}{t}}$$

$$= 1 - e^{-\frac{\xi|x-y|}{t}}$$

$$= \mathscr{Q}\left(x, y, \frac{t}{\xi}\right).$$

Hence, the contraction conditions of Theorem 3.4 are satisfied with $\xi = \frac{2}{5}$. Consequently, the function T admits a fixed point within the space $\mathfrak{X}$.

Theorem 3.6. *Suppose $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$ is a complete IFb-MS, and an operator $T : \mathfrak{X} \to \mathfrak{X}$. Let $\xi \in \left(0, \frac{1}{b}\right)$ be a constant for which the contractive framework is satisfied:*

$$\mathscr{P}(Tx, Ty, t) \geq \min\left\{\mathscr{P}\left(x, y, \frac{t}{\xi}\right), \mathscr{P}\left(Tx, x, \frac{t}{\xi}\right), \mathscr{P}\left(Ty, y, \frac{t}{\xi}\right)\right\} \tag{3.11}$$

$$\mathscr{Q}(Tx, Ty, t) \leq \max\left\{\mathscr{Q}\left(x, y, \frac{t}{\xi}\right), \mathscr{Q}\left(Tx, x, \frac{t}{\xi}\right), \mathscr{Q}\left(Ty, y, \frac{t}{\xi}\right)\right\} \tag{3.12}$$

for all $x, y \in \mathfrak{X}$ and $t > 0$. Additionally, there exist $x_0 \in \mathfrak{X}$ and $v \in (0, 1)$ such that

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} \mathscr{P}\left(x_0, Tx_0, \frac{t}{v^i}\right) = 1, \quad t > 0$$

$$\lim_{n\to\infty} \mu_{i=n}^{\infty} \mathscr{Q}\left(x_0, Tx_0, \frac{t}{v^i}\right) = 0, \quad t > 0$$

Then the function T admits a unique fixed point within the space $\mathfrak{X}$.

Proof. Consider $x_0 \in \mathfrak{X}$ and define the sequence

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}.$$

Substituting $x = x_n$ and $y = x_{n-1}$ into Eq. (3.11) yields:

$$\mathscr{P}(x_{n+1}, x_n, t) \geq \min\left\{\mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right), \mathscr{P}\left(x_{n+1}, x_n, \frac{t}{\xi}\right), \mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right)\right\}$$

$$= \min\left\{\mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right), \mathscr{P}\left(x_{n+1}, x_n, \frac{t}{\xi}\right)\right\}.$$

To establish the recursive estimate required for Lemma 3.1, we rigorously analyze the two possible cases for the minimum.

Case 1: The minimum is achieved by $\mathscr{P}\left(x_{n+1}, x_n, \frac{t}{\xi}\right)$.

In this scenario, the inequality reduces to:

$$\mathscr{P}(x_{n+1}, x_n, t) \geq \mathscr{P}\left(x_{n+1}, x_n, \frac{t}{\xi}\right).$$

By Lemma 3.3, this implies $x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Consequently, the fuzzy degree of closeness is absolute, meaning $\mathscr{P}(x_{n+1}, x_n, t) = 1$. Since any fuzzy membership degree is bounded above by 1, it trivially follows that:

$$\mathscr{P}(x_{n+1}, x_n, t) = 1 \geq \mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right).$$

Case 2: The minimum is achieved by $\mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right)$.

In this scenario, the evaluation of the minimum directly yields the desired recursive bound:

$$\mathscr{P}(x_{n+1}, x_n, t) \geq \mathscr{P}\left(x_n, x_{n-1}, \frac{t}{\xi}\right).$$

Thus, in all possible cases, the required recursive estimate for $\mathscr{P}$ holds for

all $n \in \mathbb{N}$ and $t > 0$.

