E-ISSN 2586-9027
Homepage : https://tci-thaijo.org/index.php/SciTechAsia Science & Technology Asia
Vol.31 No.3 July - September 2026 Page: [42-58]
Original research article

New φ-fixed point results in metric spaces and well-posedness property

Binayak S. Choudhury[1,2], Ranajit Jyoti[1], Nikhilesh Metiya[3,*], Sunirmal Kundu[4], Animesh Upadhyay[5]

[1]*Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur 711103, India*
[2]*Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa*
[3]*Department of Mathematics, Sovarani Memorial College, Jagatballavpur 711408, India*
[4]*Department of Mathematics, Government General Degree College, Paschim Medinipur 721516, India*
[5]*Department of BSHU (Mathematics), Asansol Engineering College, Asansol 713305, India*

Received 11 May 2026; Received in revised form 17 July 2026
Accepted 20 July 2026; Available online 30 September 2026

ABSTRACT

In this paper, we introduce two new φ-fixed point problems in the setting of complete metric spaces using α-admissible mappings. The motivation of the work is derived from the efforts for combining several trends of research in fixed point theory to create new results. Existence and uniqueness results are established under appropriate conditions. The main results are supported by two separate illustrative examples. The well-posedness property of the proposed problems are investigated. One of the theorems is effectively applied to study the solvability of a problem involving functional equations.

Keywords: α-admissible mapping; φ-fixed point; Metric space; Well-posedness property

1. Introduction and Mathematical Preliminaries

In 2014, Jleli et al. [1] first introduced the concept of φ-fixed point, that is, a fixed point which is also a zero of a given mapping. Using the above notion, they derived a φ-fixed point theorem using some control functions, which is an extension of Banach's contraction mapping principle [2]. After that, many authors produced different types of φ-fixed point results in the context of different metric spaces. Some such recent works are [3–6].

In 2012, Samet et al. [7] introduced the concept of α-admissibility. Using the α-admissibility concept they provided a new

***Corresponding author:** metiya.nikhilesh@gmail.com
doi.org/10.70730/sta.v31i3.265529

contraction mapping, which is known as $(\alpha - \psi)$- contraction, and established some fixed point results in metric spaces. Further, many works have been done in this direction. Some instances are [8–11].

We combine the above two new trends in our work. Our main findings are categorized under the φ-fixed point problem, where we utilized α-admissible mappings and the contraction condition contains rational components. Rational contraction was proposed by Das et al. [12]. The present work utilized two key advantages of the α-admissibility condition. On the one hand, it provides a flexible framework for establishing a wide class of contractive inequalities. On the other hand, it relaxes the traditional contraction requirement by allowing the contractive condition to be imposed only on α-admissible pairs, rather than throughout the entire underlying space.

We study the well-posedness property of the main theorems. It can play a crucial role in φ-fixed point theory, if further research can be done in this direction. Some research works on the well-posedness property in fixed point theory are [13–16].

In the application section, a functional equation is solved by a main theorem. Functional equations frequently arise in dynamic programming, where they play a fundamental role in modeling and solving optimization problems. Solution of functional equation by using fixed point methodology appears to work like [17–19].

Before starting the main results, we recall some basic notations and concepts that are used in the main findings. Throughout the paper, we use the following notations:

$\mathbb{N}$: the set of all natural numbers;

$\mathbb{N} \cup \{0\}$: the set of all natural numbers with zero;

$\mathbb{R}$: the set of all real numbers;

F_T : the set of all fixed points of $T = \{u \in \mathcal{H} : u = Tu\}$;

Z_φ : the set of all zeroes of $\varphi = \{u \in \mathcal{H} : \varphi(u) = 0\}$;

$F_T \cap Z_\varphi := \{u \in \mathcal{H} : u = Tu \text{ and } \varphi(u) = 0\}$, where $T : \mathcal{H} \to \mathcal{H}$ and $\varphi : \mathcal{H} \to [0, \infty)$ are any two mappings and the set $\mathcal{H}$ is non-empty.

Definition 1.1 ([1])**.** Let $\mathcal{H}$ be a non-empty set, $T : \mathcal{H} \to \mathcal{H}$ and $\varphi : \mathcal{H} \to [0, \infty)$ be two mappings. An element $u \in \mathcal{H}$ is said to be a φ-fixed point of T if and only if $u \in F_T \cap Z_\varphi$, that is, $Tu = u$ and $\varphi(u) = 0$.

Definition 1.2 ([7])**.** Let $\mathcal{H}$ be a non-empty set and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be a mapping. A self-mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be an α-admissible if $\alpha(u, v) \geq 1$, for $u, v \in \mathcal{H}$ implies $\alpha(Tu, Tv) \geq 1$.

Definition 1.3 ([20])**.** Let $(\mathcal{H}, \mu)$ be a metric space and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be a mapping. Then $\mathcal{H}$ is said to have α-regular property if for every sequence $\{u_n\}$ in $\mathcal{H}$ converging to $u \in \mathcal{H}$, $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ implies that $\alpha(u_n, u) \geq 1$, for all $n \in \mathbb{N}$.

Let $\mathcal{F}$ denote the set of all functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

$(F1)$ $\max\{a, b\} \leq F(a, b, c)$, for $a, b, c \in [0, \infty)$;

$(F2)$ $F(0, 0, 0) = 0$;

$(F3)$ F is continuous;

$(F4)$ F is non-decreasing with respect to the first component;

$(F5)$ $F(a_1 + a_2, b, c) \leq F(a_1, b, d) + F(a_2, d, c)$, for all $a_1, a_2, b, c, d \in [0, \infty)$.

In this paper, we take a class of functions Θ which is the set of all non-decreasing and

continuous functions $\theta : [0,\infty) \to [0,\infty)$ satisfying $\sum_{n=0}^{\infty} \theta^n(t) < \infty$, where $t > 0$ is any real number and θ^n is the n-th iterate of θ.

Lemma 1.4 ([5]). *If $\theta \in \Theta$, then $\theta(t) < t$ for every $t > 0$ and $\theta(0) = 0$.*

2. Main Results

Theorem 2.1. *Let $(\mathcal{H}, \mu)$ be a complete metric space with α-regular property, where $\alpha : \mathcal{H} \times \mathcal{H} \to [0,\infty)$. Let $F \in \mathcal{F}$, $\theta \in \Theta$ and $\varphi : \mathcal{H} \to [0,\infty)$ be a lower semi-continuous mapping. Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping such that T is α-admissible and there exists $u_0 \in \mathcal{H}$ for which $\alpha(u_0, Tu_0) \geq 1$. Suppose that for all $u, v \in \mathcal{H}$,*

$$\begin{aligned}
&\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)) \\
&\leq \theta\Bigg(\max\Bigg\{F(M(u,v),\varphi(u),\varphi(v)), \\
&\quad F(N(u,v),\varphi(v),\varphi(Tv)), \\
&\quad \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Bigg\}\Bigg), \qquad (2.1)
\end{aligned}$$

where

$$\begin{aligned}
M(u,v) = \max\{&\mu(u,v), \mu(u,Tu), \\
&\mu(v,Tu), \frac{\mu(u,Tu)(1+\mu(v,Tv))}{1+\mu(Tu,Tv)}, \\
&\frac{\mu(v,Tu)(1+\mu(u,Tv))}{1+\mu(u,v)}\}
\end{aligned}$$

and

$$\begin{aligned}
N(u,v) = \max\{&\mu(v,Tv), \\
&\frac{\mu(v,Tv)(1+\mu(u,Tu))}{1+\mu(u,v)}\}.
\end{aligned}$$

Then T has a φ-fixed point in $\mathcal{H}$. Further, if $\alpha(z_*, w_*) \geq 1$, for all $z_*, w_* \in F_T$, then the φ-fixed point is unique.

Proof. Let $u_0 \in \mathcal{H}$ such that $\alpha(u_0, Tu_0) \geq 1$. We construct a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $u_{n+1} = Tu_n$, for all $n \in \mathbb{N} \cup \{0\}$. As $\alpha(u_0, Tu_0) = \alpha(u_0, u_1) \geq 1$, applying α-admissibility of T, we have $\alpha(u_1, u_2) \geq 1$. By induction, it follows that $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. If possible, suppose that $u_{i+1} = u_i$, for some $i \in \mathbb{N} \cup \{0\}$. Then $u_i = u_{i+1} = Tu_i$, that is, u_i is a fixed point of T. As $\alpha(u_i, u_i) = \alpha(u_i, u_{i+1}) \geq 1$, applying Eq. (2.1) with $u = v = u_i$, we have

$$\begin{aligned}
&F(0,\varphi(u_i),\varphi(u_i)) \\
&= F(\mu(u_i,u_i),\varphi(u_i),\varphi(u_i)) \\
&= F(\mu(Tu_i,Tu_i),\varphi(Tu_i),\varphi(Tu_i)) \\
&\leq \alpha(u_i,u_i)F(\mu(Tu_i,Tu_i),\varphi(Tu_i),\varphi(Tu_i)) \\
&\leq \theta\Bigg(\max\Bigg\{F(M(u_i,u_i),\varphi(u_i),\varphi(u_i)), \\
&\quad F(N(u_i,u_i),\varphi(u_i),\varphi(Tu_i)), \\
&\quad \frac{1}{2}F(\mu(u_i,Tu_i),\varphi(u_i),\varphi(Tu_i))\Bigg\}\Bigg) \\
&= \theta\Bigg(\max\Bigg\{F(M(u_i,u_i),\varphi(u_i),\varphi(u_i)), \\
&\quad F(N(u_i,u_i),\varphi(u_i),\varphi(u_i)), \\
&\quad \frac{1}{2}F(\mu(u_i,u_i),\varphi(u_i),\varphi(u_i))\Bigg\}\Bigg), \qquad (2.2)
\end{aligned}$$

where

$$\begin{aligned}
&M(u_i,u_i) \\
&= \max\{\mu(u_i,u_i), \mu(u_i,Tu_i), \\
&\quad \mu(u_i,Tu_i), \\
&\quad \frac{\mu(u_i,Tu_i)(1+\mu(u_i,Tu_i))}{1+\mu(Tu_i,Tu_i)}, \\
&\quad \frac{\mu(u_i,Tu_i)(1+\mu(u_i,Tu_i))}{1+\mu(u_i,u_i)}\} = 0
\end{aligned}$$

and

$$\begin{aligned}
N(u_i,u_i) = \max\{&\mu(u_i,Tu_i), \\
&\frac{\mu(u_i,Tu_i)(1+\mu(u_i,Tu_i))}{1+\mu(u_i,u_i)}\} = 0.
\end{aligned}$$

Therefore, from Eq. (2.2), we have

$$F(0,\varphi(u_i),\varphi(u_i)) \leq \theta(F(0,\varphi(u_i),\varphi(u_i))),$$

which leads to a contradiction by a property of θ unless $F(0, \varphi(u_i), \varphi(u_i)) = 0$. Therefore,

$$F(0, \varphi(u_i), \varphi(u_i)) = 0, \qquad (2.3)$$

which, by the property $(F1)$ of F, implies that $\varphi(u_i) = 0$. Thus, $u_i \in F_T \cap Z_\varphi$ and hence the theorem holds trivially. Therefore, we assume that $u_{n+1} \neq u_n$, for all $n \in \mathbb{N} \cup \{0\}$. As $\mu(u_n, u_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$, using the property $(F1)$ of F, we have $F(\mu(u_n, u_{n+1}), \varphi(u_n), \varphi(u_{n+1})) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Again $\alpha(u_{n-1}, u_n) \geq 1$, for $n \geq 1$. Applying Eq. (2.1) with $u = u_{n-1}, v = u_n$, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
&F(\mu(u_n, u_{n+1}), \varphi(u_n), \varphi(u_{n+1})) \\
&= F(\mu(Tu_{n-1}, Tu_n), \varphi(Tu_{n-1}), \varphi(Tu_n)) \\
&\leq \alpha(u_{n-1}, u_n) F(\mu(Tu_{n-1}, Tu_n), \\
&\quad \varphi(Tu_{n-1}), \varphi(Tu_n)) \\
&\leq \theta\Bigg(\max\Bigg\{ F(M(u_{n-1}, u_n), \\
&\quad \varphi(u_{n-1}), \varphi(u_n)), F(N(u_{n-1}, u_n), \varphi(u_n), \\
&\quad \varphi(Tu_n)), \frac{1}{2} F(\mu(u_{n-1}, Tu_n), \varphi(u_{n-1}), \\
&\quad \varphi(Tu_n)) \Bigg\} \Bigg) \\
&= \theta\Bigg(\max\Bigg\{ F(M(u_{n-1}, \\
&\quad u_n), \varphi(u_{n-1}), \varphi(u_n)), F(N(u_{n-1}, u_n), \\
&\quad \varphi(u_n), \varphi(u_{n+1})), \frac{1}{2} F(\mu(u_{n-1}, \\
&\quad u_{n+1}), \varphi(u_{n-1}), \varphi(u_{n+1})) \Bigg\} \Bigg),
\end{aligned} \qquad (2.4)$$

where

$$\begin{aligned}
&M(u_{n-1}, u_n) \\
&= \max\{\mu(u_{n-1}, u_n), \mu(u_{n-1}, Tu_{n-1}), \\
&\quad \mu(u_n, Tu_{n-1}), \\
&\quad \frac{\mu(u_{n-1}, Tu_{n-1})(1 + \mu(u_n, Tu_n))}{1 + \mu(Tu_{n-1}, Tu_n)}, \\
&\quad \frac{\mu(u_n, Tu_{n-1})(1 + \mu(u_{n-1}, Tu_n))}{1 + \mu(u_{n-1}, u_n)} \} \\
&= \max\{\mu(u_{n-1}, u_n), \mu(u_{n-1}, u_n), \mu(u_n, u_n), \\
&\quad \frac{\mu(u_{n-1}, u_n)(1 + \mu(u_n, u_{n+1}))}{1 + \mu(u_n, u_{n+1})}, \\
&\quad \frac{\mu(u_n, u_n)(1 + \mu(u_{n-1}, u_{n+1}))}{1 + \mu(u_{n-1}, u_n)} \} \\
&= \mu(u_{n-1}, u_n)
\end{aligned}$$

and

$$\begin{aligned}
&N(u_{n-1}, u_n) \\
&= \max\{\mu(u_n, Tu_n), \\
&\quad \frac{\mu(u_n, Tu_n)(1 + \mu(u_{n-1}, Tu_{n-1}))}{1 + \mu(u_{n-1}, u_n)} \} \\
&= \max\{\mu(u_n, u_{n+1}), \\
&\quad \frac{\mu(u_n, u_{n+1})(1 + \mu(u_{n-1}, u_n))}{1 + \mu(u_{n-1}, u_n)} \} \\
&= \mu(u_n, u_{n+1}).
\end{aligned}$$

Therefore, from Eq. (2.4), we get

$$\begin{aligned}
&F(\mu(u_n, u_{n+1}), \varphi(u_n), \varphi(u_{n+1})) \\
&\leq \theta\Bigg(\max\Bigg\{ F(\mu(u_{n-1}, u_n), \\
&\quad \varphi(u_{n-1}), \varphi(u_n)), F(\mu(u_n, u_{n+1}), \varphi(u_n), \\
&\quad \varphi(u_{n+1})), \frac{1}{2} F(\mu(u_{n-1}, \\
&\quad u_{n+1}), \varphi(u_{n-1}), \varphi(u_{n+1})) \Bigg\} \Bigg).
\end{aligned}$$

It follows by the properties $(F4)$ and $(F5)$ of F and the non-decreasing property of θ

that

$$
\begin{aligned}
&F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))\\
&\le \theta\Bigg(\max\Bigg\{F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\\
&\quad \varphi(u_n)),F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1})),\\
&\quad \frac{1}{2}F(\mu(u_{n-1},u_n)+\mu(u_n,u_{n+1}),\\
&\quad \varphi(u_{n-1}),\varphi(u_{n+1}))\Bigg\}\Bigg)\\
&\le \theta\Bigg(\max\Bigg\{F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\\
&\quad \varphi(u_n)),F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1})),\\
&\quad \frac{1}{2}\Bigg(F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\varphi(u_n))+\\
&\quad F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))\Bigg)\Bigg\}\Bigg)\\
&= \theta\Bigg(\max\Bigg\{F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\\
&\quad \varphi(u_n)),F(\mu(u_n,u_{n+1}),\varphi(u_n),\\
&\quad \varphi(u_{n+1}))\Bigg\}\Bigg).
\end{aligned}
\tag{2.5}
$$

If possible, suppose that $F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\varphi(u_n)) < F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))$. Then from Eq. (2.5), we have

$$
\begin{aligned}
&F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))\\
&\le \theta(F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1})))\\
&< F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1})),
\end{aligned}
$$

which is a contradiction. Therefore, $F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1})) \le F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\varphi(u_n))$. Thus from Eq. (2.5), we get

$$
\begin{aligned}
&F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))\\
&\le \theta(F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\\
&\quad \varphi(u_n))),\ \text{for all } n\in\mathbb{N}.
\end{aligned}
$$