To independently verify the recursive estimate for the non-membership degree $\mathscr{Q}$, we substitute $x = x_n$ and $y = x_{n-1}$ into Eq. (3.12), yielding:

$$\begin{aligned}\mathscr{Q}(x_{n+1}, x_n, t) \leq \max\Big\{&\mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right),\\ &\mathscr{Q}\left(x_{n+1}, x_n, \frac{t}{\xi}\right),\\ &\mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right)\Big\}\\ = \max\Big\{&\mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right),\\ &\mathscr{Q}\left(x_{n+1}, x_n, \frac{t}{\xi}\right)\Big\}.\end{aligned}$$

Because this inequality utilizes the maximum operator alongside a $\leq$ condition, we must explicitly evaluate the two distinct possibilities:

Case 1: The maximum is achieved by $\mathscr{Q}\left(x_{n+1}, x_n, \frac{t}{\xi}\right)$.
In this scenario, the inequality reduces to:

$$\mathscr{Q}(x_{n+1}, x_n, t) \leq \mathscr{Q}\left(x_{n+1}, x_n, \frac{t}{\xi}\right).$$

By Lemma 3.3, this implies $x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Consequently, the fuzzy degree of non-separation evaluates to its absolute minimum, meaning $\mathscr{Q}(x_{n+1}, x_n, t) = 0$. Since any fuzzy non-membership degree is strictly bounded below by 0, it trivially follows that:

$$\mathscr{Q}(x_{n+1}, x_n, t) = 0 \leq \mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right).$$

Case 2: The maximum is achieved by $\mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right)$.
In this scenario, the evaluation of the maximum directly yields the recursive bound:

$$\mathscr{Q}(x_{n+1}, x_n, t) \leq \mathscr{Q}\left(x_n, x_{n-1}, \frac{t}{\xi}\right).$$

Therefore, in all possible cases, the exact recursive estimate required by Lemma 3.1 is rigorously established for both $\mathscr{P}$ and $\mathscr{Q}$.

Application of Lemma 3.1 guarantees that the sequence $\{x_n\}$ exhibits the Cauchy property. Hence, there exists an element $x \in \mathfrak{X}$ satisfying

$$\begin{aligned}&\lim_{n\to\infty} x_n = x,\\ &\lim_{n\to\infty} \mathscr{P}(x, x_n, t) = 1, \quad t > 0,\end{aligned} \tag{3.13}$$

$$\lim_{n\to\infty} \mathscr{Q}(x, x_n, t) = 0, \quad t > 0. \tag{3.14}$$

Let us demonstrate that x is a fixed point for T. Let $\delta_1 \in (\xi b, 1)$, and define $\delta_2 = 1 - \delta_1$. In view of Eqs. (3.11)-(3.12), the following relation holds:

$$\begin{aligned}\mathscr{P}(Tx, x, t) \geq \tau\Big(&\mathscr{P}\left(Tx, Tx_n, \frac{\delta_1 t}{b}\right),\\ &\mathscr{P}\left(x_{n+1}, x, \frac{\delta_2 t}{b}\right)\Big)\\ \geq \tau\Big(&\min\Big\{\mathscr{P}\left(x, x_n, \frac{\delta_1 t}{b\xi}\right),\\ &\mathscr{P}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right),\\ &\mathscr{P}\left(x_n, x_{n+1}, \frac{\delta_1 t}{b\xi}\right)\Big\},\\ &\mathscr{P}\left(x_{n+1}, x, \frac{\delta_2 t}{b}\right)\Big)\end{aligned}$$

$$\begin{aligned}\mathscr{Q}(Tx, x, t) \leq \mu\Big(&\mathscr{Q}\left(Tx, Tx_n, \frac{\delta_1 t}{b}\right),\\ &\mathscr{Q}\left(x_{n+1}, x, \frac{\delta_2 t}{b}\right)\Big)\\ \leq \mu\Big(&\max\Big\{\mathscr{Q}\left(x, x_n, \frac{\delta_1 t}{b\xi}\right),\end{aligned}$$

$$\mathscr{Q}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right),$$
$$\mathscr{Q}\left(x_n, x_{n+1}, \frac{\delta_1 t}{b\xi}\right)\Bigg\},$$
$$\mathscr{Q}\left(x_{n+1}, x, \frac{\delta_2 t}{b}\right)\Bigg)$$