By repeated application of the above inequality, we have

$$
\begin{aligned}
&F(\mu(u_n,u_{n+1}),\varphi(u_n),\varphi(u_{n+1}))\\
&\le \theta(F(\mu(u_{n-1},u_n),\varphi(u_{n-1}),\varphi(u_n)))\\
&\le \theta^2(F(\mu(u_{n-2},u_{n-1}),\varphi(u_{n-2}),\varphi(u_{n-1})))\\
&\ \vdots\\
&\le \theta^n(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1))).
\end{aligned}
$$

Using the property $(F1)$ of F, we have

$$
\begin{aligned}
&0<\mu(u_n,u_{n+1})\\
&\quad \le \theta^n(F(\mu(u_0,u_1),\\
&\qquad \varphi(u_0),\varphi(u_1)))\ \ and\\
&0\le \varphi(u_n)\\
&\quad \le \theta^n(F(\mu(u_0,u_1),\\
&\qquad \varphi(u_0),\varphi(u_1))).
\end{aligned}
\tag{2.6}
$$

By a property of θ, we have $\lim_{n\to\infty}\theta^n(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1))) = 0$. Taking the limit as $n\to\infty$ in Eq. (2.6), we get

$$
\lim_{n\to\infty}\mu(u_n,u_{n+1}) = 0\ and\ \lim_{n\to\infty}\varphi(u_n) = 0.
\tag{2.7}
$$

Using Eq. (2.6), we have for $m>n$ ($m,n\in\mathbb{N}$),

$$
\begin{aligned}
&\mu(u_n,u_m)\\
&\le \mu(u_n,u_{n+1})+\mu(u_{n+1},u_{n+2})+\ldots\\
&\quad +\mu(u_{m-2},u_{m-1})+\mu(u_{m-1},u_m)\\
&\le \theta^n(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1)))\\
&\quad +\theta^{n+1}(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1)))\\
&\quad +\ldots+\theta^{m-2}(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1)))\\
&\quad +\theta^{m-1}(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1)))\\
&\le \sum_{i=n}^{m-1}\theta^i(F(\mu(u_0,u_1),\varphi(u_0),\varphi(u_1))),
\end{aligned}
\tag{2.8}
$$

which, by a property of θ, implies that $\lim_{m,n\to\infty}\mu(u_n,u_m) = 0$, that is, $\{u_n\}$ is a

Cauchy sequence in $\mathcal{H}$. As the metric space $(\mathcal{H}, \mu)$ is complete, there exists $z_0 \in \mathcal{H}$ such that $u_n \to z_0$ as $n \to \infty$.

Next we prove that z_0 is a fixed point of T and finally we will prove $z_0 \in F_T \cap Z_\varphi$. Since φ is lower semi-continuous, we have

$$\varphi(z_0) \leq \liminf_{n\to\infty} \varphi(u_n) = 0.$$

Therefore, we have $\varphi(z_0) = 0$. Since $\mathcal{H}$ is α-regular and $\{u_n\}$ is a sequence in $\mathcal{H}$ satisfying $u_n \to z_0$ as $n \to \infty$ and $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $\alpha(u_n, z_0) \geq 1$. Applying Eq. (2.1) with $u = u_n, v = z_0$,

$$\begin{aligned}
&F(\mu(u_{n+1}, Tz_0), \varphi(u_{n+1}), \varphi(Tz_0)) \\
&= F(\mu(Tu_n, Tz_0), \varphi(Tu_n), \varphi(Tz_0)) \\
&\leq \alpha(u_n, z_0) F(\mu(Tu_n, Tz_0), \\
&\quad \varphi(Tu_n), \varphi(Tz_0)) \\
&\leq \theta\Bigg(\max\Bigg\{ F(M(u_n, z_0), \varphi(u_n), \\
&\quad \varphi(z_0)), F(N(u_n, z_0), \varphi(z_0), \varphi(Tz_0)), \\
&\quad \frac{1}{2} F(\mu(u_n, Tz_0), \varphi(u_n), \varphi(Tz_0)) \Bigg\} \Bigg),
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
&M(u_n, z_0) \\
&= \max\{\mu(u_n, z_0), \mu(u_n, Tu_n), \\
&\quad \mu(z_0, Tu_n), \frac{\mu(u_n, Tu_n)(1 + \mu(z_0, Tz_0))}{1 + \mu(Tu_n, Tz_0)}, \\
&\quad \frac{\mu(z_0, Tu_n)(1 + \mu(u_n, Tz_0))}{1 + \mu(u_n, z_0)} \} \\
&= \max\{\mu(u_n, z_0), \mu(u_n, u_{n+1}), \mu(z_0, u_{n+1}), \\
&\quad \frac{\mu(u_n, u_{n+1})(1 + \mu(z_0, Tz_0))}{1 + \mu(u_{n+1}, Tz_0)}, \\
&\quad \frac{\mu(z_0, u_{n+1})(1 + \mu(u_n, Tz_0))}{1 + \mu(u_n, z_0)} \}.
\end{aligned}$$

and

$$\begin{aligned}
N(u_n, z_0) = \max\{&\mu(z_0, Tz_0), \\
&\frac{\mu(z_0, Tz_0)(1 + \mu(u_n, u_{n+1}))}{1 + \mu(u_n, z_0)} \}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(u_n, z_0) \\
&\quad = 0 \; and \; \lim_{n\to\infty} N(u_n, z_0) = \mu(z_0, Tz_0).
\end{aligned}$$

Taking the limit as $n \to \infty$ in Eq. (2.9), we get

$$\begin{aligned}
&F(\mu(z_0, Tz_0), 0, \varphi(Tz_0)) \\
&\leq \theta\Bigg(\max\Bigg\{ F(0, 0, 0), F(\mu(z_0, Tz_0), 0, \\
&\quad \varphi(Tz_0)), \frac{1}{2} F(\mu(z_0, Tz_0), 0, \varphi(Tz_0)) \Bigg\} \Bigg),
\end{aligned}$$

that is,

$$\begin{aligned}
&F(\mu(z_0, Tz_0), 0, \varphi(Tz_0)) \\
&\leq \theta(F(\mu(z_0, Tz_0), 0, \varphi(Tz_0))),
\end{aligned}$$

which, by a property of θ, is a contradiction unless $F(\mu(z_0, Tz_0), 0, \varphi(Tz_0)) = 0$. Hence, $F(\mu(z_0, Tz_0), 0, \varphi(Tz_0)) = 0$. Using the property $(F1)$ of F, we have

$$\mu(z_0, Tz_0) = 0, \text{ which implies that } z_0 = Tz_0.$$

Thus we have $z_0 = Tz_0$ and $\varphi(z_0) = 0$, that is, $z_0 \in F_T \cap Z_\varphi$.

Next we prove the uniqueness of φ-fixed point under the assumption that $\alpha(z_*, w_*) \geq 1$, whenever $z_*, w_* \in F_T$. If possible, let $z_0, w_0 \in F_T \cap Z_\varphi$. Then $Tz_0 = z_0, Tw_0 = w_0$, $\varphi(z_0) = 0 = \varphi(w_0)$ and $\alpha(z_0, w_0) \geq 1$. Applying Eq. (2.1) with $u = z_0$ and $v = w_0$

$$\begin{aligned}
&F(\mu(z_0, w_0), 0, 0) \\
&= F(\mu(z_0, w_0), \varphi(z_0), \varphi(w_0)) \\
&= F(\mu(Tz_0, Tw_0), \varphi(Tz_0), \varphi(Tw_0)) \\
&\leq \alpha(z_0, w_0) F(\mu(Tz_0, Tw_0), \\
&\quad \varphi(Tz_0), \varphi(Tw_0)) \\
&\leq \theta\Bigg(\max\Bigg\{ F(M(z_0, w_0), \varphi(z_0), \varphi(w_0)), \\
&\quad F(N(z_0, w_0), \varphi(w_0), \varphi(Tw_0)),
\end{aligned}$$

$$\frac{1}{2}F(\mu(z_0,Tw_0),\varphi(z_0),\varphi(Tw_0))\Big\}\Big), \tag{2.10}$$

where

$$\begin{aligned}
&M(z_0,w_0)\\
&=\max\{\mu(z_0,w_0),\mu(z_0,Tz_0),\mu(w_0,Tz_0),\\
&\frac{\mu(z_0,Tz_0)(1+\mu(w_0,Tw_0))}{1+\mu(Tz_0,Tw_0)},\\
&\frac{\mu(w_0,Tz_0)(1+\mu(z_0,Tw_0))}{1+\mu(z_0,w_0)}\}\\
&=\max\{\mu(z_0,w_0),0,\mu(w_0,z_0),0,\mu(w_0,z_0)\}\\
&=\mu(z_0,w_0),
\end{aligned}$$

and

$$\begin{aligned}
N(z_0,w_0)&=\max\{\mu(w_0,Tw_0),\\
&\qquad\frac{\mu(w_0,Tw_0)(1+\mu(z_0,Tz_0))}{1+\mu(z_0,w_0)}\}\\
&=0.
\end{aligned}$$

Therefore, from Eq. (2.10), we have

$$F(\mu(z_0,w_0),0,0)\le\theta(F(\mu(z_0,w_0),0,0)),$$

which, by a property of θ, leads to a contradiction unless $F(\mu(z_0,w_0),0,0)=0$. Therefore, $F(\mu(z_0,w_0),0,0)=0$, which by the property $(F1)$ of F implies that $\mu(z_0,w_0)=0$ and consequently $z_0=w_0$. This completes the proof.