In the limit as $n \to \infty$, and by application of Eq. (3.13), it yields:

$$\begin{aligned}\mathscr{P}(Tx, x, t) &\geq \tau\left(\min\left\{1, \mathscr{P}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right), 1\right\}, 1\right)\\ &= \tau\left(\mathscr{P}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right), 1\right)\\ &= \mathscr{P}\left(x, Tx, \frac{t}{v}\right)\end{aligned}$$

where $v = \dfrac{b\xi}{\delta_1} \in (0, 1)$. So,

$$\mathscr{P}(Tx, x, t) \geq \mathscr{P}\left(Tx, x, \frac{t}{v}\right)$$

Taking the limit $n \to \infty$ and applying Eq. (3.14), the following result is obtained:

$$\begin{aligned}\mathscr{Q}(Tx, x, t) &\leq \mu\left(\max\left\{0, \mathscr{Q}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right), 0\right\}, 0\right)\\ &= \mu\left(\mathscr{Q}\left(x, Tx, \frac{\delta_1 t}{b\xi}\right), 0\right)\\ &= \mathscr{Q}\left(x, Tx, \frac{t}{v}\right)\end{aligned}$$

where $v = \dfrac{b\xi}{\delta_1} \in (0, 1)$. So,

$$\mathscr{Q}(Tx, x, t) \leq \mathscr{Q}\left(Tx, x, \frac{t}{v}\right)$$

By applying Lemma 3.3, it is clear that $Tx = x$.

Assume x and y are fixed points for T, i.e., $Tx = x$ and $Ty = y$. From Eq. (3.11), it follows that

$$\begin{aligned}\mathscr{P}(Tx, Ty, t) &\geq \min\left\{\mathscr{P}\left(x, y, \frac{t}{\xi}\right),\right.\\ &\qquad \mathscr{P}\left(x, Tx, \frac{t}{\xi}\right),\\ &\qquad \left.\mathscr{P}\left(y, Ty, \frac{t}{\xi}\right)\right\}\\ &= \min\left\{\mathscr{P}\left(x, y, \frac{t}{\xi}\right), 1, 1\right\}\\ &= \mathscr{P}\left(x, y, \frac{t}{\xi}\right)\\ &= \mathscr{P}\left(Tx, Ty, \frac{t}{\xi}\right).\end{aligned}$$

Similarly, by Eq. (3.12):

$$\begin{aligned}\mathscr{Q}(Tx, Ty, t) &\leq \max\left\{\mathscr{Q}\left(x, y, \frac{t}{\xi}\right),\right.\\ &\qquad \mathscr{Q}\left(x, Tx, \frac{t}{\xi}\right),\\ &\qquad \left.\mathscr{Q}\left(y, Ty, \frac{t}{\xi}\right)\right\}\\ &= \max\left\{\mathscr{Q}\left(x, y, \frac{t}{\xi}\right), 0, 0\right\}\\ &= \mathscr{Q}\left(x, y, \frac{t}{\xi}\right)\\ &= \mathscr{Q}\left(Tx, Ty, \frac{t}{\xi}\right).\end{aligned}$$

By applying Lemma 3.3, it follows that $Tx = Ty$, i.e., $x = y$. □

Remark 3.7. *It is important to note the structural relationship between our two main results: Theorem 3.4 is a direct special case of Theorem 3.6. If a mapping T satisfies the contractive conditions of Theorem 3.4, it trivially satisfies the generalized conditions of Theorem 3.6 because:*

$$\begin{aligned}\mathscr{P}(Tx, Ty, t) &\geq \mathscr{P}\left(x, y, \frac{t}{\xi}\right)\\ &\geq \min\left\{\mathscr{P}\left(x, y, \frac{t}{\xi}\right),\right.\end{aligned}$$

$$\mathscr{P}\left(Tx, x, \frac{t}{\xi}\right),$$
$$\mathscr{P}\left(Ty, y, \frac{t}{\xi}\right)\Bigg\},$$

and analogously for the non-membership degree $\mathscr{Q}$ utilizing the maximum operator. Therefore, Theorem 3.6 serves as a genuine analytical generalization of the Banach-type contraction presented in Theorem 3.4.

Example 3.8. Consider the set $\mathfrak{X} = [0, 1]$. We construct an IFb-MS expressed as the ordered quintuple $(\mathfrak{X}, \mathscr{P}, \mathscr{Q}, \tau, \mu)$, where we explicitly define the continuous t-norm as the product $\tau(\alpha, \beta) = \alpha \cdot \beta$ and the continuous t-conorm as the probabilistic sum $\mu(\alpha, \beta) = \alpha + \beta - \alpha\beta$. The fuzzy sets are defined by the standard exponential metric indicator:

$$\mathscr{P}(x, y, t) = e^{-\frac{|x-y|}{t}},$$
$$\mathscr{Q}(x, y, t) = 1 - e^{-\frac{|x-y|}{t}}.$$

Before testing the contractive operator, we rigorously verify that this space satisfies the fundamental hypotheses of Definition 2.12 for the parameter $b = 2$. By definition, the condition $\mathscr{P}(x, y, t) = 1$ evaluates to:

$$e^{-\frac{|x-y|}{t}} = 1.$$

This equality holds true if and only if the exponent is zero, meaning:

$$|x - y| = 0 \Longleftrightarrow x = y.$$

Symmetrical reasoning proves the non-membership separation condition

$$\mathscr{Q}(x, y, t) = 0 \Longleftrightarrow x = y,$$

thereby satisfying axioms (*IFbM*3) and (*IFbM*8). Furthermore, because the defined fuzzy sets rely on the absolute value metric and the exponential function, their composition is strictly continuous across the domain $t \in (0, \infty)$, satisfying axioms (*IFbM*6) and (*IFbM*11).

To establish the b-triangle inequality for the membership degree, we apply the classical metric triangle inequality:

$$|x - z| \leq |x - y| + |y - z|,$$

alongside the standard algebraic property:

$$\frac{A + B}{t + s} \leq \frac{A}{t} + \frac{B}{s}.$$

Combining these yields the intermediate bound:

$$\mathscr{P}(x, z, t + s) \geq \exp\left(-\frac{|x - y|}{t} - \frac{|y - z|}{s}\right)$$
$$= \mathscr{P}(x, y, t) \cdot \mathscr{P}(y, z, s).$$

Because $\mathscr{P}$ is non-decreasing with respect to the time parameter, scaling the time denominators by a factor of $b = 2$ strictly relaxes the bound. Substituting this scaled time into our product t-norm τ confirms the structural axiom (*IFbM*5):

$$\mathscr{P}(x, z, t + s) \geq \mathscr{P}\left(x, y, \frac{t}{2}\right) \cdot \mathscr{P}\left(y, z, \frac{s}{2}\right)$$
$$= \tau\left(\mathscr{P}\left(x, y, \frac{t}{b}\right), \mathscr{P}\left(y, z, \frac{s}{b}\right)\right).$$

Substituting $\mathscr{Q} = 1 - \mathscr{P}$ into the probabilistic sum t-conorm μ mathematically mirrors this bound, confirming the dual requirement for (*IFbM*10). Finally, because the underlying space $\mathfrak{X} = [0, 1]$ is a closed, bounded, and complete subset of the Real numbers, the induced IFb-MS naturally inherits this completeness, guaranteeing that all Cauchy sequences converge within $\mathfrak{X}$.

Now, we define an operator $T : \mathfrak{X} \to \mathfrak{X}$ acting as $T(x) = x^2$, with parameters $\xi = \frac{1}{4}$ and $b = 2$.