□

Example 2.2. Take the complete metric space $(\mathcal{H},\mu)$, where $\mathcal{H}=[0,1]$ and $\mu(u,v)=|u-v|$, for all $u,v\in\mathcal{H}$. Let $F\in\mathcal{F}$ and $\theta:[0,\infty)\to[0,\infty)$ be defined respectively by $F(a,b,c)=a+b+c$ and $\theta(t)=\frac{4}{5}t$. Let $T:\mathcal{H}\to\mathcal{H}$ be defined by

$$Tu=\begin{cases}\frac{7}{10} & if\ 0\le u<\frac{1}{2},\\ \frac{3}{4} & if\ \frac{1}{2}\le u\le 1.\end{cases}$$

Define $\varphi:\mathcal{H}\to[0,\infty)$ and $\alpha:\mathcal{H}\times\mathcal{H}\to[0,\infty)$ respectively as follows:

$$\varphi(u)=\begin{cases}0 & if\ 0\le u\le\frac{3}{4},\\ 1 & if\ \frac{3}{4}<u\le 1.\end{cases}\ \text{and}$$

$$\alpha(u,v)=\begin{cases}\frac{1}{3} & if\ 0\le u,v<\frac{1}{2},\\ 2 & otherwise.\end{cases}$$

Clearly, the mapping $\varphi:\mathcal{H}\to[0,\infty)$ is lower semi-continuous, T is α-admissible and $\mathcal{H}$ has α-regular property. When either $0\le u<\frac{1}{2}$, $0\le v<\frac{1}{2}$ or $\frac{1}{2}\le u\le 1$, $\frac{1}{2}\le v\le 1$, then the quantity $F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv))$ becomes identically zero and hence the inequality Eq. (2.1) holds trivially. We discuss the following cases:

Case I. Let $0\le u<\frac{1}{2}$ and $\frac{1}{2}\le v\le\frac{3}{4}$. Then $Tu=\frac{7}{10}$, $Tv=\frac{3}{4}$ and $\varphi(u)=\varphi(Tu)=\varphi(v)=\varphi(Tv)=0$. In this case, we have

$$\begin{aligned}
&\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv))\\
&=\frac{1}{10},\ M(u,v)\ge\frac{1}{5}\ and\ N(u,v)\ge 0.
\end{aligned}$$

Now,

$$\begin{aligned}
&\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)),\\
&F(N(u,v),\varphi(v),\varphi(Tv)),\frac{1}{2}F(\mu(u,Tv),\\
&\varphi(u),\varphi(Tv))\Big\}\Big)\\
&=\frac{4}{5}\Big(\max\Big\{M(u,v),N(u,v),\frac{1}{2}\mu(u,\frac{3}{4})\Big\}\Big)\\
&\ge\frac{4}{25}>\frac{1}{10}\\
&=\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)).
\end{aligned}$$

Case II. Let $0\le u<\frac{1}{2}$ and $\frac{3}{4}<v\le 1$. Then $Tu=\frac{7}{10}$, $Tv=\frac{3}{4}$, $\varphi(u)=\varphi(Tu)=\varphi(Tv)=0$ and $\varphi(v)=1$. In this case, we have

$$\begin{aligned}
&\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv))\\
&=\frac{1}{10},\ M(u,v)\ge\frac{1}{4}\ and\ N(u,v)\ge 0.
\end{aligned}$$

Now,

$$\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)),\ F(N(u,v),\varphi(v),\varphi(Tv)),\ \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big) = \frac{4}{5}\Big(\max\Big\{M(u,v)+1, N(u,v)+1, \frac{1}{2}\mu(u,\frac{3}{4})\Big\}\Big) \geq 1 > \frac{1}{10} = \alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)).$$

Case III. Let $\frac{1}{2} \leq u \leq \frac{3}{4}$ and $0 \leq v < \frac{1}{2}$ and . Then $Tu = \frac{3}{4}$, $Tv = \frac{7}{10}$ and $\varphi(u) = \varphi(Tu) = \varphi(v) = \varphi(Tv) = 0$. In this case, we have

$$\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)) = \frac{1}{10},\ M(u,v) \geq \frac{1}{4} \text{ and } N(u,v) \geq \frac{1}{5}.$$

Now,

$$\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)),\ F(N(u,v),\varphi(v),\varphi(Tv)),\ \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big) = \frac{4}{5}\Big(\max\Big\{M(u,v), N(u,v), \frac{1}{2}\mu(u,\frac{7}{10})\Big\}\Big) \geq \frac{1}{5} > \frac{1}{10} = \alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)).$$

Case IV. Let $\frac{3}{4} < u \leq 1$ and $0 \leq v < \frac{1}{2}$. Then $Tu = \frac{3}{4}$, $Tv = \frac{7}{10}$, $\varphi(Tu) = \varphi(v) = \varphi(Tv) = 0$ and $\varphi(u) = 1$. In this case, we have

$$\alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)) = \frac{1}{10},\ M(u,v) \geq \frac{1}{4} \text{ and } N(u,v) \geq \frac{1}{5}.$$

Now,

$$\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)),\ F(N(u,v),\varphi(v),\varphi(Tv)),\ \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big) = \frac{4}{5}\Big(\max\Big\{M(u,v)+1, N(u,v), \frac{1}{2}(\mu(u,\frac{7}{10})+1)\Big\}\Big) \geq 1 > \frac{1}{10} = \alpha(u,v)F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)).$$

Here, all the conditions of Theorem 2.1 are satisfied, and it is seen that $\frac{3}{4}$ is the only φ-fixed point of T.

In the following, we state Theorem 2.1 in another approach.

Theorem 2.3. *Let $(\mathcal{H},\mu)$ be a complete metric space with α-regular property and $T : \mathcal{H} \to \mathcal{H}$ be an α-admissible mapping, where $\alpha : \mathcal{H}\times\mathcal{H} \to [0,\infty)$. Let $F \in \mathcal{F}$, $\theta \in \Theta$ and $\varphi : \mathcal{H} \to [0,\infty)$ be a lower semi-continuous mapping. Suppose there exists $u_0 \in \mathcal{H}$ for which $\alpha(u_0,Tu_0) \geq 1$ and for all $u,v \in \mathcal{H}$ with $\alpha(u,v) \geq 1$,*

$$F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)) \leq \theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)),\ F(N(u,v),\varphi(v),\varphi(Tv)),\ \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big), \quad (2.11)$$

where

$$M(u,v) = \max\{\mu(u,v),\mu(u,Tu),\mu(v,Tu), \frac{\mu(u,Tu)(1+\mu(v,Tv))}{1+\mu(Tu,Tv)}, \frac{\mu(v,Tu)(1+\mu(u,Tv))}{1+\mu(u,v)}\}$$