Let us verify the contractivity conditions of Theorem 3.6 for this mapping. Eqs. (3.11)-(3.12) are examined with these parameters: $\xi = \frac{1}{4}$ and $b = 2$:

$$\begin{aligned}\mathscr{P}(Tx, Ty, t) &= e^{-\frac{|x^2-y^2|}{t}} \\ &= e^{-\frac{|x+y||x-y|}{t}} \\ &\geq e^{-\frac{2|x-y|}{t}} \quad \text{when } |x+y| \leq 2.\end{aligned}$$

For all $x, y \in [0, 1]$, the inequality $|x + y| \leq 2$ is satisfied. Accordingly, the parameters are fixed as $b = 2$ and $\xi = \frac{1}{4}$.

$$\begin{aligned}&\min\left\{\mathscr{P}\left(x, y, \frac{t}{4}\right),\right. \\ &\mathscr{P}\left(x^2, x, \frac{t}{4}\right), \\ &\left.\mathscr{P}\left(y^2, y, \frac{t}{4}\right)\right\} \\ &= \min\left\{e^{-\frac{4|x-y|}{t}},\right. \\ &e^{-\frac{4|x^2-x|}{t}}, \\ &\left.e^{-\frac{4|y^2-y|}{t}}\right\}.\end{aligned}$$

Since

$$\begin{aligned}e^{-\frac{|x+y||x-y|}{t}} &\geq e^{-\frac{2|x-y|}{t}} \\ &\geq e^{-\frac{4|x-y|}{t}}\end{aligned}$$

for $t > 0$ and $|x + y| \leq 2$, the contractivity condition for $\mathscr{P}$ is satisfied.

Similarly,

$$\begin{aligned}\mathscr{Q}(Tx, Ty, t) &= 1 - e^{-\frac{|x^2-y^2|}{t}} \\ &= 1 - e^{-\frac{|x+y||x-y|}{t}} \\ &\leq 1 - e^{-\frac{2|x-y|}{t}} \\ &\leq 1 - e^{-\frac{4|x-y|}{t}}.\end{aligned}$$

$$\begin{aligned}&\max\left\{\mathscr{Q}\left(x, y, \frac{t}{4}\right),\right. \\ &\mathscr{Q}\left(x^2, x, \frac{t}{4}\right), \\ &\left.\mathscr{Q}\left(y^2, y, \frac{t}{4}\right)\right\} \\ &= \max\left\{1 - e^{-\frac{4|x-y|}{t}},\right. \\ &1 - e^{-\frac{4|x^2-x|}{t}}, \\ &\left.1 - e^{-\frac{4|y^2-y|}{t}}\right\}.\end{aligned}$$

The contractivity condition for $\mathscr{Q}$ is also satisfied:

$$\begin{aligned}&\mathscr{P}(Tx, Ty, t) \\ &\geq \min\left\{\mathscr{P}\left(x, y, \frac{t}{4}\right),\right. \\ &\mathscr{P}\left(x^2, x, \frac{t}{4}\right), \\ &\left.\mathscr{P}\left(y^2, y, \frac{t}{4}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}&\mathscr{Q}(Tx, Ty, t) \\ &\leq \max\left\{\mathscr{Q}\left(x, y, \frac{t}{4}\right),\right. \\ &\mathscr{Q}\left(x^2, x, \frac{t}{4}\right), \\ &\left.\mathscr{Q}\left(y^2, y, \frac{t}{4}\right)\right\}.\end{aligned}$$

Therefore, a fixed point exists.