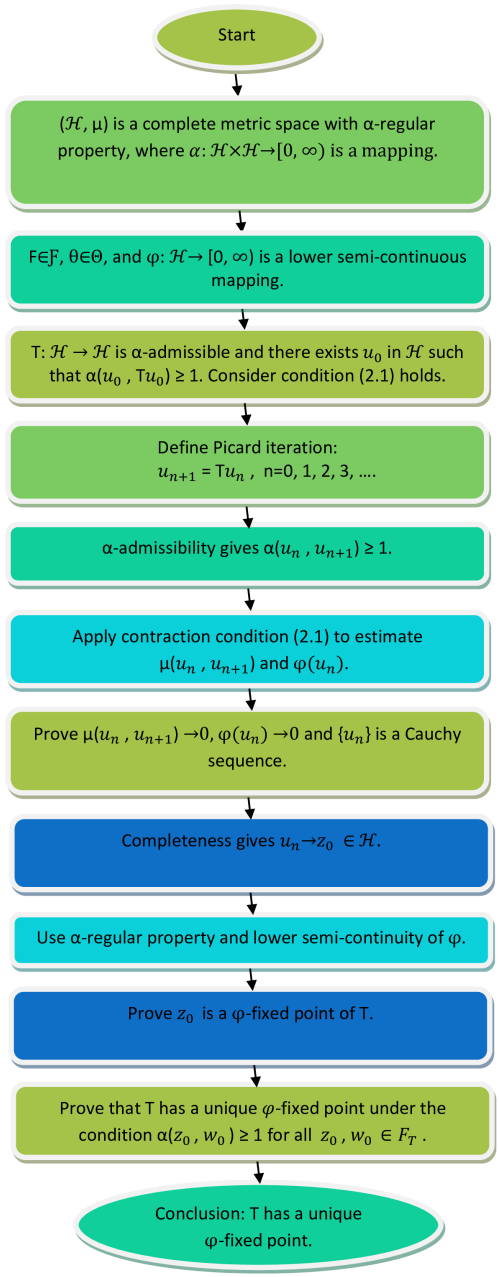


Fig. 1. Logical flowchart for the proof of Theorem 2.1.

and

$$N(u,v) = \max\{\mu(v,Tv), \frac{\mu(v,Tv)(1+\mu(u,Tu))}{1+\mu(u,v)}\}.$$

Then T has a φ-fixed point in $\mathcal{H}$. Further, if $\alpha(z_*, w_*) \geq 1$, for all $z_*, w_* \in F_T$, then the φ-fixed point is unique.

Proof. Let $u_0 \in \mathcal{H}$ such that $\alpha(u_0, Tu_0) \geq 1$. We take the same sequence $\{u_n\}$ as in the proof of Theorem 2.1. Arguing similarly to the proof of Theorem 2.1, we have $\alpha(u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. If possible, suppose that $u_{i+1} = u_i$, for some $i \in \mathbb{N} \cup \{0\}$. Then $u_i = u_{i+1} = Tu_i$, that is, u_i is a fixed point of T. As $\alpha(u_i, u_i) = \alpha(u_i, u_{i+1}) \geq 1$, applying Eq. (2.11) with $u = v = u_i$, we have

$$\begin{aligned}
&F(0, \varphi(u_i), \varphi(u_i)) \\
&= F(\mu(u_i, u_i), \varphi(u_i), \varphi(u_i)) \\
&= F(\mu(Tu_i, Tu_i), \varphi(Tu_i), \varphi(Tu_i)) \\
&\leq \theta\Bigg(\max\Bigg\{ F(M(u_i, u_i), \varphi(u_i), \varphi(u_i)), \\
&\quad F(N(u_i, u_i), \varphi(u_i), \varphi(Tu_i)), \\
&\quad \frac{1}{2} F(\mu(u_i, Tu_i), \varphi(u_i), \varphi(Tu_i)) \Bigg\} \Bigg) \\
&= \theta\Bigg(\max\Bigg\{ F(M(u_i, u_i), \varphi(u_i), \varphi(u_i)), \\
&\quad F(N(u_i, u_i), \varphi(u_i), \varphi(u_i)), \\
&\quad \frac{1}{2} F(\mu(u_i, u_i), \varphi(u_i), \varphi(u_i)) \Bigg\} \Bigg),
\end{aligned}$$

where

$$\begin{aligned}
&M(u_i, u_i) \\
&= \max\{\mu(u_i, u_i), \mu(u_i, Tu_i), \mu(u_i, Tu_i), \\
&\quad \frac{\mu(u_i, Tu_i)(1 + \mu(u_i, Tu_i))}{1 + \mu(Tu_i, Tu_i)}, \\
&\quad \frac{\mu(u_i, Tu_i)(1 + \mu(u_i, Tu_i))}{1 + \mu(u_i, u_i)} \} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&N(u_i, u_i) \\
&= \max\{\mu(u_i, Tu_i), \\
&\quad \frac{\mu(u_i, Tu_i)(1 + \mu(u_i, Tu_i))}{1 + \mu(u_i, u_i)} \} = 0.
\end{aligned}$$

It follows from the above inequality that $F(0, \varphi(u_i), \varphi(u_i)) = 0$, which by a property of F implies that $\varphi(u_i) = 0$. Therefore, $u_i \in F_T \cap Z_\varphi$. Hence, we assume that $u_{n+1} \neq u_n$, for all $n \in \mathbb{N} \cup \{0\}$. Then $F(\mu(u_n, u_{n+1}), \varphi(u_n), \varphi(u_{n+1})) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. As $\alpha(u_{n-1}, u_n) \geq 1$ $(n \geq 1)$, applying Eq. (2.11) with $u = u_{n-1}$ and $v = u_n$, and arguing similarly as in the proof of Theorem 2.1, we prove that $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}$ and T has a φ-fixed point in $\mathcal{H}$.

We now prove that the uniqueness of φ-fixed point of T under the assumption that $\alpha(z_*, w_*) \geq 1$, whenever $z_*, w_* \in F_T$. If possible, let $z_0, w_0 \in F_T \cap Z_\varphi$. Then $Tz_0 = z_0, Tw_0 = w_0$, $\varphi(z_0) = 0 = \varphi(w_0)$ and $\alpha(z_0, w_0) \geq 1$. Applying Eq. (2.11) with $u = z_0$ and $v = w_0$, arguing similarly as in the proof of Theorem 2.1, we prove that T has a unique φ-fixed point in $\mathcal{H}$. □

Example 2.4. Let $(\mathcal{H}, \mu)$ be a complete metric space, where $\mathcal{H} = [0, 1]$ and $\mu(u, v) = |u - v|$, for all $u, v \in \mathcal{H}$. Let $F \in \mathcal{F}$ and $\theta : [0, \infty) \to [0, \infty)$ be defined respectively by $F(a, b, c) = a + b + c$ and $\theta(t) = \frac{1}{3}t$. Let $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$Tu = \begin{cases} \frac{u}{4} & \text{if } 0 \leq u \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if } \frac{1}{2} < u \leq 1. \end{cases}$$

Consider $\varphi : \mathcal{H} \to [0, \infty)$, where $\varphi(u) = \frac{u}{5}$, for all $u \in \mathcal{H}$. Again, we define a function $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 1.7 & \text{if } 0 \leq u, v \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the mapping $\varphi : \mathcal{H} \to [0, \infty)$ is lower semi-continuous, T is α-admissible and $\mathcal{H}$ has α-regular property. Again $\alpha(u, v) \geq 1$ when $0 \leq u, v \leq \frac{1}{2}$. When $0 \leq u, v \leq \frac{1}{2}$, then $\varphi(u) = \frac{u}{5}$, $\varphi(Tu) = \frac{u}{20}$, $\varphi(v) = \frac{v}{5}$ and $\varphi(Tv) = \frac{v}{20}$. As a result, we have

$$F(\mu(Tu, Tv), \varphi(Tu), \varphi(Tv))$$

$$= |\frac{u}{4} - \frac{v}{4}| + \frac{u}{20} + \frac{v}{20},$$
$$M(u,v) \geq |u-v| \text{ and } N(u,v) \geq 0.$$

Now,

$$\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)), F(N(u,v), \varphi(v),\varphi(Tv)), \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big)$$
$$= \frac{1}{3}\Big(\max\Big\{F(M(u,v),\frac{u}{5},\frac{v}{5}), F(N(u,v), \frac{v}{5},\frac{v}{20}), \frac{1}{2}F(\mu(u,\frac{v}{4}),\frac{u}{5},\frac{v}{20})\Big\}\Big)$$
$$= \frac{1}{3}\Big(\max\Big\{M(u,v) + \frac{u}{5} + \frac{v}{5}, N(u,v) + \frac{v}{5} + \frac{v}{20}, \frac{1}{2}(\mu(u,\frac{v}{4}) + \frac{u}{5} + \frac{v}{20})\Big\}\Big)$$
$$\geq \frac{1}{3}\Big(\max\Big\{|u-v| + \frac{u}{5} + \frac{v}{5}, 0 + \frac{v}{5} + \frac{v}{20}, \frac{1}{2}(|u-\frac{v}{4}| + \frac{u}{5} + \frac{v}{20})\Big\}\Big)$$
$$\geq \frac{1}{3}(|u-v| + \frac{u}{5} + \frac{v}{5}) \geq \frac{1}{4}(|u-v| + \frac{u}{5} + \frac{v}{5})$$
$$= F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)).$$

Thus the inequality Eq. (2.11) holds, whenever $\alpha(u,v) \geq 1$, for all $u, v \in \mathcal{H}$. Here, all the conditions of Theorem 2.3 are satisfied, and it is seen that 0 is the only φ-fixed point of T.