Example 3.9. To demonstrate the genuine generality of our results over existing literature, we construct a scenario where previous contractive conditions completely fail to apply. Let $\mathfrak{X} = [0, 1]$ and consider the IFb-MS defined by:

$$\mathscr{P}(x, y, t) = e^{-\frac{2(x-y)^2}{t}},$$
$$\mathscr{Q}(x, y, t) = 1 - e^{-\frac{(x-y)^2}{t}}.$$

First, we verify the fundamental intuitionistic condition $\mathscr{P} + \mathscr{Q} \leq 1$. Let $u = \frac{(x-y)^2}{t} \geq 0$. We require:

$$e^{-2u} + 1 - e^{-u} \leq 1 \Longleftrightarrow e^{-2u} \leq e^{-u},$$

which is strictly true for all $u \geq 0$. Crucially, the hesitation margin for this space is non-zero, since:

$$\pi(x, y, t) = 1 - \mathscr{P} - \mathscr{Q}$$
$$= e^{-u} - e^{-2u} > 0 \quad (\text{for } x \neq y).$$

It is essential to observe why existing fixed-point theorems cannot be applied to this specific space. Classical Intuitionistic Fuzzy Metric Spaces (IFMS) require the standard triangle inequality where $b = 1$. Because the exponential argument here relies on the squared distance $(x - y)^2$, this space inherently forms a b-metric with $b = 2$, rendering standard IFMS fixed-point theorems inapplicable. Furthermore, standard Fuzzy b-Metric Spaces (Fb-MS), such as those utilized by Rakić et al. [19], operate under the strict assumption that no hesitation exists (i.e., non-membership is either undefined or strictly $1 - \mathscr{P}$). Because our defined space possesses a strictly positive hesitation margin ($\pi > 0$), the mathematical framework of Fb-MS is fundamentally incapable of evaluating it.

In contrast, our proposed framework effortlessly resolves this scenario. We define the mapping $T(x) = \frac{x}{2}$ and evaluate the contractive conditions with parameters $b = 2$ and $\xi = \frac{1}{4}$. For the membership degree, we obtain:

$$\mathscr{P}(Tx, Ty, t) = e^{-\frac{2\left(\frac{x}{2}-\frac{y}{2}\right)^2}{t}}$$
$$= e^{-\frac{2(x-y)^2}{4t}}$$
$$= \mathscr{P}\left(x, y, \frac{t}{1/4}\right).$$

Similarly, the non-membership degree evaluates to:

$$\mathscr{Q}(Tx, Ty, t) = 1 - e^{-\frac{\left(\frac{x}{2}-\frac{y}{2}\right)^2}{t}}$$
$$= 1 - e^{-\frac{(x-y)^2}{4t}}$$
$$= \mathscr{Q}\left(x, y, \frac{t}{1/4}\right).$$

Since the contraction parameter $\xi = \frac{1}{4}$ is strictly less than the required upper bound $\frac{1}{b} = \frac{1}{2}$, all hypotheses of Theorem 3.4 are definitively satisfied. This successfully demonstrates that our IFb-MS framework provides rigorous fixed-point conclusions for complex uncertainty spaces that are entirely unavailable from existing literature.

4. Conclusion

In this manuscript, we have rigorously established a mathematical framework for fixed-point theory within Intuitionistic Fuzzy b-Metric Spaces (IFb-MS). Specifically, we proved sufficient analytical criteria under which an iterative sequence satisfies the Cauchy property in this generalized topology. By incorporating both a continuous t-norm for membership (closeness) and a continuous t-conorm for non-membership (separation), we derived and proved the existence and uniqueness of fixed points for self-mappings under generalized contraction conditions. We

have also explicitly demonstrated, via constructed examples, that these theorems successfully evaluate spaces with non-zero hesitation margins and relaxed triangle inequalities ($b \geq 1$) that cannot be resolved by standard fuzzy b-metric or intuitionistic fuzzy frameworks.

While the results established in this paper are strictly theoretical, the mathematical structures explored herein, specifically those dealing with multi-criteria uncertainty and imprecision, hold significant potential for future applied research. Future studies may explore the application of these IFb-MS fixed-point structures as theoretical models for algorithms in soft computing, optimization, and control theory.

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