Note Take $u = 0$ and $v = 1$. Then $Tu = 0$, $Tv = \frac{1}{2}$, $\varphi(u) = \varphi(Tu) = 0$, $\varphi(v) = \frac{1}{5}$ and $\varphi(Tv) = \frac{1}{10}$. Now

$$F(\mu(Tu,Tv),\varphi(Tu),\varphi(Tv)) = F(\mu(0,\frac{1}{2}),0,\frac{1}{10}) = \frac{3}{5},$$

$$M(u,v) = \max\{\mu(u,v),\mu(u,Tu),\mu(v,Tv), \frac{\mu(u,Tu)(1+\mu(v,Tv))}{1+\mu(Tu,Tv)}, \frac{\mu(v,Tu)(1+\mu(u,Tv))}{1+\mu(u,v)}\}$$
$$= \max\{\mu(0,1),\mu(0,0),\mu(1,0), \frac{\mu(0,0)(1+\mu(1,\frac{1}{2}))}{1+\mu(0,\frac{1}{2})}, \frac{\mu(1,0)(1+\mu(0,\frac{1}{2}))}{1+\mu(0,1)}\} = 1$$

and

$$N(u,v) = \max\{\mu(v,Tv), \frac{\mu(v,Tv)(1+\mu(u,Tu))}{1+\mu(u,v)}\}$$
$$= \max\{\mu(1,\frac{1}{2}), \frac{\mu(1,\frac{1}{2})(1+\mu(0,0))}{1+\mu(0,1)}\} = \frac{1}{2}.$$

Therefore,

$$\theta\Big(\max\Big\{F(M(u,v),\varphi(u),\varphi(v)), F(N(u,v), \varphi(v),\varphi(Tv)), \frac{1}{2}F(\mu(u,Tv),\varphi(u),\varphi(Tv))\Big\}\Big)$$
$$= \frac{1}{3}\Big(\max\Big\{1+0+\frac{1}{5}, \frac{1}{2}+\frac{1}{5}+\frac{1}{10}, \frac{1}{2}(\frac{1}{2}+0+\frac{1}{10})\Big\}\Big) = \frac{2}{5} < \frac{3}{5}.$$

Thus the inequality Eq. (2.11) is not satisfied for $u = 0$ and $v = 1$. In Example 2.4, it is seen that the inequality Eq. (2.11) is not satisfied throughout the whole space. So, the use of α-admissibility in Theorem 2.3 established that the contraction condition is not required to be satisfied throughout the whole space.

Remark 2.5. *If the third and fifth components from $M(u,v)$ are extracted and placed within $N(u,v)$, then both the Theorems* 2.1-2.3 *are remain valid.*

3. Well-Posedness Property

In this section, we discuss the well-posedness property of φ-fixed point of the following problem.

Problem (H): Let $(\mathcal{H}, \mu)$ be a metric space. Let $T : \mathcal{H} \to \mathcal{H}$ and $\varphi : \mathcal{H} \to [0, \infty)$ be two mappings. We consider the problem of finding a φ-fixed point of T, that is, the problem of finding $u \in \mathcal{H}$ such that $Tu = u$ and $\varphi(u) = 0$.

Definition 3.1. ***Problem (H)*** is called well-posed if the following conditions hold:

- T has a unique φ-fixed point z_0 in $\mathcal{H}$;
- $\mu(u_n, z_0) \to 0$ as $n \to \infty$, whenever $\{u_n\}$ is a sequence in $\mathcal{H}$ such that $\{\mu(u_n, z_0)\}$ and $\{\mu(Tu_n, z_0)\}$ are bounded and $\mu(u_n, Tu_n) \to 0$, $\varphi(u_n) \to 0$ as $n \to \infty$.

In the next theorem, we take the following condition to ensure the well-posedness of **Problem (H)** via α-admissible mapping.

(A_1) : If z_0 is any solution of **Problem (H)** and $\{u_n\}$ is any sequence in $\mathcal{H}$ with $\mu(u_n, Tu_n) \to 0$ and $\varphi(u_n) \to 0$ as $n \to \infty$, then $\alpha(u_n, z_0) \geq 1$, for all $n \in \mathbb{N}$.

Theorem 3.2. *In addition to the hypothesis of Theorem* 2.1*, suppose that* $F(a, 0, 0) = a$*, for all* $a \in [0, \infty)$ *and the assumption* (A_1) *hold. Then* **Problem (H)** *is well-posed.*

Proof. By Theorem 2.1, T has a unique φ-fixed point z_0 in $\mathcal{H}$. Then $Tz_0 = z_0$ and $\varphi(z_0) = 0$. Let $\{u_n\}$ be a sequence in $\mathcal{H}$ such that $\mu(u_n, Tu_n) \to 0$ and $\varphi(u_n) \to 0$ as $n \to \infty$. By the assumption (A_1), we have $\alpha(u_n, z_0) \geq 1$, for all n. Using Eq. (2.1), we have

$$
\begin{aligned}
&\mu(u_n, z_0) \\
&\leq \mu(u_n, Tu_n) + \mu(Tu_n, z_0) \\
&\leq \mu(u_n, Tu_n) + F(\mu(Tu_n, Tz_0), \\
&\quad \varphi(Tu_n), \varphi(Tz_0)) \\
&\leq \mu(u_n, Tu_n) + \alpha(u_n, z_0) F(\mu(Tu_n, Tz_0), \\
&\quad \varphi(Tu_n), \varphi(Tz_0)) \\
&\leq \mu(u_n, Tu_n) + \theta\Bigg(\max\Bigg\{ F(M(u_n, z_0), \\
&\quad \varphi(u_n), \varphi(z_0)), F(N(u_n, z_0), \varphi(z_0), \varphi(Tz_0)), \\
&\quad \frac{1}{2} F(\mu(u_n, Tz_0), \varphi(u_n), \varphi(Tz_0)) \Bigg\} \Bigg),
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
&M(u_n, z_0) \\
&= \max\{\mu(u_n, z_0), \mu(u_n, Tu_n), \mu(z_0, Tu_n), \\
&\quad \frac{\mu(u_n, Tu_n)(1 + \mu(z_0, Tz_0))}{1 + \mu(Tu_n, Tz_0)}, \\
&\quad \frac{\mu(z_0, Tu_n)(1 + \mu(u_n, Tz_0))}{1 + \mu(u_n, z_0)} \} \\
&= \max\{\mu(u_n, z_0), \mu(u_n, Tu_n), \mu(Tu_n, z_0), \\
&\quad \frac{\mu(u_n, Tu_n)}{1 + \mu(Tu_n, z_0)}, \\
&\quad \frac{\mu(Tu_n, z_0)(1 + \mu(u_n, z_0))}{1 + \mu(u_n, z_0)} \},
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&N(u_n, z_0) \\
&= \max\{\mu(z_0, Tz_0), \\
&\quad \frac{\mu(z_0, Tz_0)(1 + \mu(u_n, Tu_n))}{1 + \mu(u_n, z_0)} \}.
\end{aligned}
\tag{3.3}
$$

Taking the limit supremum in Eqs. (3.2)-(3.3), we get

$$
\begin{aligned}
&\limsup_{n \to \infty} M(u_n, z_0) = \max\{r, 0, s, 0, s\} \\
&= \max\{r, s\} \text{ and} \\
&\limsup_{n \to \infty} N(u_n, z_0) = 0,
\end{aligned}
$$

where $r = \limsup_{n \to \infty} \mu(u_n, z_0) < \infty$ and $s = \limsup_{n \to \infty} \mu(Tu_n, z_0) < \infty$. Taking the limit

supremum in Eq. (3.1), we have

$$r \le s \le \theta\Big(\max\Big\{F(\max\{r,s\},0,0), F(0,0,0), \frac{1}{2}F(r,0,0)\Big\}\Big) = \theta(\max\{r,s\}),$$

which leads to a contradiction by a property of θ unless $r = s = 0$. Therefore $r = 0$, i.e., $\limsup_{n\to\infty} \mu(u_n, z_0) = 0$ and hence $\lim_{n\to\infty} \mu(u_n, z_0) = 0$. This completes the proof. □

Theorem 3.3. *In addition to the hypothesis of Theorem* 2.3*, suppose that* $F(a,0,0) = a$*, for all* $a \in [0,\infty)$ *and the assumption* (A_1) *hold. Then* **Problem (H)** *is well-posed.*

Proof. The proof is similar as that of Theorem 3.2. □

4. Application

In this section, we present an application of one of our main theorems to a problem of functional equations. It is well known that functional equations appear in different domains of mathematics like [21–23]. For a detail study on functional equations we refer to books [21, 24]. Further, more contemporary research on this topic is noted in [17, 25, 26].

Let $B(S)$ denotes be the set of all real valued bounded functions defined on a Banach space S, that is, $x \in B(S)$ if and only if $x : S \to \mathbb{R}$ is a bounded function. We consider the following functional equation

$$x(\kappa) = \sup_{\mu\in D}\{f(\kappa,\ \mu) + g(\kappa,\ \mu,\ x(\eta(\kappa,\ \mu)))\}, \tag{4.1}$$

where $\eta : S \times D \to S$, $f : S \times D \to \mathbb{R}$ and $g : S \times D \times \mathbb{R} \to \mathbb{R}$, S and D are Banach spaces under some suitable conditions on f and g. Such types of equations find their application in mathematical optimization, computer programming and in dynamic programming, which gives tools for solving boundary value problems arising in engineering and physical sciences.
It is well known that $B(S)$ endowed with the metric

$$d(x,\ y) = \sup_{\kappa\in S} |\ x(\kappa) - y(\kappa)\ |, \quad \text{for all } x,\ y \in B(S), \tag{4.2}$$

is a complete metric space. Also assume that this metric space endowed with a partial ordered relation '$\prec$'. Let in $B(S)$, the relation $x \prec y$ holds if $x(t) \le y(t)$, for all $t \in S$. Let us define a mapping $\alpha : B(S)\times B(S) \to [0,\ \infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \prec y, \\ 0 & \text{otherwise.} \end{cases}$$

Now we designate the following assumptions by F_1, F_2 and F_3 :

F_1 : $f : S \times D \to \mathbb{R}$ and
$g : S \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions.

F_2 : for $r_1,\ r_2 \in \mathbb{R}$, with $r_1 \le r_2$ and
$\kappa \in S,\ \mu \in D$ imply
$0 \le g(\kappa,\ \mu,\ r_2) - g(\kappa,\ \mu,\ r_1)$
$\le \xi\ [r_2 - r_1]$, where $\xi \in [0,\ 1)$.

F_3 : there exists a $x_0 \in B(S)$ such that
for all $\mu \in D$
$x_0(\kappa) \le f(\kappa,\ \mu) + g(\kappa,\ \mu,\ x_0(\eta(\kappa,\ \mu)))$,
where $\kappa \in S$.

Theorem 4.1. *Let* $(X,\ d) = (B(S),\ d)$*. Assume that* f *and* g *satisfy all the assumptions* F_1, F_2 *and* F_3*. Then the functional Eq. (4.1) has a unique solution.*

Proof. Define a mapping $F : B(S) \to B(S)$ by

$$F(\omega)(\kappa) = \sup_{\mu \in D}\{f(\kappa,\ \mu) + g(\kappa,\ \mu,\ \omega(\eta(\kappa,\ \mu)))\},$$
$$\text{where } \kappa \in S,\ \ \omega \in B(S). \tag{4.3}$$

By assumption F_1, the mapping $F : X \to X$ given by Eq. (4.3) is well defined.

Let $\{x_n\}$ be a increasing sequence in $B(S)$ with limit $x \in B(S)$ and $\alpha(x_n,\ x_{n+1}) \geq 1$, then by definition of α, we have $x_n \prec x_{n+1}$, for all n. As $x_n \to x$, we have $x_n \prec x$, for all n. The definition of α implies that $\alpha(x_n, x) \geq 1$, for all n. Therefore $B(S)$ has α- regular property.

Let $x,\ y \in B(S)$ with $\alpha(x, y) \geq 1$. Then $x(s) \leq y(s)$, hence by the assumption F_2, we have $g(\kappa,\ \mu,\ x(\eta(\kappa,\ \mu))) \leq g(\kappa,\ \mu,\ y(\eta(\kappa,\ \mu)))$. Therefore,

$$\begin{aligned} &F(x)(\kappa)\\ &= \sup_{\mu \in D}\{f(\kappa,\ \mu) + g(\kappa,\ \mu,\ x(\eta(\kappa,\ \mu)))\}\\ &\leq \sup_{\mu \in D}\{f(\kappa,\ \mu) + g(\kappa,\ \mu,\ y(\eta(\kappa,\ \mu)))\}\\ &= F(y)(\kappa),\ \text{for all } \kappa \in S, \end{aligned} \tag{4.4}$$

which implies $Fx \prec Fy$. From definition of α we have $\alpha(Fx, Fy) \geq 1$ and hence F is α-admissible.

Let $x,\ y \in B(S)$ and $\alpha(x,\ y) \geq 1$ holds. By definition of α, we have $x \prec y$, i.e., $x(s) \leq y(s)$. From definition of F, for $x,\ y \in B(S)$ and an arbitrary $\epsilon > 0$, there exist $\mu_1,\ \mu_2 \in D$ such that

$$F(x)(\kappa) < f(\kappa,\ \mu_1) + g(\kappa,\ \mu_1,\ x(\eta(\kappa,\ \mu_1))) + \epsilon$$

and

$$F(y)(\kappa) < f(\kappa,\ \mu_2) + g(\kappa,\ \mu_2,\ y(\eta(\kappa,\ \mu_2))) + \epsilon.$$

Again,

$$F(x)(\kappa) \geq f(\kappa,\ \mu_2) + g(\kappa,\ \mu_2,\ x(\eta(\kappa,\ \mu_2)))$$

and

$$F(y)(\kappa) \geq f(\kappa,\ \mu_1) + g(\kappa,\ \mu_1,\ y(\eta(\kappa,\ \mu_1))).$$

From the first and fourth inequalities, we get

$$\begin{aligned} &F(x)(\kappa) - F(y)(\kappa)\\ &\leq g(\kappa,\ \mu_1,\ x(\eta(\kappa,\ \mu_1))\\ &\quad - g(\kappa,\ \mu_1,\ y(\eta(\kappa,\ \mu_1))) + \epsilon \end{aligned}$$

and also from the second and third inequalities, we get

$$\begin{aligned} &F(y)(\kappa) - F(x)(\kappa)\\ &\leq g(\kappa,\ \mu_2,\ y(\eta(\kappa,\ \mu_2))\\ &\quad - g(\kappa,\ \mu_2,\ x(\eta(\kappa,\ \mu_2))) + \epsilon. \end{aligned}$$

So, using (F_2), we have

$$\begin{aligned} &F(x)(\kappa) - F(y)(\kappa)\\ &\leq [x(\eta(\kappa,\ \mu_1)) - y(\eta(\kappa,\ \mu_1))] + \epsilon, \end{aligned}$$

and

$$\begin{aligned} &F(y)(\kappa) - F(x)(\kappa)\\ &\leq [y(\eta(\kappa,\ \mu_2)) - x(\eta(\kappa,\ \mu_2))] + \epsilon. \end{aligned}$$

As, $\kappa \in S,\ \mu_1 \in D,\ \mu_2 \in D, \eta : S \times D \to S$ be a mapping, so $\eta(\kappa,\ \mu_1) = \kappa_1 \in S$ and $\eta(\kappa,\ \mu_2) = \kappa_2 \in S$, for some $\kappa_1, \kappa_2 \in S$. Therefore, we have

$$\begin{aligned} &F(x)(\kappa) - F(y)(\kappa)\\ &\leq \xi\ [x(\kappa_1) - y(\kappa_1)] + \epsilon\\ &\leq \xi \sup_{\kappa \in S} |\ x(\kappa) - y(\kappa)\ | + \epsilon\\ &= \xi\ d(x,\ y) + \epsilon. \end{aligned}$$

and

$$\begin{aligned} &F(y)(\kappa) - F(x)(\kappa)\\ &\leq \xi\ |y(\kappa_2) - x(\kappa_2)| + \epsilon\\ &\leq \xi \sup_{\kappa \in S} |\ y(\kappa) - x(\kappa)\ | + \epsilon\\ &= \xi\ d(x,\ y) + \epsilon. \end{aligned}$$

It follows that

$$|F(x)(\kappa) - F(y)(\kappa)| = \xi\, d(x,\, y) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, taking the supremum in both sides in the above for all $k \in S$, we obtain

$$d(Fx,\, Fy) = \sup_{\kappa \in S}\{|\, F(x)(\kappa) - F(y)(\kappa)\,|\} \leq \xi\, d(x,\, y).$$

Let us take the mapping $\Gamma : [0,\, \infty)^3 \to [0,\, \infty)$ as $\Gamma(a,\, b,\, c) = a + b$, for all $(a,\, b,\, c) \in [0,\, \infty)^3$, then clearly $\Gamma \in \mathcal{F}$. Let $\varphi : B(S) \to [0,\, \infty)$ be defined as $\varphi(x) = 0$, for all $x \in B(S)$ and $\theta : [0,\, \infty) \to [0,\, \infty)$ be defined as $\theta(t) = \xi\, t$, for all $t \in [0,\, \infty)$ and $\xi \in [0,\, 1)$. Then clearly $\sum_{n=0}^{\infty} \theta^n(t) = \sum_{n=0}^{\infty} \xi^n t = \dfrac{t}{1-\xi} < \infty$.

Therefore, we have $[d(Fx,\, Fy) + \varphi(Fx)] \leq \xi\, [d(x,\, y) + \varphi(x)] = \theta(F(d(x,\, y),\, \varphi(x),\, \varphi(y)))$.

From definition of Γ, we have for $x,\, y \in B(S)$ with $\alpha(x,\, y) \geq 1$,

$$\Gamma(d(Fx,\, Fy),\, \varphi(Fx),\, \varphi(Fy)) \leq \theta\Big(\max\{\Gamma(M(x,\, y),\, \varphi(x),\, \varphi(y)),\, \Gamma(N(x,\, y),\, \varphi(y),\, \varphi(Ty)),\, \frac{1}{2}\Gamma(d(x,\, Ty),\, \varphi(x),\, \varphi(Ty))\}\Big).$$

Hence Eq. (2.11) of Theorem 2.3 is satisfied.

By F_3 there exists $x_0 \in X$ such that $x_0(\kappa) \leq f(\kappa,\, \mu) + g(\kappa,\, \mu,\, x_0(\eta(\kappa,\, \mu)))$, where $\kappa \in S$, which implies that $x_0(\kappa) \leq \sup_{\mu \in D} f(\kappa,\, \mu) + g(\kappa,\, \mu,\, x_0(\eta(\kappa,\, \mu)))$, for all $\kappa \in S$. This implies that there exists $x_0 \in X$ such that $\alpha(x_0,\, Fx_0) \geq 1$. Therefore, all the assumptions of Theorem 2.3 are satisfied. Therefore, Theorem 2.3 applies to F, which guarantees the existence of a φ-fixed point w in X. That is, w is a solution of the functional Eq. (4.1) in $B(S)$.

5. Conclusion

In this work, the main findings are established in the domain of φ-fixed point theory by employing α-admissibility conditions. The elegance of the α-admissibility condition is that the contraction condition is not necessarily required to be satisfied throughout the whole space. Using these conditions, further new results are supposed to be established in the recently emerging domain of φ-fixed point theory. One of our main theorems is applied to finding the solution of a functional equation. Similar such applications are possible with the development of this line of research. We are hopeful that many classical results in metric fixed point theory may be generalized as more general spaces, like b-metric spaces, partial metric spaces, G-metric spaces, S-metric spaces, by employing the method introduced herein in the future.

Acknowledgement

The second author acknowledges UGC for supporting him as SRF (NTA Ref. No.: 211610073023). The authors sincerely thank the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality and clarity of this manuscript.

References

[1] Jleli M, Samet B, Vetro C. Fixed point theory in partial metric spaces via ϕ-fixed point's concept in metric spaces. Journal of Inequalities and Applications.

2014;2014(1):1-9.

[2] Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fundamenta mathematicae. 1922;3(1):133-81.

[3] Fan Y, Zhu C, Wu Z. Some ϕ-coupled fixed point results via modified F-control function's concept in metric spaces and its applications. Journal of Computational and Applied Mathematics. 2019;349:70-81.

[4] Işık H, Sezen MS, Vetro C. ϕ-Best proximity point theorems and applications to variational inequality problems. Journal of Fixed Point Theory and Applications. 2017;19(4):3177-89.

[5] Kumrod P, Sintunavarat W. A new contractive condition approach to ϕ-fixed point results in metric spaces and its applications. Journal of Computational and Applied Mathematics. 2017;311:194-204.

[6] Roy S, Saha P, Choudhury BS. Extensions of ϕ-Fixed Point Results via w-Distance. Sahand Communications in Mathematical Analysis. 2024;21(2):219-34.

[7] Samet B, Vetro C, Vetro P. Fixed point theorems for α–ψ-contractive type mappings. Nonlinear Analysis. 2012;75(4):2154-65.

[8] Budhia L, Aydi H, Ansari AH, Gopal D. Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations. Nonlinear Analysis: Modelling and Control. 2020;25(4):580-97.

[9] Choudhury BS, Bhaskar TG, Metiya N, Kundu S. Existence and stability of coupled fixed point sets for multivalued mappings. Fixed Point Theory. 2021;22(2):571-86.

[10] Lakzian H, Gopal D, Sintunavarat W. New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations. Journal of Fixed Point Theory and Applications. 2016;18:251-66.

[11] Metiya N, Choudhury BS, Kundu S. Fixed point sets of multivalued contractions and stability analysis. Communications in Advanced Mathematical Sciences. 2018;1(2):163-71.

[12] Dass BK, Gupta S. An extension of Banach contraction principle through rational expression. Indian Journal of Pure and Applied Mathematics. 1975;6(12):1455-8.

[13] Almalki Y, Radhakrishnan B, Jayaraman U, Tamilvanan K. Some common fixed point results in modular ultra- metric space using various contractions and their application to well-posedness. Mathematics. 2023;11(19):4077.

[14] Kumar DR, Pitchaimani M. Set-valued contraction mappings of Prešić–Reich type in ultrametric spaces. Asian-European Journal of Mathematics. 2017;10(04):1750065.

[15] Kumar M, Kumar P, Mutlu A, Ramaswamy R, Abdelnaby OAA, Radenović S. Ulam–Hyers stability and well-posedness of fixed point problems in C^*-algebra valued bipolar b-metric spaces. Mathematics. 2023;11(10).

[16] Sofonea M, Tarzia DA. Convergence criteria, well-posedness concepts and applications. Annals: Series on Mathematics & its Applications. 2023;15.

[17] Choudhury BS, Metiya N, Kundu S. Existence, uniqueness and well-posedness results for relation theoretic coupled fixed points problem using C-class function with some consequences and an application. The Journal of Analysis. 2021;29(1):227-45.

[18] Liu Z, Dong H, Kang SM. Solving a class of functional equations using fixed point theorems. Journal of Inequalities and Applications. 2013;2013(1):516.

[19] Wang J, Zhu C, Fečkan M. Solvability of fully nonlinear functional equations involving Erdélyi-Kober fractional integrals on the unbounded interval. Optimization. 2014;63(8):1235-48.

[20] Choudhury BS, Metiya N, Kundu S. Existence, error estimation, rate of convergence, Ulam-Hyers stability, well-posedness and limit shadowing property related to a fixed point problem. Boletim da Sociedade Paranaense de Matematica. 2022;40:1-10.

[21] Bellman R. Dynamic Programming. Princeton University Press; 1957.

[22] Liu Z, Xu Y, Ume JS, Kang SM. Solutions to two functional equations arising in dynamic programming. Journal of Computational and Applied Mathematics. 2006;192(2):251-69.

[23] Petryshyn WV. Approximation-solvability of nonlinear functional and differential equations. CRC Press; 2017.

[24] Kuczma M. An introduction to the theory of functional equations and inequalities: Cauchy's equation and Jensen's inequality. Springer; 2009.

[25] Saha P, Mondal P, Choudhury BS. Application of contraction mapping principle to the Hyers-Ulam-Rassias stability problem of pexiderized quadratic functional equation in modular spaces. Acta Mathematica Univer- sitatis Comenianae. 2025;94(1):7-20.

[26] Saha P, Samanta TK, Mondal P, Choudhury BS. Stability of two variable pexiderized quadratic functional equation in intuitionistic fuzzy Banach spaces. Proyecciones (Antofagasta). 2019;38(3):447-68.